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A CONSTANT CHROMOMAGNETIC FIELD LEADS NOWHERE^{*)}

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ABSTRACT

The computation of the effective action for classically unstable field configurations such as a constant chromomagnetic field is shown to be a completely non-perturbative problem. We show in particular that such configurations do not dominate in any sense the functional integral. Ultra-violet properties of the effective action are examined in detail. We do not find any anomalous scaling behaviour in contrast to a claim recently appeared in the literature.

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1. - Introduction

A constant and static chromomagnetic field is a non trivial classical solution of the Yang-Mills equations of motion. As such, it is a very interesting object to study, because it could provide the leading approximation to a quantum state which may lie lower in energy than the trivial perturbative vacuum state, $A_\mu = 0$.

Early investigation of this problem⁽¹⁾ led to the result that the effective potential, computed at the one loop level, shows a minimum at a non vanishing value of the chromomagnetic field thus suggesting that quantum fluctuations could indeed generate spontaneously a constant magnetic field, in analogy to the Goldstone phase in a scalar theory. Subsequent analysis⁽²⁾, however, showed that the situation is more complicated. Although a solution of the equations of motion, a constant chromomagnetic field is classically unstable and there is a whole class of quantum runaway fluctuations, corresponding to modes with negative frequency in the quadratic approximation around the classical solution itself. Thus the problem becomes intrinsically non perturbative : the existence of unstable modes forbids the small fluctuation approximation in the functional integral around the classical solution.

In the series of papers of ref.(2), the conjecture has been advanced that the non-perturbative treatment could be restricted to the functional space spanned by the unstable modes only. To this aim, one writes down the classical action, in the presence of the external chromomagnetic field, restricted to the unstable modes only, and one tries to find the configuration which minimizes this restricted action. A number of interesting field configurations have been studied, all of which, however, correspond simply to stationary configurations and can be shown to be still unstable. Nonetheless, it is clear that finding the absolute minimum of

the restricted action is a much simpler problem than solving the full Yang-Mills equations of the motion, and in fact an, at least qualitative, understanding of this minimum configuration can be provided⁽²⁾.

From a completely different point of view, the case of a constant chromomagnetic field has been reconsidered in ref.(3). These authors compute the expectation value of the hamiltonian for a class of trial vacuum functionals, using the amplitudes of the unstable modes as variational parameters, while leaving the dependence from the stable mode amplitudes as dictated by perturbation theory. The minimum of the energy with respect to the strength of the chromomagnetic field is obtained for a non trivial value, H^* , a result analogous to the Saviddy one. It is claimed, however, that gH^* does not scale as expected from renormalization group arguments, when the ultraviolet cut-off Λ , needed to regularize the theory, goes to infinity. They find:

$$gH^* = (\text{const}) \Lambda^2 e^{-1/2bg} \quad (1.1)$$

where g is the bare QCD coupling constant, while the renormalization group prediction is:

$$gH^* = (\text{const}) \Lambda^2 e^{-1/bg} \quad (1.2)$$

In both equations, b is the one loop coefficient of the β -function, i.e.

$$1/g_R^2 = 1/g^2 - b \ln(\Lambda^2/gH) \quad (1.3)$$

and the renormalized coupling, g_R , is defined at an energy scale of order gH .

The result in eq.(1.1) would signal a severe pathology in the pure Yang-Mills gauge theory.

Prompted by the claim made in ref.(3), we have reexamined the whole issue, with the results presented in this paper.

Before explaining the results, we need to introduce some notations. We denote by $S(A_\mu)$ the classical, Yang-Mills action, and by $H_\mu(x)$ the vector potential describing the constant chromomagnetic field, H . For a general field configuration, we write (color indices are always understood unless explicitly needed):

$$A_\mu = H_\mu + \eta_\mu = H_\mu + \eta_\mu^{(u)} + \eta_\mu^{(s)} \quad (1.4)$$

where the suffixes label the components over the unstable (u) and stable (s) modes, corresponding to negative or positive eigenvalues of the quadratic part of the action:

$$\begin{aligned} S(H_\mu + \eta_\mu) &= S(H_\mu) + \\ 1/2 \int dx dy \eta_\mu(x) \mathcal{Q}_{\mu\nu}(x,y,H) \eta_\nu(y) &+ \\ &+ O(\eta^3) \end{aligned} \quad (1.5)$$

We shall use throughout this paper the background gauge⁽⁴⁾ defined by H_μ , and denote by $\Gamma(H)$ the corresponding effective potential.

Our results are both deluding and reassuring.

i) - We discuss the construction of $\Gamma(\varphi)$ for those field configurations which are unstable extrema of the classical action. For the case of stable configurations, the functional integral which appears in the definition of $\Gamma(\varphi)$ is dominated by field configurations which differ by φ to order $\hbar$. At variance with this, we point out that in the unstable case the relevant field configurations differ from φ by amounts which cannot be controlled by any small parameter. This makes a reliable calculation of $\Gamma(\varphi)$ a full-fledged non perturbative one, well outside the present capabilities.

ii) - Following the suggestion of ref.(2) we have explored the possibility that the field configurations relevant to describe the QCD vacuum differ from H_μ only in the unstable-mode manifold. Upon integration over the stable modes with the saddle point method, we find that the remaining effective action does not reduce to the restriction of the original action in the unstable sector. Rather, it contains additional non local terms, corresponding to tree diagrams where s-modes are propagated between vertices with external u-modes

$$S_{\text{eff}}(\eta_\mu^{(u)}) = S(H_\mu + \eta_\mu^{(u)}) + \text{Non-local terms} \quad (1.6)$$

With respect to the strategy outlined in ref.(2) we conclude that the minimum of the restricted action, $S(H_\mu + \eta_\mu^{(u)})$, is of no help. More generally, we find that any classical solution of the equations of motion can be relevant (in the limit $\hbar \rightarrow 0$) only if it corresponds to a stable minimum of the full action. Unfortunately, as proved in the Appendix, there are no solutions of the Yang-Mills equations of the form $H_\mu + \eta_\mu^{(u)}$, for non vanishing $\eta_\mu^{(u)}$. We want to stress that the authors of ref.(2) have been very cautious in underlining that their suggestion could fail, due to the unfavourable feedback arising from the needed readjustment of the "stable" sector. We find their caveats quite justified.

iii) - In spite of the difficulty mentioned in (i), if one relies on the (crucial) assumption that the ultraviolet divergences of the full theory are the same as those found in the perturbative expansion, one can display quite simply the ultraviolet properties of the effective potential. For the case under study, we find the form:

$$\frac{\Gamma(H)}{VT} = \frac{1}{2} (gH)^2 f[g_R(gH)] \quad (1.7)$$

where $g_R(gH)$ is the renormalized coupling constant, computed at the scale gH . Although we cannot tell whether $\Gamma(H)$ has a minimum for a non-vanishing H^* , an immediate consequence of eq.(1.7) is the scaling law (1.2).

iv) - To further clarify the issue of the ultraviolet structure of $\Gamma(H)$, we have examined the result of the integration over the stable modes, which can be done perturbatively for small values of the u-mode amplitude, $\eta_r^{(u)}$. The resulting effective action expressed as a power expansion in $\eta_r^{(u)}$ displays a pattern of UV divergences, so as to agree with eq.(1.2). The further integration over $\eta_\mu^{(u)}$, which cannot be done explicitly, has a natural ultraviolet cut-off gH , and is not expected to introduce any further divergence. Of course this is nothing more than an exercise, since we have no control over possible non perturbative ultraviolet divergences, which cannot be seen in the power expansion. However the exercise is interesting in that it closely parallels the approximations needed in ref.(3) to perform the variational calculation. Given the equivalence of the approximations involved in the two analyses, we think that the variational calculation should be carefully reconsidered.

