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A QUANTUM COSMOLOGICAL APPROACH TO KALUZA-KLEIN THEORY
AND THE BOUNDARY CONDITION OF "NO BOUNDARY"

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A B S T R A C T

A quantum cosmological model of the Kaluza-Klein theory is investigated. The Wheeler-De Witt equation of $(D+1)$ -dimensional gravity with a $(D-4)^{\text{th}}$ rank antisymmetric tensor field (ATF) is analyzed, restricting ourselves to a minisuperspace of topology $R \times S^3 \times S^n$. The initial condition problem is discussed and the boundary condition of "no boundary" for a Universe corresponding to a manifold with topology of Kaluza-Klein type is formulated. The analysis, using the WKB approximation, shows that a subclass of classical paths which contribute to the wave function consists of bounce solutions. Furthermore, it is found that initial values of the classical paths, leading to Universes of large scale and a compactified internal space of Planck size, can be localized in a very restricted region.

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1. - INTRODUCTION

One of the central problems in cosmological models is the choice of the initial conditions which are necessary for solving the dynamical equations of the system at hand. It is a well-known difficulty that in a model which should describe the Universe there is no "outside of the Universe" to pass their specification off to. The question of whether there is a principle from which we can deduce these initial conditions is still open.

A very appealing proposal to this problem has recently been put forward by Hawking and further developed by Hartle and Hawking in an attempt to describe quantum cosmology¹⁾. Their proposal of the boundary conditions is that the Universe has no boundary. They formulate the wave function of the Universe through the functional integral over Euclidean metrics $g_{\alpha\beta}$ and matter fields

$$\Psi(h_{ij}, \hat{\Phi}) = \int_C [dg_{\alpha\beta}] [d\Phi] \exp[-I(g_{\alpha\beta}, \Phi)] \quad (1)$$

and the boundary condition is applied by specifying the domain C of the path integral. The domain C is defined from the condition of "no boundary" as all regular compact Euclidean four geometries, the boundary of which is a three-manifold with the induced three-metric h_{ij} , and the regular matter field configuration, the value of which is given by $\hat{\Phi}$ on the three manifold.

Although their proposal does not give an explanation of the choice of the boundary conditions, it is very attractive because it leads to a self-contained model of cosmology.

An alternative boundary condition has also been proposed from a different point of view of the very early Universe in the context of the creation of the Universe from "nothing"^{2),3)}. The investigation of cosmological models resulting from these two different approaches will be an interesting future problem.

The same type of difficulty regarding the initial conditions appears in the Kaluza-Klein theories⁴⁾. The key point of the Kaluza-Klein theory is to find a reasonable explanation for the large separation of the scale of our three-dimensional space from the scale of the internal one. To find a possible scenario for this separation, many authors have investigated time-dependent solutions of the Einstein equations in the higher-dimensional theory of gravity, with or without including quantum effects⁵⁾. However, all of the scenarios so far proposed depend very much on the initial conditions.

The purpose of this paper is to investigate the problem of the initial conditions in the Kaluza-Klein theory from the point of view of quantum cosmology. We investigate the simplest model which has a stable, static solution for compactification, i.e., the Einstein theory with an antisymmetric tensor field (ATF)^{6),7)}. We consider the compactification into the manifold of $R \times S^3 \times S^n$, and therefore the minisuperspace of our model is the two-dimensional space of the two scales a and b , which are the radii of S^3 and S^n , respectively. The boundary conditions will be investigated, paying attention to the applicability of Hawking's proposal to a higher dimensional theory of gravity. We formulate the boundary condition of "no boundary" for the Kaluza-Klein theories. There, the domain of the path integration is specified as the class of all regular Euclidean $(D+1)$ -geometries, the boundary of which is given by $S^3 \times S^n$.

After specifying the boundary conditions, the numerical analysis of the classical Lorentzian and Euclidean solutions is performed, which gives us a rough idea of the behaviour of the wave function at the semi-classical level. We are able to specify a subclass of those classical solutions which contribute to the wave function.

The investigation of this subclass of classical paths shows the following features.

If the Universe starts with initial values in a certain region, i.e., around the local maximum of the potential $W(b)$ in the (a,b) plane, it undergoes an inflation and a long expansion period of the scale of S^3 , a , follows. Starting from a point slightly distant from this particular region, the solutions of the equations of motion also exhibit an expansion, however, in this case the expansion time is too short and the corresponding model Universe soon collapses. It is also found that the chaotic scenario is unlikely to be realized in our Kaluza-Klein model, since in this case the radius of the internal space does not contract to Planck size but expands.

