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THE INTERACTING GAUGE COVARIANT BOSONIC STRING

A. Neveu and P. West^{*)}

CERN - Geneva

ABSTRACT

The interaction of the gauge covariant open bosonic string is presented in oscillator form, and in functional form. In the latter form, the string is described by commuting and anticommuting co-ordinates. The interaction is essentially the condition that the strings should overlap.

^{*)} Permanent address: Mathematics Department,
King's College, London WC2R 2LS, U.K.

There has recently been a considerable effort to find a gauge covariant formulation of string theories. At the free level, this problem has been solved: the essentially new step being the introduction of supplementary fields, which was first done successfully in Ref. 1). A gauge covariant formulation of all known string theories was given in Ref. 2), and enlarged formulations found in Refs. 3) and 4). Alternative gauge covariant formulations were also found^{5),6)} by working back from the gauge fixed and BRS covariant formulation of Ref. 7). The latter was a second quantized generalization of Ref. 8). More recently⁹⁾, a new gauge covariant formulation of strings was given. It contains as gauge choices all the previously mentioned ones, and admits a very simple functional form.

To complete the field theory of strings, one requires the interactions. The purpose of this paper is to give the order g interaction term for the open bosonic string. We will proceed in a series of stages; first, we present a vertex which solves the so-called "duality problem". This involves assigning lengths to strings. Their interaction depends on the lengths of the incoming three strings, although on-shell quantities are independent of these lengths. This vertex is taken to be the vertex for the ϕ^3 interaction, where ϕ is the original string field. The second step is to find a set of identities which relate the action of conformal transformations on one leg of the vertex to such transformations on the other two legs. These identities, together with the vertex completely specify an algorithm which uniquely determines the interaction for all supplementary fields as well as their non-Abelian gauge transformation laws. This algorithm is carried out explicitly at the lowest level. It is then shown that the interaction found by the above algorithm is essentially given by the overlap function for three things, when the strings are expressed in term of the functional form given in Ref. 9). This latter formulation involves $x^\mu(\sigma)$ and anticommuting co-ordinates $c(\sigma)$ and $\bar{c}(\sigma)$. The overlap for these anticommuting co-ordinates leads to an oscillator vertex which determines the couplings of the supplementary fields. It is clear that this geometrical picture requires a four open string interaction.

In Ref. 1), the interacting theory was given at the first level using a kind of Noether technique, to find both the transformations of the fields and the interaction at the same time. Although one could extend this work to higher levels, it is better to begin from a vertex which automatically solves the duality problem, which can be stated as follows. In the dual model theory of strings, a single diagram such as the four-point function contained both s and t channel poles. The full Veneziano amplitude could be obtained by summing either

over s channel or t channel poles. More generally, all topologically equivalent (in the sense of the string world sheet) diagrams could be obtained from a single one of them by duality transformations. Consequently, a given amplitude is given by the sum of diagrams of a distinct topology. These combinatoric rules are to be contrasted with those found in field theory, where one joins lines in all possible ways. For a field theory of strings, one constructs Green functions from a Feynman path integral, and so one must use the usual combinatoric rules of field theory. If we took as our vertex the Sciuto¹⁰⁾ or Caneschi-Schwimmer-Veneziano¹¹⁾ vertex, which do possess duality, and so poles for more than one channel, and attempted to use them as vertices in string field theory, we would not obtain the correct four point or higher point functions.

In the second quantized light-cone formulation of string field theory, this problem has been solved¹²⁾⁻¹⁵⁾. In this case, we have a time $x^+ = \tau$, which can be used to label the interaction points. One finds¹²⁾ diagrams such as those given in Fig. 1, which contribute to the four-point function. Integrating over the interaction times, one finds that the first diagram gives rise to only s channel poles, while the second gives rise to only t channel poles.

The solution to the duality problem for covariant string theory is found by extracting certain desirable features from the light-cone solution. We therefore suppose that strings have a length, denoted by α , which is conserved in the interactions. Strings for which $\alpha < 0$ are outgoing, while those for $\alpha > 0$ are incoming. With this rule, the three-point vertex can contain two possible distinct diagrams, corresponding to whether two α 's are positive or negative. In an interaction, one must sum over all possible string configurations consistent with keeping the lengths of the external string fixed. We require that S-matrix elements do not depend on the lengths of the external strings. In this sense, the lengths are gauge parameters which can be changed by conformal transformations. This will be discussed below.