In conclusion, our analysis shows that a classically unstable solution, such as a constant chromomagnetic field,

is really useless. Not only it does not dominate the vacuum, but it does not play any important role even in the determination of the effective potential for the unstable configuration itself.

2. -Classically unstable field configurations: effective potential and relevance for the vacuum structure

We consider here the construction of the effective potential for field configurations which are extrema but not local minima of the classical action. For simplicity of notation, we use the symbol φ to denote collectively the relevant fields, with a classical action $S(\varphi)$. We denote by φ_c an unstable solution of the classical equations of motion:

$$\left. \frac{\delta S}{\delta \varphi(x)} \right|_{\varphi=\varphi_c} = 0 \quad (2.1)$$

with:

$$\left. \frac{\delta^2 S}{\delta \varphi(x) \delta \varphi(y)} \right|_{\varphi=\varphi_c} \equiv \mathcal{O}(x,y) \quad (2.2)$$

$\mathcal{O}$ being not positive definite, we define the stable and unstable modes of $\mathcal{O}$ according to:

$$\begin{aligned} \int dy \mathcal{O}(x,y) \eta_N^{(u)}(y) &= \lambda_N^{(u)} \eta_N^{(u)}(x) & \lambda_N^{(u)} < 0 \\ \int dy \mathcal{O}(x,y) \eta_P^{(s)}(y) &= \lambda_P^{(s)} \eta_P^{(s)}(x) & \lambda_P^{(s)} > 0 \end{aligned} \quad (2.3).$$

The normalized stable and unstable eigenfunctions of $\mathcal{O}$ induce a natural decomposition of the functional space of the fluctuations around φ_c . Setting

$$\varphi = \varphi_c + \eta \quad (2.4)$$

we decompose η according to:

$$\begin{aligned} \eta &= \eta^{(u)} + \eta^{(s)} = \\ &= \sum_N C_N \eta_N^{(u)}(x) + \sum_P C_P \eta_P^{(s)}(x) \end{aligned} \quad (2.5)$$

The measure of functional integration can be correspondingly split as follows:

$$\mathcal{D}\eta = \prod_N dC_N \prod_P dC_P \equiv \mathcal{D}\eta^{(u)} \mathcal{D}\eta^{(s)} \quad (2.6)$$

For the particular case where φ_c represents a constant chromomagnetic field, a complete classification of the stable and unstable modes can be given, corresponding to the well known Landau levels for charged particles in a magnetic field. The relevant formulae are reported in the Appendix.

To define the effective potential, one starts with the connected Green-function generating functional, $W(J)$:

$$e^{-\frac{1}{\hbar} W(J)} = \int \mathcal{D}\varphi e^{-\frac{1}{\hbar} [S(\varphi) - \int J\varphi dx]} \quad (2.7)$$

$\Gamma(\varphi_c)$ is defined⁽⁵⁾ as the Legendre transform of $W(J)$, which in general is given by:

$$\Gamma(\varphi_c) = \min_{\{J\}} \left[W(J) + \int J \varphi_c dx \right] \quad (2.8)$$

the minimum being taken in J at fixed φ_c . For stable configurations, the definition (2.8) reduces to the usual one, where the minimum is achieved for a value of J such that:

$$\frac{\delta W}{\delta J(x)} = -\varphi_c(x) \quad (2.9)$$

For unstable configurations, eq.(2.9) has in general many solutions, and eq.(2.8) instructs us to take the one corresponding to the absolute minimum.

In the case where $S(\varphi)$ is a convex functional, the classical limit of $W(J)$ poses no problem. In the limit $\hbar \rightarrow 0$, the integral in eq.(2.7) is dominated by the solution of the minimum equation:

$$\left. \frac{\delta S}{\delta \varphi(x)} \right|_{\varphi=\varphi_J} - J(x) = 0 \quad (2.10)$$

and the minimum in eq.(2.8) is obtained for a value of J such that:

$$\varphi_J = \varphi_c + o(\hbar) \quad (2.11)$$

and

$$\Gamma(\varphi_c) = S(\varphi_c) + o(\hbar) \quad (2.12)$$

When $S(\varphi)$ is not convex, and φ_c corresponds to an unstable configuration, the situation is completely diffe-

rent. Eq.(2.10) has still a solution in the vicinity of φ_c , for small J . However, since φ_c is not even a local minimum of $S(\varphi)$ we are sure that such φ_J will not dominate $W(J)$. Indeed, since for some directions in functional space φ_J is a local maximum, the relevant trajectories for $W(J)$ will tend to move away from this particular φ_J . As a consequence, both eqs.(2.11) and (2.12) are not valid and we lose the possibility of using $\hbar$ as a perturbative parameter.

A residual hope may still remain, at this level, that the non perturbative treatment can be restricted to the much smaller manifold of the unstable fluctuations. However, in the case of local interactions this is not possible. A local lagrangian always couples unstable to stable fluctuations, thus propagating back non perturbative effects from the unstable to the stable sector. This makes the calculation of the effective potential for classically unstable field configurations a fully non-perturbative problem.

The previous discussion throws some light on the conjecture made in ref.(2), namely that the instability of the constant chromomagnetic field leads to large fluctuations in the unstable sector only. Stated differently, this amounts to say that the QCD vacuum is dominated by classical configurations of the form $H_\mu + \eta_\mu^{(u)}$. The exact form of $\eta_\mu^{(u)}$ has to be obtained from a minimization of an effective action, which in tree approximation reduces to the classical action, restricted to the unstable sector:

$$S_{\text{eff}} = S(H_\mu + \eta_\mu^{(u)}) + O(\hbar) \quad (2.13)$$

We can examine this question in general terms, for any classically unstable solution, φ_c of a given action $S(\varphi)$. To determine the trajectories relevant to the vacuum structure,

one has to consider the generating functional $W(J)$ for $J=0$.