In the minisuperspace model of the Kaluza-Klein theory with topology $R \times S^3 \times S^n$, a singularity occurs when both scales a and b go to zero. This singular behaviour causes difficulties in our attempt to calculate explicitly the wave function of our model Universe.

This paper is organized as follows. In Section 2, we introduce our model and give the Wheeler-De Witt equation. In Section 3, we formulate the boundary condition of "no boundary" in Kaluza-Klein theory and analyze the classical paths

which contribute to the wave function in the WKB approximation. Section 4 is devoted to discussions and conclusions.

2. - THE MODEL

The model we consider is the (D+1)-dimensional Einstein theory with a (D-4)th rank ATF and a cosmological term⁶⁾. The action is

$$S^{D+1} = \int d^{D+1}x \sqrt{-\hat{g}} \left\{ \frac{1}{16\pi G_{D+1}} (R + 2\Lambda) - \frac{1}{2n!} \hat{F}_{M_1, \dots, M_n} \hat{F}^{M_1, \dots, M_n} \right\} + (\text{surface terms}) \quad (2)$$

where R is the (D+1)-dimensional scalar curvature, Λ is the cosmological constant, $\hat{g}_{MN}$ is the (D+1)-dimensional metric with signature $(-, +, +, \dots, +)$ and $\hat{g} = \det(\hat{g}_{MN})$, $(M, N = 0, 1, \dots, D)$. G_{D+1} is the (D+1)-dimensional gravitational constant. $\hat{F}_{M_1, \dots, M_n}$ is the field strength of the ATF: $\hat{F}_{M_1, \dots, M_n} = n \partial_{[M_1} \hat{A}_{M_2, \dots, M_n]}$. We introduced the notation $n = D-3$.

We formulate our theory in the Arnowitt-Deser-Misner (ADM) parametrization⁸⁾ in (D+1) dimensions. The Einstein action becomes

$$S_E^{D+1} = \frac{1}{16\pi G_{D+1}} \int dt d^Dx \sqrt{-\hat{g}} \left({}^D R + K_{\hat{m}\hat{n}} K^{\hat{m}\hat{n}} - (K_{\hat{m}}^{\hat{m}})^2 \right) \quad (3)$$

${}^D R$ is the D-dimensional scalar curvature and $K_{\hat{m}\hat{n}}$ and $K_{\hat{m}}^{\hat{m}}$ are the extrinsic curvature and its trace, respective ($\hat{m}, \hat{n} = 1, \dots, D$).

The action given by Eq. (3) has an infinite number of degrees of freedom. To reduce the degrees of freedom to a finite number we restrict ourselves to a minisuperspace, taking the metric to be that of $R \times S^3 \times S^n$:

$$ds^2 = \sigma^2 \left(-\hat{N}^2 dt^2 + \hat{a}^2(t) d\Omega_3^2 + \hat{b}^2(t) d\Omega_n^2 \right) \quad , \quad (4)$$

where $d\Omega_p^2$ is the metric of a unit p-sphere, $\hat{N}$ is the lapse function, $\hat{a}(t)$ and $\hat{b}(t)$ are the time-dependent scale factors of external and internal space, respectively. σ is the unit of the scale and will be defined in the following.

In the given metric the extrinsic curvature term in Eq. (3) reads

$$K_{\hat{m}\hat{n}} K^{\hat{m}\hat{n}} - (K_{\hat{m}}^{\hat{m}})^2 = (\hat{N}\sigma)^{-2} \left(-6 \frac{\dot{\hat{a}}^2}{\hat{a}^2} - n(n-1) \frac{\dot{\hat{b}}^2}{\hat{b}^2} - 3n \frac{\dot{\hat{a}}\dot{\hat{b}}}{\hat{a}\hat{b}} \right), \quad (5)$$

where an overdot denotes a derivative with respect to the time. The scalar curvature is split into two parts as

$${}^D R = {}^3 R + {}^n R = \frac{6}{\sigma^2 \hat{a}^2} + \frac{n(n-1)}{\sigma^2 \hat{b}^2}. \quad (6)$$

The cosmological term in the parametrization of our minisuperspace is given by

$$S^{D+1} = \frac{12\sigma^{D-1}}{16\pi G_{D+1}} 2\pi^2 V(S^n) \int dt \hat{a}^3 \hat{b}^n \hat{N} \left(-\frac{\lambda}{2} \right), \quad (7)$$

where $\lambda = (\sigma^2/3)\Lambda$ and $V(S^n)$ is the volume of a unit n -sphere (recall that $n = D-3$).