For our algorithm, to be explained below, we must also specify what the interaction vertex is between three $|\psi\rangle$ states, where ψ is the original string functional. In analogy with the light-cone formulation^{12),14),15)}, we take this to be given by

$$V^c = \langle 0_{123} | \exp \left[\frac{1}{2} \sum_{n,m} \alpha_{n2}^\mu N_{nm;rs} \alpha_{ms}^\mu + \sum_m N_{m0;rs} \alpha_{m2}^\mu \alpha_{0s}^\mu \right], \quad (1)$$

where

$$N_{mn;rs} = -\frac{\alpha_1 \alpha_2 \alpha_3}{\alpha_2 \alpha_s} \frac{mn}{m\alpha_0 + n\alpha_2} f_m \left(-\frac{\alpha_{r+1}}{\alpha_2} \right) f_n \left(-\frac{\alpha_{s+1}}{\alpha_s} \right),$$

$$N_{m0;rs} = -\frac{\alpha_1 \alpha_2}{\alpha_2 \alpha_s} f_m \left(-\frac{\alpha_{r+1}}{\alpha_2} \right) c_s, \quad (2)$$

$$c_1 = 1, \quad c_2 = -1, \quad c_3 = 0, \quad \alpha_{0s}^\mu = p_s^\mu \sqrt{2\alpha'},$$

$$f_m(\gamma) = \frac{(m\gamma-1)!}{m!(m\gamma-m)!}, \quad \alpha_1 + \alpha_2 + \alpha_3 = 0. \quad (3)$$

This vertex is a direct Lorentz covariant generalization of the light cone vertex. We have replace the sum over $i = 1$ to $D-2$ by $\mu = 0, 1, \dots, D-1$, and we no longer identify α_r with p_r^+ , but now regard α_r as parameters. This choice has several advantages. Firstly, it solves the duality problem discussed above. In the covariant formulation, we can express the propagators in the form

$$\int_0^\infty dT \exp -(L_0-1)T. \quad (4)$$

The T integration can then be compared with the corresponding light-cone integration on the interaction times. The classification of the covariant diagrams with string lengths α is the same as the light-cone classification, and some thought shows that they contain only poles of the explicit channels. In this way, the covariant formulation recovers the solution of duality found for the second quantized light-cone field theory. Secondly, for V^c to be a good three-point function, it should give the same couplings as the Ganeschi-Schwimmer-Veneziano vertex for three arbitrary external on-shell physical states. On-shell physical states are described in a Lorentz covariant way by

$$(L_0-1)|\psi\rangle = L_n|\psi\rangle = 0, \quad n \geq 1, \quad (5)$$

with all supplementary fields vanishing.

This means that the two vertices may only differ by gauges. In particular, the dependence on the length parameters $\alpha_1, \alpha_2, \alpha_3$ should be a pure gauge artifact. Indeed, one has

$$\begin{aligned}
 V^c = & V^{CSV} \exp \left[-\frac{1}{\gamma} L_1^{(1)} - \frac{\gamma-1}{2\gamma^2} L_2^{(1)} - \frac{(\gamma-1)(4\gamma-5)}{12\gamma^3} L_3^{(1)} + \dots \right] \\
 & \times \exp \left[\frac{\gamma}{1-\gamma} L_1^{(2)} + \frac{\gamma}{2(1-\gamma)^2} L_2^{(2)} + \frac{\gamma(\gamma+4)}{12(1-\gamma)^3} L_3^{(2)} + \dots \right]_{(6)} \\
 & \times \exp \left[(\gamma-1) L_1^{(3)} + \frac{\gamma(\gamma-1)}{2} L_2^{(3)} - \frac{\gamma(1-\gamma)(5\gamma-1)}{12} L_3^{(3)} + \dots \right], \\
 & \gamma = -\frac{\alpha_1}{\alpha_3}.
 \end{aligned}$$

Proof of this identity goes beyond the scope of this paper, and will be presented later¹⁶⁾. It shows that the two vertices differ only by conformal transformations on the external lines, and so give rise to the same S-matrix elements. Intuitively, it is the presence of these conformal transformations which solve the duality problem, as we shall explain later¹⁶⁾; when computing the four-point function with V^c , one ends with the same integrand as with V^{CSV} , which is the integrand of the Euler beta function, but the integration range is different, and depends on the external α 's.