Assuming that φ_c has some relevance for the ground state structure, it is natural to write $W(0)$ according to:

$$\begin{aligned}
 e^{-\frac{1}{\hbar} W(0)} &= \int \mathcal{D}\eta e^{-\frac{1}{\hbar} S(\varphi_c + \eta)} \\
 &= \int \mathcal{D}\eta^{(u)} \int \mathcal{D}\eta^{(s)} e^{-\frac{1}{\hbar} S(\varphi_c + \eta^{(u)} + \eta^{(s)})} \\
 &\equiv \int \mathcal{D}\eta^{(u)} e^{-\frac{1}{\hbar} S_{\text{eff}}(\varphi_c + \eta^{(u)})}
 \end{aligned} \tag{2.14}$$

The conjecture of ref.(2) amounts to say that the integration over the stable modes is controlled, in the limit $\hbar \rightarrow 0$, by the configuration $\eta^{(s)} = 0$. If this is the case:

$$S_{\text{eff}}(\varphi_c + \eta^{(u)}) = S(\varphi_c + \eta^{(u)}) + o(\hbar) \tag{2.15}$$

In the same limit, the integration over the unstable modes is dominated by the configuration $\hat{\eta}^{(u)}$ which minimizes the effective action:

$$\begin{aligned}
 \hat{\eta}^{(u)}_{(x)} &= \sum_N \hat{c}_N \eta_N^{(u)}(x) \\
 \left. \frac{\partial S}{\partial c_N} \right|_{\hat{c}_N} &= \int dx \eta_N^{(u)}(x) \left. \frac{\delta S}{\delta \varphi(x)} \right|_{\varphi_c + \hat{\eta}^{(u)}} = 0
 \end{aligned} \tag{2.16}$$

Eq.(2.16) translated back into the QCD language, exactly corresponds to the equation considered in ref.(2), for which a varieties of non trivial solutions have been found. However, we can now see clearly where the program may fail. Eq.(2.15) does not take into account the feedback that a non

vanishing $\eta^{(u)}$ induces in the stable sector. In the tree approximation, the correct expression of S_{eff} is rather^(f):

$$S_{\text{eff}} = S(\varphi_c + \eta^{(u)} + \bar{\eta}^{(s)}) + O(\hbar) \quad (2.17)$$

where $\bar{\eta}^{(s)} = \bar{\eta}^{(s)}(\varphi_c + \eta^{(u)})$ is the solution of:

$$\int dx \, \eta_p^{(s)}(x) \frac{\delta S}{\delta \varphi(x)} \Big|_{\varphi_c + \eta^{(u)} + \bar{\eta}^{(s)}} = 0 \quad (2.18)$$

In the general case, where $\bar{\eta}^{(s)}(\eta^{(u)} + \varphi_c)$ is non vanishing, it is not difficult to give a diagrammatic interpretation of S_{eff} , corresponding to an expansion into powers of $\eta^{(u)}$. For example, restricting to lowest order terms, one finds from eq.(2.18):

$$\bar{\eta}_s^{(s)} = -\frac{1}{2} \int \Theta_s^{-1}(x, y) \frac{\delta^3 S}{\delta \varphi(y) \delta \varphi(u) \delta \varphi(v)} \Big|_{\varphi_c} \eta^{(u)}(u) \eta^{(v)}(v) dy du dv \quad (2.19)$$

where Θ_s^{-1} is the restriction of Θ^{-1} to the s-sector:

$$\Theta_s^{-1}(x, y) = \sum_p \eta_p^{(s)}(x) \frac{1}{\lambda_p^{(s)}} \eta_p^{(s)*}(y) \quad (2.20)$$

(f) Strictly speaking, the situation is more complicated than implied by our explicit expressions. For non vanishing $\eta^{(u)}$ one should adapt the definition of stable directions in terms of the eigenfunctions of $\Theta(\varphi_c + \eta^{(u)})$ which, in general, do not span the same functional space as those of $\Theta(\varphi_c)$.

This leads to:

$$S_{\text{eff}} = S(\varphi_c + \eta^{(u)}) + \quad (2.21)$$

$$+ \frac{1}{8} \int dx dy dz dx' dy' dz' \eta^{(u)}(x) \eta^{(u)}(y) \frac{\delta^3 S}{\delta \varphi(x) \delta \varphi(y) \delta \varphi(z)} \Big|_{\varphi_c} \times$$

$$\times \Theta_S^{-1}(z, z') \eta^{(u)}(x') \eta^{(u)}(y') \frac{\delta^3 S}{\delta \varphi(x') \delta \varphi(y') \delta \varphi(z')} \Big|_{\varphi_c}$$

The first correction to S_{eff} in eq.(2.21) corresponds to a tree diagram where two vertices with external u-lines are joined by an s-mode propagator.

Solutions of eq.(2.16) are relevant only if, at the same time:

$$\bar{\eta}^{(s)}(\varphi_c + \hat{\eta}^{(u)}) = 0 \quad (2.22)$$

Stated in trivial terms, this gives what had to be expected, namely that eq.(2.16) is relevant only if $\varphi_c + \hat{\eta}^{(u)}$ is a stable solution of the complete equations of the motion:

$$\frac{\delta S}{\delta \varphi(x)} \Big|_{\varphi_c + \hat{\eta}^{(u)}} = 0 \quad (2.23)$$

Unfortunately, in the QCD case this does not work. It is simple to prove (see the Appendix) that no solution of the Yang-Mills equations exists of the form $\varphi_c + \eta^{(u)}$ with non vanishing $\eta^{(u)}$.

3. - The ultraviolet structure of the effective potential

We review in this Section the scaling properties of $\Gamma(\varphi_c)$ which derive from the assumption that the full theory has the same ultraviolet divergences as those found in the perturbative expansion⁽⁶⁾. Of course, there exists at present no proof of this statement, which is usually accepted without discussion. The results which follow have to be considered as the standard expectation about the scaling behaviour of $\Gamma(\varphi_c)$, which does apply if truly non perturbative pathologies are absent.

Quite in general, one may consider the generating functional $W(J)$, defined in eq.(2.7), as a cut-off dependent functional of J and of the bare coupling constant^(f):

$$W(J) = W_0(J, g, \Lambda) \quad (3.1)$$

Guided by perturbation theory, one assumes that the cut-off dependence can be removed from $W(J)$ by making a rescaling of J , and redefining a renormalized coupling constant, g_R , defined at some energy scale, μ :

$$W(J) = W_R(\sqrt{Z_3}J, g_R(\mu), \mu) \quad (3.2)$$

where the r.h.s. is a cut-off independent functional of its arguments.