It is known that the Einstein gravity (with a cosmological term) does not possess a stable static solution realizing compactification of the extra space (except in the five-dimensional theory). The introduction of the ATF provides a local minimum with respect to the scale $\hat{b}$ of the internal space which is necessary for the existence of a static classical solution⁹⁾. For the ATF we take the ansatz⁶⁾

$$\begin{aligned} \hat{F}_{i_1 \dots i_n} &= f \sigma^{n-1} \sqrt{12\tilde{g}} \epsilon_{i_1 \dots i_n}, \quad \text{for the internal space} \\ \hat{F}_{M_1 \dots M_n} &= 0, \quad \text{otherwise.} \end{aligned} \quad (8)$$

Here, a latin suffix is for the internal space, i.e., $i, j = D-n, \dots, D$. $\tilde{g}$ is the determinant of the n -dimensional metric of a unit n -sphere $\tilde{g}_{ij}$. $\epsilon_{i_1, \dots, i_n}$ is the n -dimensional Levi-Civita symbol. This ansatz satisfies the equation of motion of the ATF. f has to be a constant so that the Bianchi identity for the ATF is satisfied.

Then, the total action in ADM parametrization is given by

$$S = \frac{12}{16\pi G_{D+1}} 2\pi^2 V(S^n) \int dt \frac{1}{2} \hat{N} \hat{a}^3 \dot{b}^n \left\{ - \left(\frac{\dot{\hat{a}}}{\hat{N} \hat{a}} \right)^2 - \frac{n(n-1)}{6} \left(\frac{\dot{\hat{b}}}{\hat{N} \hat{b}} \right)^2 + \frac{n(n-1)}{6 \hat{b}^2} - n \left(\frac{\hat{a} \dot{\hat{b}}}{\hat{N}^2 \hat{a} \hat{b}} \right) + \frac{1}{\hat{a}^2} - \lambda - f^2 \hat{b}^{1-2n} \right\}. \quad (9)$$

We choose $\sigma = (2G_{D+1}/3V(S^n)\pi)^{1/(D-1)}$, in order to make unity the factor in front of the integral in Eq. (9).

For the later purpose of the calculation, we diagonalize the kinetic term of the action, choosing the following variables,

$$\hat{a}^2 = b^{-n} a^2, \quad \hat{b}^2 = b^2, \quad \hat{N}^2 = b^{-n} N^2 \quad (10)$$

The choice of this parametrization corresponds to a Weyl rescaling, which is performed in order to get the canonical coefficient for the Einstein scalar curvature term of the effective action in four dimensions. We choose $N = 1$ in the following. The action now reads

$$S(a, b) = \frac{1}{2} \int dt \left(\frac{1}{a} \right) \left\{ - (a \dot{a})^2 + \frac{n(n+2)}{12} a^4 \frac{\dot{b}^2}{b^2} - V(a, b) \right\}, \quad (11)$$

where the "potential" $V(a, b)$ is given by

$$V(a, b) = -a^2 + a^4 W(b) \quad (12)$$

with

$$W(b) = \lambda b^{-n} - \frac{n(n-1)}{6} b^{-(n+2)} + f^2 b^{-3n}. \quad (13)$$

Note that after the Weyl rescaling the kinetic term of a is negative whereas that of b is positive in Eq. (11). The action $S(a, b)$ yields the following equations of motion

$$\ddot{a} = -\frac{1}{2} \frac{\dot{a}^2}{a} - \frac{n(n+2)}{8} a \frac{\dot{b}^2}{b^2} - \frac{1}{2a} + \frac{3}{2} a W(b), \quad (14)$$

$$\ddot{b} = \frac{\dot{b}^2}{b} - 3 \frac{\dot{a}}{a} \dot{b} - \frac{6}{n(n+2)} b^2 \frac{\partial W(b)}{\partial b}. \quad (15)$$

The Hamiltonian constraint is given by

$$H = -\frac{1}{2a} \Pi_a^2 + \frac{6}{n(n+2)} \frac{b^2}{a^3} \Pi_b^2 - \frac{1}{2a} V(a,b) = 0, \quad (16)$$

where the $\Pi_a = -\dot{a}a$ and $\Pi_b = (n(n+2)/12)(a^3/b^2)\dot{b}$ are the canonical conjugate momenta of a and b , respectively.