A further attractive feature of the proposed vertex is that it is equivalent to the overlap integral^{14),15)} between three strings, namely

$$\begin{aligned}
 V^c | \psi_1 \rangle | \psi_2 \rangle | \psi_3 \rangle = & \int \mathcal{D} x_1^\mu(\eta_1) \mathcal{D} x_2^\mu(\eta_2) \mathcal{D} x_3^\mu(\eta_3) \times \\
 & \psi_1 [x_1^\mu(\eta_1)] \psi_2 [x_2^\mu(\eta_2)] \psi_3 [x_3^\mu(\eta_3)] \times \\
 & \delta (x_3^\mu(\eta_3) - \theta(\beta_2 \pi - \eta_3) x_2^\mu(\eta_2) - \theta(\eta_3 - \beta_1 \pi) x_1^\mu(\eta_1)),^{(7)}
 \end{aligned}$$

where

$$\eta_2 = \frac{1}{\alpha_2} \sigma_2, \quad 0 \leq \sigma_2 \leq \pi \alpha_2, \quad \beta_2 = -\frac{\alpha_3}{\alpha_2}, \quad \beta_1 = -\frac{\alpha_1}{\alpha_3}.$$

This fact is established by a careful study of the analogous proof by

Gremmer and Gervais¹⁴⁾ for the light-cone case.

Having decided on the ϕ^3 vertex and equipped with the free action of Ref. 9), we can consider completing the vertex to obtain the coupling of the supplementary fields. Following Ref. 1), the variation of the ϕ^3 term under the order $1/g$, i.e., the field independent, transformation of ϕ is given by

$$g V^c L_{-n}^{(1)} |\Lambda_n^{(1)}\rangle | \psi_2 \rangle | \psi_3 \rangle, \quad |\Lambda_n\rangle \sim O(\frac{1}{g}). \quad (8)$$

This term must be cancelled against the order g^0 variations of the fields coming from the free action, as well as the order g^{-1} variations of interaction terms involving supplementary fields. Clearly, this requires the covariant vertex V^c to satisfy identities relating the action of the L 's on the external legs. We now give a heuristic derivation of "suitable" identities using functional language. The Virasoro operators are given by

$$L_n = \frac{\alpha'}{2i} \oint \frac{dz}{z} z^n P^2(\gamma), \quad (9)$$

where

$$P^2(\gamma) = \left[-i\hbar \frac{\delta}{\delta x^\mu(\gamma)} + \frac{1}{2\pi\alpha'} \frac{dx^\mu(\gamma)}{d\sigma} \right]^2, \quad z = e^{-i\gamma} \quad (10)$$

Using the relation

$$\frac{\delta}{\delta x_3^\mu(\gamma_3)} = \frac{1}{\beta_2} \frac{\delta}{\delta x_2^\mu(\gamma_2)} \quad (11)$$

and similar identities, in the appropriate integration regions, we find that

$$\begin{aligned} & V \oint \frac{dz_3}{z_3} f(z_3) P^2(z_3) + \frac{1}{\beta_2} V \oint \frac{dz_2}{z_2} f(z_3(z_2)) P^2(z_2) \\ & + \frac{1}{\beta_1} V \oint \frac{dz_1}{z_1} f(z_3(z_1)) P^2(z_1) = 0 \end{aligned} \quad (12)$$

where V is the vertex in x space, namely the δ functions given in Eq. (7).

At first sight, it might seem that one could choose any function f and obtain a valid identity. This is not the case and it illustrates a point that we wish to stress; namely functional expressions are generally ill-defined and it

is rather easy to carry out manoeuvres that lead to incorrect results. The only true criterion for establishing a correct answer is to transform the functional equations to corresponding operator expressions and then establish the desired result. An illustration of this is given if we take to be $\cos n\eta_3$; when transformed to operator language, this leads to a meaningless, i.e., divergent identity, a fact that is obvious in oscillator form.

In fact, the correct and unique functions to take are rational functions of Z which vanish at the point common to the three strings,

with

$$Z = \frac{e^{-\gamma(z_3)}}{z_3}, \quad z_1 = (-z_3)^{\frac{\alpha_3}{\alpha_1}}, \quad z_2 = -z_3^{\frac{\alpha_3}{\alpha_2}},$$

$$\gamma(z_3) = -\frac{\alpha_1}{\alpha_3} \ln(1 + z_3 e^{\gamma(z_3)}). \quad (13)$$

This choice possesses convergent expansions simultaneously in z_1 , z_2 and z_3 . For example:

$$Z = \frac{1}{z_3} - \gamma - \gamma(\gamma-1) \sum_{n=1}^{\infty} z_3^n \frac{\gamma^n}{n+1} f_n(\gamma), \quad \gamma = -\frac{\alpha_1}{\alpha_3},$$

$$Z = -1 - z_1 \sum_{n=0}^{\infty} \frac{\gamma'(n+1)}{\gamma' - (1-\gamma')^n} z_1^n f_{n+1}(\gamma'), \quad \gamma' = -\frac{\alpha_2}{\alpha_1},$$

$$Z = \sum_{n=1}^{\infty} \frac{\gamma^n}{n\gamma^n-1} (1-\gamma^n) f_n(\gamma^n) z_2^n, \quad \gamma'' = -\frac{\alpha_3}{\alpha_2}. \quad (14)$$