Following the definition given in eq.(2.8), we obtain:

(f) For simplicity we consider here a massless theory with only one coupling constant, such as QCD, and ignore all subtleties related to gauge invariance, to be taken fully into account in the next Section.

$$\begin{aligned}
\Gamma(\varphi_c) &= \min_{\{J\}} \left[W_0(J, g, \Lambda) + \int J \varphi_c dx \right] = \\
&= \min_{\{\sqrt{Z_3} J\}} \left[W_R(\sqrt{Z_3} J, g_R(\mu), \mu) + \int (\sqrt{Z_3} J) (\varphi_c / \sqrt{Z_3}) \right] = \\
&= \Gamma_R(\varphi_c / \sqrt{Z_3}, g_R(\mu), \mu)
\end{aligned}
\tag{3.3}$$

where Γ_R is again a finite functional of its arguments. We may restrict to constant field configurations, which are those relevant for the vacuum state of the theory. In this case, we may easily derive from eq.(3.3) the scaling properties of any particular value of φ_c which corresponds to an extremum of Γ . The first line of eq.(3.3) informs us that any such value of φ_c , call it φ_c^* , has the form:

$$\varphi_c^* = f(g) \Lambda \tag{3.4}$$

On the other hand, the last line of the same equation implies that $\varphi_c^* / \sqrt{Z_3}$ is finite, i.e. cut-off independent. Therefore:

$$\varphi_c^* / \sqrt{Z_3} = \Lambda e^{-\int^g \frac{dg'}{\beta(g')}} \tag{3.5}$$

where β is the usual function:

$$\beta(g) = \Lambda \left. \frac{\partial g}{\partial \Lambda} \right|_{g_R, \mu} \tag{3.6}$$

An explicit way of deriving eq.(3.5) is to work out directly the equation which follows from the cut-off independence

of $\varphi_c^*/\sqrt{z_3}$:

$$\begin{aligned} 0 &= \Lambda \frac{\partial}{\partial \Lambda} \left(\varphi_c^*/\sqrt{z_3} \right)_{g_R, r} = \\ &= \frac{\Lambda}{\sqrt{z_3}} \left[(1 - \gamma(g)) f(g) - \beta(g) f'(g) \right] \end{aligned} \quad (3.7)$$

This yields:

$$f(g) = e^{-\int^g \frac{dg'}{\beta(g')}} \quad (3.8)$$

with⁽⁶⁾:

$$\tilde{\beta}(g) = \frac{\beta(g)}{1 - \gamma(g)} \quad (3.9)$$

where γ is the anomalous dimension of φ , i.e.:

$$\Lambda \frac{\partial}{\partial \Lambda} (\log \sqrt{z_3})_{g_R, r} = \gamma(g) \quad (3.10)$$

On the other hand, from eq.(3.10) we have:

$$\sqrt{z_3} = e^{\int^g dg' \frac{\gamma(g')}{\beta(g')}} \quad (3.11)$$

so that, putting together eqs.(3.4), (3.8) and (3.11) we obtain eq.(3.5). Eq.(3.5) is the analogous of the scaling law eq.(1.2), which we will derive shortly.

4. - The effective potential for gauge fields

There is a particular choice of the gauge condition which makes the definition and general properties of the effective action for Yang-Mills theories particularly transparent. This choice corresponds to the so-called background gauge⁽⁴⁾. We review here the basic properties of $\Gamma(H)$ in this particular gauge, in order to recover the results stated in eqs.(1.7) and (1.2).

One starts with a particular background field configuration, A_μ^B . In the end, A_μ^B will be the argument for which Γ is computed. Next, one splits the gauge field into A_μ^B plus a quantized component, η_μ :

$$A_\mu = A_\mu^B + \eta_\mu \quad (4.1)$$

The gauge is fixed by requiring:

$$D^\Gamma(A^B) \eta_\mu = 0 \quad (4.2)$$

where $D^\Gamma(A^B)$ denotes the covariant derivative in the background field.

The generating functional, is defined according to:

$$e^{-\frac{1}{\hbar} W(Y, A^B)} = \int \mathcal{D}\eta_\mu \mu(A^B, \eta) e^{-\frac{1}{\hbar} [S(A_\mu^B + \eta_\mu) - \int \int_\mu \eta^\mu dx]} \quad (4.3)$$

$$\mu(A^B, \eta) = \Delta(A^B, \eta) \delta(D^\Gamma(A^B) \eta_\mu)$$

where Δ is the Faddeev-Popov factor induced by the gauge fixing.

It is well known that $W(J, A^B)$ is invariant under the combined set of transformations:

$$\begin{aligned} J_\mu &\rightarrow J_\mu^\Omega = \Omega J_\mu \Omega^+ \\ A_\mu^B &\rightarrow (A_\mu^B)^\Omega = \Omega A_\mu^B \Omega^+ + \frac{i}{g} (\partial_\mu \Omega) \Omega^+ \end{aligned} \quad (4.4)$$

In analogy with eq.(2.8), the Legendre transform with respect to J is given by:

$$\tilde{\Gamma}(Q, A^B) = \min_{\{J\}} \left[W(J, A^B) + \int J_\mu Q^\mu dx \right] \quad (4.5)$$

Finally, as discussed in ref.(4),

$$\tilde{\Gamma}(0, A^B) = \Gamma(A^B) \quad (4.6)$$

where $\Gamma(A^B)$ is the effective potential for A^B , computed in the gauge:

$$D^\mu \Gamma(A^B) \eta_\mu - \partial_\mu A^B = 0 \quad (4.7)$$

The usefulness of this construction lies in the fact that $\Gamma(A^B)$ thus defined is a gauge invariant functional, as it follows immediately from the invariance of W and from eq.(4.6):

$$\Gamma((A^B)^{\mathcal{N}}) = \Gamma(A^B) \quad (4.8)$$

Using again perturbation theory as a guide, the renormalization properties can be summarized as follows:

$$\begin{aligned} W(J, A^B) &= W_0(J, g, \Lambda, A^B) = \\ &= W_R(J \sqrt{Z_3}, g_R(\mu), \mu, \sqrt{Z_A} A^B) \end{aligned} \quad (4.9)$$

whence:

$$\tilde{\Gamma}(0, A^B) = \Gamma(A^B) = \Gamma_R(\sqrt{Z_A} A^B, g_R(\mu), \mu) \quad (4.10)$$

Moreover, the gauge invariance of Γ implies that⁽⁴⁾:

$$\sqrt{Z_A} = g/g_R \quad (4.11)$$

For the case we are interested in, namely the one where A^B describes a constant chromomagnetic field, the structure of Γ is particularly simple. Due to gauge invariance and to eqs.(4.10) and (4.11), Γ must be translationally invariant, and such that:

$$\Gamma/VT = F(gH, g_R(\mu), \mu) \quad (4.12)$$

Furthermore, the μ -independence of the r.h.s. of eq.(4.12) allows us to choose $\mu = gH$, so that we recover eq.(1.7),

namely:

$$\Gamma_{VT} = 1/2 (g_H)^2 f(g_R(g_H)) \quad (4.13)$$

A non trivial value of g_H which minimizes Γ must be found among the solutions of the equation:

$$\begin{aligned} f(g_R) + \frac{1}{2} (g_H) f'(g_R) \frac{\partial g_R}{\partial g_H} &= \\ = f(g_R) + \frac{1}{2} \beta(g_R) f'(g_R) &= 0 \end{aligned} \quad (4.14)$$

If we call g^* a solution of eq.(4.14) and recall the general relation:

$$\ln(\Lambda^2/g_H) = 2 \int_{g_R}^g \frac{dg'}{\beta(g')} \quad (4.15)$$

we obtain:

$$g_H^* = \Lambda^2 e^{-2 \int_{g^*}^g \frac{dg'}{\beta(g')}} \quad (4.16)$$

which reduces to eq.(1.2), in the one loop approximation.