Canonical quantization leads us to the Wheeler-De Witt equation of our minisuperspace model

$$\left[\frac{1}{a^p} \frac{\partial}{\partial a} a^p \frac{\partial}{\partial a} - a^2 - \frac{12}{n(n+2)} \frac{1}{a^2 b^{q-2}} \frac{\partial}{\partial b} b^q \frac{\partial}{\partial b} + a^4 W(b) \right] \Psi = 0. \quad (17)$$

The p and q represent some ambiguity in the factor ordering. Since we restrict ourselves to the semi-classical approximation this factor is not important and we do not deal with the problem of factor ordering here.

We introduce new variables with which the causal structure of our system will become more transparent:

$$x = \frac{1}{2} a (b^{-\beta} - b^{+\beta}), \quad y = \frac{1}{2} a (b^{-\beta} + b^{+\beta}) \quad (18)$$

with $\beta = (n(n+2)/12)^{\frac{1}{2}}$. In these variables the Wheeler-De Witt equation with $p = 1, q = 1$ reads

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + V(x,y) \right] \Psi(x,y) = 0. \quad (19)$$

The wave function of Eq. (19) is determined by specifying the boundary conditions on the light cone $y = |x|$ in the (x,y) plane. So far, the dimension of space-time is arbitrary except that $D+1 > 6$. In the following numerical computation, $D+1 = 6$ will be chosen for definiteness. The qualitative features of the results will be the same for other dimensions $D+1 > 6$. The following values of the parameters are used in the numerical analysis: $f^2 = 1/(36\lambda)$ and $\lambda = 5$.

When we investigate the Wheeler-De Witt equation and the wave function, we need to know the properties of the potential. The total potential $V(a,b)$ is shown in Fig. 1. For fixed b the potential consists of two terms, one is the curvature term and the other, $a^4 W(b)$, acts effectively like a cosmological term. Along a line of fixed a , $W(b)$ has a local minimum due to the presence of the ATF. We require that the value of $W(b)$ vanishes at its minimum, so that the static

classical solution of Eqs. (14) and (15) represents a product manifold of the flat four-dimensional space-time and the n -dimensional sphere, $M_4 \times S^n$.

Following Hawking, we define the Euclidean and Lorentzian region depending on whether the wave function has exponential behaviour or is oscillating in the scale factor a of the four-dimensional space-time. As was already pointed out in the four-dimensional case¹⁰⁾, these two regions are approximately separated by the line $V(a,b) = 0$ and since the same criterion is useful in our model as well, we show the line $V = 0$ in the (a,b) plane in Fig. 2 and in the (x,y) plane in Fig. 3.

3. - SEMI-CLASSICAL ANALYSIS OF THE WAVE FUNCTION

To study the behaviour of the wave function of the Wheeler-De Witt equation of the minisuperspace, we use the WKB approximation. For this purpose, first we specify the boundary condition of the wave function and give the initial conditions of the classical Euclidean and Lorentzian paths. We apply the boundary condition of "no boundary" and deduce the corresponding initial conditions.

Using these initial conditions, we perform the numerical analysis of the Lorentzian equations of motion in Eqs. (14) and (15) and the Euclidean equations, which are obtained by substituting $t = -i\tau$ into the Lorentzian equations.

3.1. - The boundary condition and the Euclidean region

Hartle and Hawking formulated the wave function by a path integral over all compact and regular Euclidean metrics¹⁾. A direct extension of their proposal to a higher dimensional theory would require the path integral to be performed over all compact, regular Euclidean $(D+1)$ -dimensional metrics.

However, to apply their proposal to a Kaluza-Klein theory, one has to keep in mind that the topology of our model is the direct product of the time, the three-dimensional space and the internal space, i.e., $R \times S^3 \times S^n$. Therefore, the boundary condition of "no boundary" means that the path integral has to be performed over all regular compact Euclidean $(D+1)$ -geometries, the boundary of which is given by $S^3 \times S^n$.

To illustrate this problem, let us consider a simplified model in three dimensions. The product manifold in question is $R \times S^1 \times S^1$, where one of the two radii is represented by the scale a and the other by the scale b . In this case, the integration domain is a class of regular compact three-geometries, the boundary of which is $S^1 \times S^1$. The simplest geometry of this type is an annulus, i.e., $D^2 \times S^1$ (D^2 is the two-dimensional disc. See Fig. 4).