From Ref. 14), we recognize in the function Z of Eq. (13) the mapping of the three interacting string diagram onto the upper half plane. Taking $f = Z$, we see from Eq. (12) that the expansions (15) give identities between series of L_n 's on the three external lines. Explicitly, one obtains:

$$V^c \left[L_{-1}^{(3)} + (1-2\gamma) L_0^{(3)} + \frac{1}{2} \gamma(1-\gamma) L_1^{(3)} + \frac{1}{3} \gamma(1-\gamma)(2\gamma-1) L_2^{(3)} + \dots \right]$$

$$= \frac{\alpha_3}{\alpha_2} V^c \left[L_1^{(2)} + \frac{\gamma}{1-\gamma} L_2^{(2)} + \dots \right] + V^c L_0^{(2)} \quad (15a)$$

$$+ \frac{\alpha_3}{\alpha_1} V^c \left[L_1^{(1)} + \frac{\gamma-1}{\gamma} L_2^{(1)} + \dots \right] - V^c L_0^{(1)}, \quad \gamma = -\frac{\alpha_1}{\alpha_3}.$$

The identities (15b) and (15c) obtained with $f = 1/Z$ and $f = 1/Z-1$ differ from this one by cyclic permutations of the external lines, (15b) by $123 \rightarrow 231$ and (15c) by $123 \rightarrow 312$. When cycling we must, of course, also cycle the string

lengths. We note that Eq. (15a) is just the transform of the corresponding equation for the Caneschi-Schwimmer-Veneziano vertex¹⁷⁾ by gauge transformations of Eq. (6).

$$V^{CSV} \left(L_{-1}^{(3)} - L_0^{(3)} + L_1^{(2)} - L_0^{(2)} + L_0^{(1)} \right) = 0 \quad (16)$$

Similarly, taking $f = Z^2$ gives an identity involving L_{-2} , in which the space-time dimension $D = 26$ explicitly occurs, which is the gauge transform of the corresponding identity of V^{CSV} :

$$V^{CSV} \left(L_{-2}^{(3)} - L_0^{(3)} - L_2^{(2)} + 3L_1^{(2)} - 2L_0^{(2)} - \sum_{p=1}^{\infty} L_p^{(1)} + L_0^{(1)} \right) = 0 \quad (17)$$

We also note the similarity of Eq. (15a) with the gauge identities of the fermion emission vertex¹⁸⁾; our covariant vertex V^C acts as a junction between waveguides: a gauge operator L_{-1} incident on line 3 is partly transmitted on lines 1 and 2 as series of L_n 's, and partly reflected back on line 3 as another series of L_n 's. A more detailed treatment of the derivation of all these identities will be given in Ref. 16), using the oscillator formalism. It is remarkable that the identities which are well-defined are precisely of the required form to enable the construction of the gauge covariant vertex, as we shall see below.

Equipped with the ϕ^3 vertex of Eq. (1) and the identities of Eq. (12) with the correct choice of functions f , discussed above, we have in effect uniquely determined the entire vertex, i.e., terms involving the supplementary fields. This is done through the use of the algorithm given in Ref. 1). The algorithm works by first finding the transformations of all the fields, including the supplementary fields from the identities. Given the field transformations, we can find the variation of the free action which in turn is cancelled by the variation of the interaction term and so uniquely determines the interaction.

We now illustrate this procedure at the lowest level. At this level, we require only the fields $\psi = \psi_0^0$, $\zeta = \zeta_1^1$ and $\phi = \phi^1$ and the free action is given by

$$\begin{aligned} & \frac{1}{2} \langle \psi | (L_{-1}) | \psi \rangle + \langle \phi | L_1 | \psi \rangle + \langle \zeta | L_1 | \phi \rangle \\ & + \langle \phi | \phi \rangle - \frac{1}{2} \langle \zeta | (L_{+1}) | \zeta \rangle \end{aligned} \quad (18)$$

and is invariant under

$$\begin{aligned}\delta\langle\psi| &= \langle\wedge|L_1, \quad , \delta\langle\phi| = -\langle\wedge|L_0, \\ \delta\langle\zeta| &= -\langle\wedge|L_1.\end{aligned}\tag{19}$$