One can observe that even if we knew $f(g_R)$ as a power series in g_R , we could not obtain any reliable information about the existence and location of a non trivial minimum g_H^* . Indeed eq.(4.14) requires an illegal compensation of terms of different order in g_R . However this is only a minor obstacle compared with the result discussed in Sect. 2, where we have seen that $f(g_R)$ is not accessible to perturbative techniques anyhow.

5. - Comparison with the variational approach

In this Section, we would like to make contact with the computation presented in ref.(3).

These authors give an upper bound to the true QCD vacuum energy, by estimating the energy of a trial state centered around the (unstable) constant chromomagnetic field, H_c . The best value they find for the field H scales with the ultraviolet cut-off Λ according to the anomalous law expressed by eq.(1.1). This means that if the cutoff is removed as suggested by the renormalization group, gH^* would remain infinite, implying an ultraviolet structure of QCD dramatically different from the one inferred from perturbation theory (see Sect.4).

Translated into the language of the effective potential, the variational approach can be seen as the implementation of the strategy described in Sect. 2. One considers $W(J, A_\mu)$ defined in eq.(4.3), and performs perturbatively the integration over the stable modes, $\eta^{(s)}$, at fixed values of $\eta^{(u)}$ and J . The residual $\eta^{(u)}$ integration cannot be performed, since it is an essentially non perturbative calculation while at best we can express the result of the first integration as a power expansion in $\eta^{(u)}$ and J . However, the $\eta^{(s)}$ integration cannot give rise to new ultraviolet divergences, since it has a natural cutoff of the order of gH . Thus, the ultraviolet structure of the result of the first integration over $\eta^{(s)}$ is already sufficient to determine the ultraviolet structure of $\Gamma(H)$.

Our approach bears the same relation with that of ref.(3) as the covariant Feynman diagram approach with the old-fashioned perturbation theory, and therefore both method

are bound to give the same result. None of the two methods, of course, can give information about possible non analytic terms in the bare coupling constant, which could spoil the ultraviolet structure thus found.

We start from the formal expression given in eq.(4.3), which we rewrite according to:

$$e^{-\frac{1}{\hbar} W(J, H)} = \int \mathcal{D}\eta_r^{(u)} e^{\frac{1}{\hbar} \int J^r \eta_r^{(u)} dx} \int \mathcal{D}\eta_r^{(s)} \mu(H, \eta_r^{(s)} + \eta_r^{(u)}) \times e^{-\frac{1}{\hbar} [S(H_r + \eta_r^{(s)} + \eta_r^{(u)}) - \int J^r \eta_r^{(s)} dx]} \quad (5.1)$$

Notice that the result of the $\eta^{(s)}$ integration is not simply a function of $H_r + \eta_r^{(u)}$, because of the gauge fixing measure. Therefore H_r and $\eta_r^{(u)}$ will be renormalized with different renormalization constants. For the sake of orientation, we discuss first the case where we set $J = 0$, and define:

$$e^{-\frac{1}{\hbar} X(\eta^{(u)})} \equiv \int \mathcal{D}\eta^{(s)} \mu(H, \eta^{(s)} + \eta^{(u)}) e^{-\frac{1}{\hbar} S(H_r + \eta_r^{(s)} + \eta_r^{(u)})} \quad (5.2)$$

$$X(\eta^{(u)}) = \sum_n \frac{1}{n!} \int X_n(x_1, \dots, x_n) \eta^{(u)}(x_1) \dots \eta^{(u)}(x_n)$$

It is not difficult to give a diagrammatic interpretation to X_n . To one loop X_0 is simply the real part of Saviddy's result⁽¹⁾:

$$X_0 = \frac{1}{2} \frac{(g_H)^2}{g_R^2(g_H)} + \mathcal{O}(\hbar) \quad (5.3)$$

For $n > 0$, X_n is the sum of all connected diagrams which are

one-particle irreducible with respect to $\eta^{(u)}$, but not with respect to $\eta^{(s)}$ which runs in the internal lines. Propagators for internal $\eta^{(s)}$ and ghost lines are to be taken in the background field H_μ .

In a gauge invariant regularization (e.g. on the lattice), no quadratic divergences arise, and the ultraviolet divergences are determined by the asymptotic form of the propagators, which is not affected by the presence of H_μ . Thus, we find:

$$X_m = (\sqrt{Z_3})^{-m} \alpha_m(g_R, g_H) \quad (5.4)$$

where α_n is cut-off independent (we have suppressed the space time arguments $x_1, \dots, x_n$) and $\sqrt{Z_3}$ is the wave function renormalization of the quantized field. Notice that through vertices and propagators, H enters only in the natural combination gH , which is unrenormalized in the background gauge⁽⁴⁾. The factors $\sqrt{Z_3}$ can be eliminated by the rescaling:

$$\eta^{(u)} / \sqrt{Z_3} \rightarrow \eta^{(u)} \quad (5.5)$$

leading to the result that $W(0, H)$ is a finite function of $g_R(gH)$ and gH .

For $J \neq 0$ we may expand in J as well as in $\eta^{(u)}$ the result of the $\eta^{(s)}$ integration in eq.(5.1). We define:

$$\begin{aligned} & \int \delta\eta^{(s)} \mu[H, \eta^{(s)} + \eta^{(u)}] \eta^{(s)}(x_1) \dots \eta^{(s)}(x_n) e^{-\frac{1}{\hbar} S} = \\ & = \sum_{l=0}^{\infty} \frac{1}{l!} \int G^{(n,l)}(x_1, \dots, x_n; y_1, \dots, y_l) \eta^{(u)}(y_1) \dots \eta^{(u)}(y_l) dy_1 \dots dy_l e^{-\frac{1}{\hbar} X(\eta^{(u)})} \end{aligned} \quad (5.6)$$

and one sees that the functions $G^{(n,l)}$ are the $(n+l)$ point non-connected Green functions, with n external $\eta^{(s)}$ and l external $\eta^{(u)}$ lines, one particle irreducible with respect to $\eta^{(u)}$ lines only.