We now find the non-Abelian transformations of these fields by utilizing the following observation. The commutator of the non-Abelian transformation and that of Eq. (19) must again yield a transformation of the type of Eq. (19). When translated in terms of identities, this requires the existence of an identity which has only suitable L's on one leg in terms of other L's on the other legs. For the case of $\langle\phi|$, this means only L_{+n} 's on one leg and at this level only L_{+1} . Manipulating Eqs. (15), we find the unique identity to be given, at this level, by

$$\begin{aligned}V^c \left[L_1^{(3)} + L_{-1}^{(1)} + L_{-1}^{(2)} + 2 \left(\frac{\alpha_2}{\alpha_1} L_0^{(1)} - \frac{\alpha_1}{\alpha_2} L_0^{(2)} \right) \right. \\ \left. + \frac{1}{2} \frac{\alpha_1}{\alpha_2} (-3\alpha_2 - \alpha_1) L_1^{(2)} + \frac{1}{2} \frac{\alpha_2}{\alpha_1^2} (-3\alpha_1 - \alpha_2) L_1^{(1)} \right] = 0\end{aligned}\tag{20}$$

From this equation we find the transformation of $|\phi\rangle$ to be

$$\begin{aligned}\delta\langle\psi_3| &= \langle\wedge_3|L_1^{(3)} + gV^c [|\psi_1\rangle|\wedge_2\rangle - (1\leftrightarrow 2)] \\ &\quad - 2gV^c \left[\frac{\alpha_2}{\alpha_1} |\phi_1\rangle|\wedge_2\rangle + (1\leftrightarrow 2) \right] \\ &\quad + \frac{1}{2} gV^c \left[\frac{3\alpha_1 + \alpha_2}{\alpha_1^2} \alpha_2 |\phi_1\rangle|\wedge_2\rangle - (1\leftrightarrow 2) \right]\end{aligned}\tag{21}$$

Using the same technique we find that

$$\begin{aligned}\delta\langle\phi_3| &= -\langle\wedge_3|L_0^{(3)} + gV^c \left[-\frac{\alpha_2}{\alpha_3} |\psi_1\rangle|\wedge_2\rangle \right. \\ &\quad \left. + \frac{\alpha_1}{\alpha_3} |\wedge_1\rangle|\psi_2\rangle \right] + gV^c \left[\frac{2\alpha_2^2 + \alpha_1\alpha_2 + \alpha_1^2}{\alpha_1\alpha_3} |\phi_1\rangle|\wedge_2\rangle \right]\end{aligned}\tag{22}$$

$$\begin{aligned}
 & - \left(\frac{2\alpha_1^2 + \alpha_2\alpha_1 + \alpha_2^2}{\alpha_3\alpha_2} \right) |\Lambda_1\rangle |\phi_2\rangle \Big] \\
 & + g V^c \left[\left(\frac{1}{2} \frac{\alpha_2^2}{\alpha_1^2} + \frac{\alpha_2}{\alpha_3} \right) |\phi_1\rangle |\Lambda_2\rangle + (1 \leftrightarrow 2) \right] \quad (22) \\
 & \text{cont.}
 \end{aligned}$$

and

$$\begin{aligned}
 \delta \langle \phi_3 | &= - \langle \Lambda_3 | L_{-1}^{(3)} + g V^c \left[- \frac{\alpha_2}{2\alpha_3^2} (2\alpha_2 - 3\alpha_1) |\phi_1\rangle |\Lambda_2\rangle \right. \\
 & \left. + \frac{\alpha_1}{2\alpha_3^2} (2\alpha_1 - 3\alpha_2) |\Lambda_1\rangle |\phi_2\rangle \right] \\
 & + g V^c \left[\left(\frac{2\alpha_2}{\alpha_1} - \frac{5\alpha_2^2}{\alpha_3^2} \right) |\phi_1\rangle |\Lambda_2\rangle + (1 \leftrightarrow 2) \right] \quad (23) \\
 & + g V^c \left[\left(- \frac{\alpha_3}{\alpha_1} + \frac{\alpha_2}{\alpha_3} + \frac{5}{2} \frac{\alpha_2^2}{\alpha_3^2} - \frac{\alpha_1^2}{2\alpha_1^2} - \frac{5}{4} \frac{\alpha_2^2}{\alpha_3\alpha_1} \right) |\phi_1\rangle |\Lambda_2\rangle \right. \\
 & \left. - (1 \leftrightarrow 2) \right]
 \end{aligned}$$

The reader will have observed the various 1 to 2 interchanges that occur in the vertex. These interchanges are also required from Chan-Paton¹⁹⁾ considerations: For the open string, we take $|\phi\rangle$ to be a matrix which takes different forms at various levels;

$$\langle \phi |_a^b = \langle 0 | \left[\varphi_s (\mu_s)_a^b + \alpha_{-1}^\mu A_\mu^i (\lambda_i)_a^b + \dots \right] \quad (24)$$

For $SO(N)$, for example, at even levels the matrix μ is symmetric while at odd levels the matrix λ is antisymmetric [see Ref. 20)]. Clearly, $\delta \langle \phi |$ must have the same matrix structure as $\langle \phi |$. To ensure this, the transformation of an arbitrary bra $\langle s |$ into an arbitrary ket $|T\rangle$ must be of the form

$$\begin{aligned}
 \delta \langle s_3 |_a^b &= \bar{V}(1,2,3) |T_1\rangle_a^c |\Lambda_2\rangle_c^b \\
 & + (-1)^N \bar{V}(1,2,3) |\Lambda_1\rangle_a^c |T_2\rangle_c^b \quad (25)
 \end{aligned}$$

where $N = N_1 + N_2 + N_3$ and N_i is the level of the i^{th} field and $\bar{V}$ is a vertex

which satisfies

$$\bar{V}(1,2,3) = (-1)^{H_1+H_2+H_3} \bar{V}(2,1,3) \quad (26)$$

where $H = L_0 - \frac{1}{2}p^2$. Consequently, we must replace V^c in all our transformation laws by $\bar{V} = V^c(1,2,3) + (-1)^{H_1+H_2+H_3} V^c(2,1,3)$ and similarly in the vertex. For simplicity we will take this to be understood.

We now vary the free action (18) under the above transformations and collect all terms of given type. For example, one finds a term of the form

$$g V^c \left[- (L_0^{(1)} - 1) \left(\frac{\alpha_2}{\alpha_3} - \frac{1}{2} \frac{\alpha_1 \alpha_2}{\alpha_3^2} \right) - \left[\frac{1}{2} \frac{\alpha_2^2}{\alpha_3^2} + \frac{\alpha_2}{\alpha_3} \right] L_1^{(1)} \right. \\ \left. - \frac{\alpha_2}{\alpha_3} L_{-1}^{(3)} + \frac{\alpha_2}{2\alpha_3^2} (2\alpha_2 - 3\alpha_1) (L_0^{(3)} + 1) \right] | \psi_1 \rangle | \psi_2 \rangle | \psi_3 \rangle \quad (27)$$

We must be able to express this entirely in terms of $L^{(2)}$'s acting on $|\Lambda\rangle_2$ by use of the identities; fortunately, $-\alpha_2/\alpha_3$ (15c) $- \frac{1}{2} \alpha_1 \alpha_2 / (\alpha_3)^2$ (15b) is just such an identity and we find that this particular term becomes

$$g V^c \left\{ \frac{\alpha_1 \alpha_2}{2\alpha_3^2} L_{-1}^{(2)} - L_0^{(2)} \left[\frac{\alpha_2}{\alpha_3} + \frac{1}{2} \frac{\alpha_1}{\alpha_3^2} (\alpha_1 - \alpha_3) \right] \right. \\ \left. - \left(1 - \frac{1}{4} \frac{\alpha_1^2}{\alpha_2 \alpha_3} \right) L_1^{(2)} \right\} | \psi_1 \rangle | \psi_2 \rangle | \psi_3 \rangle \quad (28)$$

Such conspiracies give one confidence in the algorithm and one finds the existence of similar identities for all other terms. The cubic vertex is then the one whose variation under the transformations of Eq. (19) cancels all terms in Eq. (28) as well as its companions. This involves further non-trivial checks and it is found to be given uniquely by