The ultraviolet divergences are such that:

$$G^{(n,l)} = (\sqrt{Z_3})^n (\sqrt{Z_3})^{-l} \beta^{(n,l)}(g_R, g_H) \quad (5.7)$$

with $\beta^{(n,l)}$ cut-off independent. Note the different renormalizations in correspondence to $\eta^{(s)}$ versus $\eta^{(u)}$ lines, due to the fact that for the former we do not have to remove external propagators. The factor $(\sqrt{Z_3})^n$ in eq.(5.7) goes with the n factors of J , while the other disappears with the rescaling indicated in (5.5). Up to now the instability of $\eta^{(u)}$ has not played any particular role. The arguments used are the same as those needed to show that, in a renormalizable field theory, the partition of the integration variables in two ranges, one with low frequencies, and the other with frequencies up to the cutoff, does lead to the expected renormalizability property eq.(3.2). When an unstable sector is present, some care is required due to possible infrared singularities.

Infrared singularities may arise because the stable spectrum of Θ starts from zero, so that formally its inverse, Θ_s^{-1} , eq.(2.20) is not defined. This point is clearly illustrated by the explicit expression of the eigenvalues in the external magnetic field (see Appendix). Θ_s^{-1} is of the form:

$$\begin{aligned} (\Theta_s^{-1})_{\mu\nu}^{ab}(x,y) = & \int_{K_0^2 + K_2^2 \geq g_H} \frac{dK_0 dK_2 dK_3}{K_0^2 + K_2^2 - g_H} \psi_{\mu;(K,n=0)}^a(x) \psi_{\nu;(K,n=0)}^{b*}(y) + \\ & + (n \geq 1 \text{ terms}) \end{aligned} \quad (5.8)$$

The integral in eq.(5.8) diverges at the lower end and $\mathcal{Q}_S^{-1}$ is not defined. As a consequence all the Green functions appearing in the expansion (5.6) are divergent in the infrared. These infrared divergences, however, are not dangerous because they can be eliminated with a suitable resummation in $\eta^{(u)}$ and J . For the sake of definiteness we restrict to one loop in the $\eta^{(s)}$ variables, so that the expression of $W(J,H)$ takes the form:

$$e^{-\frac{1}{\hbar}W(J,H)} = \int \mathcal{D}\eta^{(u)} e^{-\frac{1}{\hbar} [S(H, \eta^{(u)} + \bar{\eta}^{(s)}) - \int J_\mu (\eta_\mu^{(u)} + \bar{\eta}_\mu^{(s)})] - \frac{1}{2} \text{Tr} \log [\mathcal{Q}_S, \bar{\eta}^{(s)}]} \quad (5.9)$$

As already discussed in sect.2, $\bar{\eta}^{(s)}(H, \eta^{(u)}, J)$ is the solution of the equations:

$$\int dx \, \eta_{p(x)}^{(s)} \frac{\delta S}{\delta A_\mu(x)} \Big|_{H, \eta^{(u)} + \bar{\eta}^{(s)}} = \int dx \, J_\mu(x) \eta_{p(x)}^{(s)} \quad (5.10)$$

The ghost contribution and the overall normalization constant of the functional integral are understood in the $\text{Tr} \log$ term. In the resummed form (5.9) all infrared divergences have disappeared. To see this we notice that the expression of $\bar{\eta}^{(s)}$ in terms of $\eta^{(u)}$ and J involves tree diagrams only (it is a solution of the classical equations (5.10)) so that it is free from infrared problems. The only potentially infrared divergent term is $\text{Tr} \log \mathcal{Q}_S, \bar{\eta}^{(s)}$, in which a loop integration occurs. However any zero eigenvalue of $\mathcal{Q}_S, \bar{\eta}^{(s)}$, which is not a discrete point in the spectrum generates only a logarithmic singularity which is washed out in the integration implied by the trace operation. We are in a completely analogous situation as that examined by Coleman and Weinberg⁽⁶⁾, in which infrared divergences appear only in the perturbative expansion of the determinant. The resummation

procedure does not affect, of course, the ultraviolet structure of $W(J,H)$ which depends only on the large k_0, k_z behaviour of the eigenvalues, for each n .

In conclusion, going back to eq.(5.1), we obtain the standard result of eq.(4.10). No pathology arises, within this framework where the expected scaling law, eq.(1.2) is obtained. This result casts serious doubts^(f) on the calculation of ref.(3).

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Appendix

We collect in this Appendix all formulae and technical details needed in the text.

A1 - Notations and Conventions

If one splits the gauge field into a background field, H_μ , solution of the Yang-Mills equations, and a quantum fluctuation, η_μ :

(f) In particular, one important missing element in that calculation is the Faddeev-Popov determinant which has to go together with the three-dimensional gauge fixing in the subspace of gauge invariant wave functionals.

$$A_r^a = H_r^a + \gamma_r^a \quad (\text{A.1})$$

the Euclidean Lagrangian density takes the form:

$$\begin{aligned} \mathcal{L}(A) \equiv \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a = \mathcal{L}(H) + \frac{1}{2} \gamma_r^a (-D^2 \delta_{rv} + D_r D_v) \gamma_v^b + \\ - g \varepsilon^{acb} \gamma_r^a F_{\mu\nu}^c(H) \gamma_v^b + \mathcal{L}_I(H, \gamma) \end{aligned} \quad (\text{A.2})$$

where, with obvious notations, $\mathcal{L}(H)$ and $F_{\mu\nu}(H)$ are the Lagrangian density and the field strength of H_μ , respectively. $\mathcal{L}_I(H, \gamma)$ is the part of $\mathcal{L}$ which is not quadratic in γ :

$$\begin{aligned} \mathcal{L}_I(H, \gamma) = g \varepsilon^{abc} (D_r \gamma_v)^a \gamma_r^b \gamma_v^c + \\ + g^2/4 \varepsilon^{abc} \varepsilon^{a'b'c'} \gamma_r^b \gamma_v^c \gamma_{r'}^{b'} \gamma_{v'}^{c'} \end{aligned} \quad (\text{A.3})$$

In eqs.(A.2) and (A.3) we have used the definitions:

$$\bar{F}_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g \varepsilon^{abc} A_\mu^b A_\nu^c \quad (\text{A.4a})$$

$$D_r^{ab} = \partial_r \delta^{ab} + g \varepsilon^{acb} H_r^c \quad (\text{A.4b})$$

$a, b, c, \dots$ are colour indices and we take the colour group to be $SU(2)$, so that the structure constants are the components of the three-dimensional totally antisymmetry Levi-Civita tensor.