$$\begin{aligned}
 & gV^c \left\{ | \psi_1 \rangle | \psi_2 \rangle | \psi_3 \rangle \right. \\
 & + \frac{\alpha_1 - \alpha_3}{\alpha_2} | \psi_1 \rangle | \phi_2 \rangle | \psi_3 \rangle \quad + \text{cyclic} \\
 & - \frac{\alpha_1 \alpha_3}{2 \alpha_2^2} | \psi_1 \rangle | \zeta_2 \rangle | \psi_3 \rangle \quad + \text{cyclic} \\
 & + \left[\frac{\alpha_3}{\alpha_2} + \frac{1}{2} \frac{\alpha_1}{\alpha_2} (\alpha_1 - \alpha_2) \right] | \psi_1 \rangle | \zeta_2 \rangle | \phi_3 \rangle \quad + \text{cyclic} \\
 & - \left[\frac{\alpha_1}{\alpha_2} - \frac{1}{2} \frac{\alpha_3}{\alpha_2^2} (\alpha_2 - \alpha_3) \right] | \phi_1 \rangle | \zeta_2 \rangle | \psi_3 \rangle \quad + \text{cyclic} \\
 & - \left[-1 + 2 \left(\frac{\alpha_1}{\alpha_3} + \frac{\alpha_3}{\alpha_1} \right) \right] | \phi_1 \rangle | \zeta_2 \rangle | \phi_3 \rangle \quad + \text{cyclic} \quad (29) \\
 & - \frac{2 \alpha_2^2}{\alpha_1 \alpha_3} | \phi_1 \rangle | \psi_2 \rangle | \phi_3 \rangle \quad + \text{cyclic} \\
 & - \left(1 - \frac{1}{4} \frac{\alpha_1^2}{\alpha_2 \alpha_3} \right) | \psi_1 \rangle | \zeta_2 \rangle | \zeta_3 \rangle \quad + \text{cyclic} \\
 & + \left(\frac{1}{2} \frac{\alpha_2^2}{\alpha_1^2} - \frac{1}{2} \frac{\alpha_2^2}{\alpha_3^2} + \frac{\alpha_2}{\alpha_3} - \frac{\alpha_2}{\alpha_1} + \frac{1}{4} \frac{\alpha_2}{\alpha_3} - \frac{1}{4} \frac{\alpha_2}{\alpha_1} - \frac{\alpha_1}{\alpha_2} + \frac{\alpha_3}{\alpha_2} \right) \\
 & \quad \times | \zeta_1 \rangle | \phi_2 \rangle | \zeta_3 \rangle \quad + \text{cyclic} \\
 & \left. + \left(\frac{15}{8} + \frac{1}{2} \frac{\alpha_2 \alpha_3}{\alpha_1^2} + \frac{1}{2} \frac{\alpha_1 \alpha_2}{\alpha_3^2} + \frac{1}{2} \frac{\alpha_1 \alpha_3}{\alpha_2^2} \right) | \zeta_1 \rangle | \zeta_2 \rangle | \zeta_3 \rangle \right\}
 \end{aligned}$$

We note that questions of convergence were crucial for obtaining the correct identities and it was these identities that were used successfully to find an invariant action. Had we overlooked the convergence problem, we would have found that the identities could not be used to construct an invariant action even at the lowest level.

Having found the interactions at lowest level, we now turn to their expression in closed form. To find this by the techniques given above would be tedious and it is better to consider the formulation of the free gauge covariant formulation of Ref. 9). This was formulated in terms of fields which were functionals of $x^\mu(\sigma)$, $c(\sigma)$ and $\bar{c}(\sigma)$ where the latter two co-ordinates are anticommuting. In oscillator language we then have the bosonic α^μ oscillations and the fermionic oscillators β_n , $\bar{\beta}_n$. The reader is referred to Ref. 9) for more details. A reasonable guess for the vertex is to take the overlap in δ functions in the full co-ordinate space, i.e.,

$$T \delta(x_3^\mu - x_2^\mu - x_1^\mu) \delta(c_3 - c_2 - c_1) \delta\left(\frac{\bar{c}_3}{\alpha_3} - \frac{\bar{c}_2}{\alpha_2} - \frac{\bar{c}_1}{\alpha_1}\right), \quad (30)$$

where the insertion of θ functions are to be understood. It is easily seen that this vertex for $T = 1$ would not lead to the correct ϕ^3 term. Stated another way, this vertex would have the wrong ghost number. The simplest way to remedy this is to insert a T such that the vertex contains the product of the three ghost fields $(c_1(\sigma_{int})c_2(\sigma_{int})c_3(\sigma_{int}))$ at the interaction point.

Let us consider the above vertex and evaluate it in a well-defined form, namely in oscillator language. The resulting vertex will be of the form

$$V^{CT} = \langle 0_{123} | V^C \exp(\Delta_2 + \Delta_1), \quad (31)$$

where Δ_2 and Δ_1 are of the generic form

$$\Delta_2 \sim \beta_m \bar{\beta}_m, \quad n, m > 0; \quad \Delta_1 \sim \beta_n \bar{\beta}_0, \quad n > 0.$$