Specializing H_μ to be a constant chromomagnetic field in the y -direction in space and in the 3-direction in colour, we have:

$$H_\mu^a = \delta^{a3} \delta_{\mu 2} H \times \quad (\text{A.5})$$

$$F_{\mu\nu}^a(H) = \delta^{a3} (\delta_{\mu 1} \delta_{\nu 2} - \delta_{\mu 2} \delta_{\nu 1}) H \quad (\text{A.6})$$

$$\mathcal{L}(H) = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a = \frac{H^2}{2} \quad (\text{A.7})$$

We work in the background gauge:

$$D_\mu^{ab} \eta_\mu^b = \partial_\mu \eta_\mu^a + \varepsilon^{a3b} g H \times \eta_\mu^b = 0 \quad (\text{A.8})$$

The operator $\Theta_{\mu\nu}^{ab}$ which describes the quadratic part of the quantum fluctuations of the gauge field, reduces to

$$\Theta_{\mu\nu}^{ab} = (-D^2)^{ab} \delta_{\mu\nu} - 2g \varepsilon^{acb} F_{\mu\nu}^c(H) \quad (\text{A.9})$$

$\Theta_{\mu\nu}^{ab}$ is diagonalized by solving the eigenvalue equation:

$$\Theta_{\mu\nu}^{ab} \psi_{\nu,\kappa}^b = \lambda_\kappa \psi_{\mu,\kappa}^a \quad (\text{A.10})$$

with the transversality condition:

$$D_\mu^{ab} \psi_{\mu,\kappa}^b = 0 \quad (\text{A.11})$$

$\{k\}$ is a set of indices which label the eigenfunctions of Θ .

A2. - Spectrum of the Small Fluctuations

For the sake of completeness we report here the details of this well-known calculation^(1,2,7).

For $a=3$ the pair of eqs.(A.10) and (A.11) reduces to the free field equations.

For $a=1,2$ we solve eq.(A.10) ignoring the transversality condition (A.11) which is later imposed by constructing suitable linear combinations of degenerate states.

To diagonalize colour we notice that in the subspace $a,b=1,2$ we can write eq.(A.8) in the form:

$$\begin{aligned} & [(-\partial^2 + H^2 x^2) \mathbb{1}^c \delta_{\mu\nu} + 2igHx \partial_y (\tau_z)^c \delta_{\mu\nu} + \\ & + 2igHx (\tau_z)^c (\delta_{\nu 2} \delta_{\mu 1} - \delta_{\nu 1} \delta_{\mu 2})] \psi_{\nu, \kappa} = \lambda_{\kappa} \psi_{\nu, \kappa} \quad (A.12) \end{aligned}$$

where $(\tau_z)^c = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ acts in colour space. We have then two subcases, $\mu = 3, 4$ and $\mu = 1, 2$.

For $\mu = 3, 4$ eq.(A.12) is diagonal in space-time indices and it is solved by performing the Fourier transform with respect to the variables $(y, z, t) = \underline{y}$. After that, eq.(A.12) becomes the Schrödinger equation for an harmonic oscillator with mass $1/2$ and frequency $\omega = 2gH$, centered at $\mathcal{E} K_y / gH$, with $\mathcal{E} = \pm 1$ when $\psi_{\mu, \kappa}$ belongs to the eigenvalue ± 1 of $(\tau_z)^c$. We obtain two degenerate states ($\ell = 3, 4$):

$$\psi_{\mu; (k, n)}^{+, \ell} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \otimes \chi_{\mu}^{(p)} \frac{e^{i \underline{k} \cdot \underline{y}}}{(2\pi)^{3/2}} F_n^{+}(\underline{x}) \quad (A.13a)$$

$$\psi_{\mu; (k, n)}^{-, \rho} = \left[\psi_{\mu; (-k, n)}^{+, \rho} \right]^* \quad (\text{A.13b})$$

$$\lambda_{(k, n)}^{\pm} = K_0^2 + K_z^2 + g_H(2n+1) \geq 0 \quad ; \quad n = 0, 1, 2, \dots \quad (\text{A.14})$$

where $F_m^{\varepsilon}(x)$ are the normalized harmonic oscillator wave functions:

$$F_m^{\varepsilon}(x) = (2^m m!)^{-1/2} \left(\frac{g_H}{\pi} \right)^{1/4} H_m \left[\sqrt{g_H} \left(x - \frac{\varepsilon K_y}{g_H} \right) \right] e^{-\frac{g_H}{2} \left(x - \frac{\varepsilon K_y}{g_H} \right)^2} \quad (\text{A.15})$$

and the $\chi_{\mu}^{(\rho)}$ are fourdimensional unit vectors in the directions 3 and 4.

Similarly, for $\mu = 1, 2$ we diagonalize space indices by writing eq.(A.10) in the form:

$$\left[(-\square + H^2 x^2) \mathbb{1}^c \otimes \mathbb{1}^s - 2g_H x \partial_y \mathbb{1}^s \otimes (\tau_2)^c + \right. \\ \left. - 2g_H (\tau_2)^s \otimes (\tau_2)^c \right] \psi_K = \lambda_K \psi_K \quad (\text{A.16})$$

where $(\tau_2)^s = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ acts on spin indices. The eigenstates of this equation are labelled by two discrete numbers: $\varepsilon = \pm 1$

referring to charge and $\eta = \pm 1$ referring to spin. One finds

$$\psi_{\mu; (k, n)}^{+, \eta} = \frac{1}{2} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ i\eta \\ 0 \\ 0 \end{pmatrix} \frac{e^{i\mathbf{k} \cdot \mathbf{v}}}{(2\pi)^{3/2}} F_m^+ (x) \quad (\text{A.17a})$$

$$\psi_{\mu; (k, n)}^{-\epsilon, \eta} = \left[\psi_{\mu; (-k, n)}^{+\epsilon, -\eta} \right]^* \quad (\text{A.17b})$$

$$\lambda^{\epsilon, \eta} = k_0^2 + k_z^2 + g_H(2n+1) - 2\epsilon\eta g_H \quad (\text{A.18})$$

The states with $n=0$ and $\epsilon = \eta = \pm 1$ are already transverse. Their eigenvalues are given by:

$$\lambda_k = k_0^2 + k_z^2 - g_H \quad (\text{A.19})$$

and they are negative for $k_0^2 + k_z^2 < g_H$.

These are the unstable modes discussed at length in the text. The eigenvalues corresponding to all the other states are positive (or zero).

In the Table, we summarize the pattern of harmonic oscillator states, F_n , corresponding to each value of λ and to each space-time index, for the charged sector $\epsilon = +1$. For $\epsilon = -1$, the columns labelled by + and - are to be interchanged.

The transverse eigenfunctions of $0_{\mu\nu}^{ab}$ are finally constructed by taking linear combinations of degenerate states.