The expression for Δ_2 is insensitive to the form of T and only depends on the non-zero mode part of the overlap δ functions. One finds that

$$\Delta_2 = - \sum_{\substack{n, m \\ n, m}} \beta_n^{(n)} m N_{nm}^{rs} \bar{\beta}_m^{(m)}, \quad (32)$$

where the $\bar{N}_{nm}^{rs}$ is the quantity which was defined previously in Eq. (2). We take Δ_1 at lowest level to be given by

$$\Delta_1 = - \frac{1}{\alpha_3} (\alpha_1 P^2 + \alpha_2 P^1) \beta_1^{(3)} e^{\frac{\bar{\beta}_3}{\alpha_3}} + \text{cyclic}, \quad (33)$$

where

$$P^2 = (\alpha_3 \bar{\beta}_0^{(1)} - \alpha_1 \bar{\beta}_0^{(3)}), \quad P^1 = (\alpha_2 \bar{\beta}_0^{(3)} - \alpha_3 \bar{\beta}_0^{(2)})$$

$$P^3 = (\alpha_1 \bar{\beta}_0^{(2)} - \alpha_2 \bar{\beta}_0^{(1)}).$$

We must also specify the vacuum to be used in the vertex; in the notation of Ref. 9), we take it to be $\langle 1 = \begin{pmatrix} \langle + \\ 1 \end{pmatrix} \begin{pmatrix} \langle + \\ 2 \end{pmatrix} \begin{pmatrix} \langle + \\ 3 \end{pmatrix}$. This ensures the existence of a ϕ^3 piece and is annihilated by Q .

Inserting the lowest level expansion of the string functional

$$| \psi \rangle = (\psi + \xi \beta_1 \bar{\beta}_{-1}) | - \rangle + \phi \bar{\beta}_{-1} | + \rangle \quad (34)$$

into the above vertex and evaluating the results, we find complete agreement with the interaction found in Eq. (29).

Rather than work from an explicit vertex in functional form and have to carry out functional integrations and delicate point splittings associated with the ghosts at the interaction points, we use a suitably adapted set of Goto-Naka defining equations. By suitable we do not mean taking the equations relating the Fourier modes of the three strings, but rather taking the mode analysis resulting from integrating with the functions of Eq. (13). For example for $\bar{\beta}$, one should have the same as Eq. (15a) with L 's replaced by $\bar{\beta}$ taking account of the scalings. Indeed one finds that the vertex of Eqs. (31) and (32) does satisfy this identity and this confirms the choice of Δ_2 to all levels, as well as Δ_1 at first level. The higher levels of Δ_1 are uniquely determined by the higher identities for $\bar{\beta}$, taking $f(z) = z^2, z^3$, etc., as will be explained in Ref. 16).

Having found the vertex in the oscillator formalism, it can be transformed in functional form where, as explained above, it must involve the overlap δ functions. The only ambiguity is the choice of T , i.e., the assignment of ghosts at the interaction point. This involves a delicate limiting procedure at the interaction point^{13),15)} but the starting point can only be the product of the three ghost co-ordinates at the interaction point. This is supported by the fact that we must for the $\bar{\beta}$ identities take a function that vanishes at the common point of all three strings.

The transformation laws of the fields at all levels are uniquely determined by demanding invariance of the action; the corresponding vertices will be given in Ref. 16). In functional form the vertex in the transformation laws will have different ghost co-ordinate insertions at the common point. Since the free action is of the form $\langle \chi | Q | \chi \rangle$, our identities imply elegant properties for the action of Q on the vertex.

Given the vertex of Eq. (31), we may calculate an arbitrary four-point function. This calculation closely follows that of Ref. 14). The two anti-commuting oscillator β and $\bar{\beta}$ will provide an additional determinant. The 26-dimensional orbital determinant is then reduced to the 24-dimensional light cone determinant which is crucial for duality and the recovery of the well-known four-point function. In this calculation, as for the light cone case, one will find that the cubic interaction does not complete the correct integration regions and we will require a quartic interaction in the action. No doubt, in functional form this will involve similar overlap functions to the light cone case extended to $d = 26$ and the anticommuting co-ordinates.

We would like to remark that our gauge covariant vertex retains the beautiful geometrical interpretation of the splitting and joining of strings found in the light cone gauge. We see no obstacle for the extension of our work to closed or supersymmetric strings; it is a matter of repeating the procedures given in this paper.

For supersymmetric theories the finiteness properties were apparent once the theories were formulated in a manifestly symmetric manner. The finiteness properties of string theories may also become apparent in a gauge covariant formulation. The natural context for the gauge covariant theories is the coordinate space involving the anticommuting co-ordinates. The string Feynman rules could be formulated in this superspace and these rules could lead to restrictions similar to those for the superFeynman rules of point particle theories on the type of counterterms that could arise.

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NOTE ADDED

After this work was completed, we received two papers^{22),23)} on interacting string theory. The paper of Ref. 22) agrees with many features of our paper. The relation of the work of E. Witten²³⁾ to our work and also to the standard geometrical picture of splitting and joining of strings is not clear to us.

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