This is done by rewriting eq.(A.11) in the form:

$$\begin{aligned} & \left[\partial_x + \epsilon g_H \left(x - \frac{\epsilon k_y}{g_H} \right) \right] \psi_+^{\epsilon} + \left[\partial_x - \epsilon g_H \left(x - \frac{\epsilon k_y}{g_H} \right) \right] \psi_-^{\epsilon} \\ & + \partial_z \psi_3^{\epsilon} + \partial_t \psi_4^{\epsilon} = 0 \end{aligned} \quad (\text{A.20})$$

Table

$\lambda_k - k_0^2 - k_z^2$	+	-	3	4	Mult. of transv. eigenf.	ghost
$-gH$	F_0	/	/	/	1	/
gH	F_1	/	F_0	F_0	2	F_0
$3gH$	F_2	F_0	F_1	F_1	3	F_1
$(2k+1)gH$	F_{k+1}	F_{k-1}	F_k	F_k	3	F_k

and noticing that, for $\xi = +1$ ($\xi = -1$), the operators $\partial_x - \xi gH \left(x - \frac{\xi k_y}{gH}\right)$ and $\partial_x + \xi gH \left(x - \frac{\xi k_y}{gH}\right)$ act respectively as destruction (creation) and creation (destruction) operators on the harmonic oscillator wave functions, F_n^ξ . The explicit form of the transverse wave functions is not given here. We simply note that the gauge condition gives one linear constraint, for each value of λ, ξ , which reduces by one unit the previously found multiplicity of each eigenvalue with $n > 0$. The multiplicities of the transverse states are listed in the sixth column of the Table.

A3. - Search for classical solutions

We consider here the possibility of classical solutions of the Yang-Mills equations, of the form:

$$\vec{A}_\mu^a = H_\mu^a + \eta_\mu^a$$

where η_r^a belongs to the space of the unstable modes:

$$\eta = \frac{1}{2} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \varphi(x) + c.c. \quad (A.21)$$

$$\varphi(x) = \int d^3k \frac{e^{i\mathbf{k} \cdot \mathbf{v}}}{(2\pi)^{3/2}} C(k) e^{-\frac{g_H}{2} \left(x - \frac{k_y}{g_H}\right)^2} \quad (A.22)$$

with $C(k)$ an arbitrary function and $X_\mu = (x, \underline{v})$, $\underline{v} = (y, z, t)$.

We show that the only solution is the trivial (and unstable) one, $\eta_\mu^a = 0$.

Direct computation of the field strength, $F_{\mu\nu}^a(\bar{A})$, associated to $\bar{A}_\mu$, yields:

$$\begin{aligned} F_{12}^1 &= F_{12}^2 = 0 \\ F_{12}^3 &= H - g|\varphi|^2 \end{aligned} \quad (A.23)$$

while:

$$\begin{aligned} F_{\rho\nu}^a &= \partial_\rho \eta_\nu^a \\ F_{03}^a &= 0 \end{aligned} \quad g = 3, 0; \quad \nu = 1, 2$$

It is sufficient to work out the equations of motion:

$$[D^\mu(\bar{A})]^{ab} F_{\mu\nu}^b(\bar{A}) = 0$$

for the cases $a = 3, \nu = 1, 2$, where they reduce to:

$$\partial_x |\varphi|^2 = \partial_y |\varphi|^2 = 0 \quad (\text{A.24})$$

whence:

$$\varphi = f(z, t) e^{i\alpha(x, \underline{v})} \quad (\text{A.25})$$

with α a real function of x .

Comparison of eq. (A.25) with eq. (A.22) leads to the condition:

$$f(z, t) e^{i\alpha(x, \underline{v})} e^{\frac{g_H}{2} x^2} = \int \frac{d^3 k}{(2\pi)^{3/2}} C(k) e^{-\frac{k_y^2}{2g_H}} e^{i(k_0 t + k_z z)} e^{i k_y (y - ix)} \quad (\text{A.26})$$

For non vanishing $C(K)$, the r.h.s. of eq.(A.26) depends upon x and y only through the combination $y - ix$, at variance with the l.h.s.. The only possible solution corresponds to $C(K) = f = 0$, i.e. $\eta_r = 0$.

A4. - Stable mode, one loop contribution to $\Gamma(H)$.

To obtain the Savvidy result⁽¹⁾, we start from the one loop formula:

$$\Gamma(H) = VT \frac{H^2}{2} + \frac{1}{2} \text{Tr} \log [\Theta_S] - \text{Tr} \log (-D^2) \quad (\text{A.27})$$

$$T_2 = \frac{g_H}{(2\pi)^3} V T \int d^2 k \sum_n \quad (A.28)$$

the last term in eq.(A.27) is the contribution of the Faddeev-Popov ghost, which goes together with the gauge condition (A.8). The eigenvalues and the eigenfunctions of $(-D^2)^{ab}$ are easily obtained along the same lines of reasoning employed for $0_{\mu\nu}^{ab}$, and are also reported in the Table.

Taking into account the multiplicity of each eigenvalue, we obtain:

$$\begin{aligned} \frac{\Gamma(H)}{VT} = & \frac{H^2}{2} + \frac{g_H}{(2\pi)^3} \left\{ \frac{1}{2} \int_{k^2 > g_H} d^2 k \log(k^2 - g_H) + \frac{2}{2} \int d^2 k \log(k^2 + g_H) + \right. \\ & \left. + \frac{3}{2} \int d^2 k \sum_{n=1}^{\infty} \log[k^2 + g_H(2n+1)] - \int d^2 k \sum_{n=0}^{\infty} \log[k^2 + g_H(2n+1)] \right\} \end{aligned} \quad (A.29)$$

(the factor of 2 in front of the big parentheses is due to charge degeneracy). The neutral sector of the spectrum is independent of H , so it is not included in eq.(A.29). The k -integration is regularized by using the formula:

$$\frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} ds s^{\alpha} E e^{-sE} \xrightarrow{\alpha \rightarrow 0^+} \cos t - \alpha \log E \quad (A.30)$$

With this the sum over n and the k -integration can be immediately performed and one gets, a part from an irrelevant, H independent, infinite constant and finite terms:

$$\frac{\Gamma(H)}{VT} = \frac{H^2}{2} - \frac{(g_H)^2}{(2\pi)^2} \frac{11}{12} \frac{1}{2} \int \frac{ds}{s} \quad (A.31)$$

The divergent integral has to be interpreted according to:

$$\int \frac{ds}{s} \rightarrow 2 \ln \left(\frac{\Lambda^2}{g_H} \right) \quad (\text{A.32})$$

as can be seen by computing, e.g.:

$$\int d^2k \ln(\kappa^2 + g_H) - \int_{\kappa^2 > g_H} d^2k \ln(\kappa^2 - g_H)$$

with the regularization (A.30) and with an ultraviolet cut-off Λ and equating the results. Combining eqs.(A.31) and (A.32), one is led to:

$$\frac{\Gamma(H)}{VT} = \frac{1}{2} \frac{(g_H)^2}{g_R^2} \quad (\text{A.33})$$

$$\frac{1}{g_R^2} = \frac{1}{g^2} - \frac{11}{24\pi^2} \ln \left(\frac{\Lambda^2}{g_H} \right)$$

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