In the Kaluza-Klein theory of the type considered here, the simplest regular Euclidean $(D+1)$ -geometry, which will give a dominant contribution to the path integral, is given by either $D^4 \times S^n$ or $S^3 \times D^{n+1}$, depending on whether we combine the time into the external or the internal space, respectively. Note that D^4 is the four-dimensional disc, the boundary of which is S^3 . In our model, the vacuum expectation value (v.e.v.) of the ATF resides in the internal space, which means that this space contains a topological invariant and therefore, we cannot make S^n arbitrarily small without hitting a singularity (at $b = 0$). Thus, the possibility to combine the time into the internal space is excluded, since our boundary condition of "no boundary" requires regularity of the integration domain. The only possibility to form a regular Euclidean domain is to combine the time into the external space. Thus, we get the manifold $D^4 \times S^n$. This choice defines the path integration over regular compact Euclidean $(D+1)$ -dimensional geometries, the boundary of which is $S^3 \times S^n$. We see that in this formulation the scale of the internal space behaves like a scalar field and therefore, we can investigate the present model in a way analogous to the case of the minimally coupled scalar¹¹⁾.

In the Euclidean region the semi-classical approximation of the wave function is dominated by solutions of the Euclidean equations of motion and can be approximated by

$$\Psi = \sum_j N_j \exp(-I_j) \quad , \quad (20)$$

where the exponents I_j are the actions of classical Euclidean solutions.

The initial conditions for the Euclidean solutions which have to be included in this class of metrics defined above can be deduced from the regularity of the Euclidean equations of motion. They are found to be given by

$$a=0 \quad , \quad \frac{da}{d\tau} = 1 \quad , \quad b = b_0 \quad , \quad \frac{db}{d\tau} = 0 \quad , \quad \text{for } \tau = 0 \quad (21)$$

with b_0 being a positive constant. Once the initial conditions are defined, it is straightforward to solve the equations of motion numerically.

The Euclidean solutions for various initial conditions are shown in Fig. 5. First, we observe that the Euclidean paths, which are required to start off perpendicular to the b -axis, run parallel for a certain distance. This behaviour provides a region where the approximation of $b = \text{const.}$ and the WKB approximation, dominated by a single path, work well. In particular, in the region near $a = 0$, $b = \infty$, i.e., near one side of the light cone (see Fig. 3), we expect that these approximations give a reliable result.

Therefore, we estimate the form of the wave function near $a = 0$, $b = \infty$ with the approximation of vanishing b -term. Under this approximation, the Wheeler-De Witt equation is the same as that for the pure gravity case with the effective cosmological constant $W(b)$ in four dimensions. Thus, the wave function can be written as

$$\Psi(a, b) = A \exp \left[-\frac{1}{3W(b)} (1 - (1 - a^2 W(b))^{3/2}) \right], \quad (22)$$

up to a prefactor A which depends on the factor ordering. This expression is a good approximation at large b near the light cone. We take the wave function regular in a and we put $A = 1$, which is consistent with the choice of the factor ordering, $p = 1$ and $q = 1$ ¹²⁾.

The numerical analysis of the Euclidean paths gives further insight into the behaviour of the wave function in the Euclidean region. Especially, we note in Fig. 5 the remarkable feature that, for a wide range of initial values of b , all Euclidean paths gather in an area near the local maximum of the potential $W(b)$. It was shown that this behaviour of the Euclidean paths is closely related to the behaviour of the wave function of the model Universe at hand^{1),11),14)}. In particular, the structure of the wave function around the local maximum of $W(b)$ can be estimated by using the results of the investigation of the minimally coupled scalar field with a double-well potential¹³⁾.

We have learned from the analysis of the double-well potential that in the case where the Euclidean paths gather, the area along the light cone where the wave function has the exponential behaviour, becomes narrower. It implies a recession of the exponential behaviour of the wave function from the line $V = 0$. This recession is to be interpreted as follows. The bounce points of the Lorentzian paths are shifted slightly into the Euclidean region. It indicates that the time derivative of the v.e.v. of the scalar field cannot be neglected. The suggestion from this result for the present Kaluza-Klein model is that the velocity b is not negligible at the bounce point.

Thus, a gathering of Euclidean paths does not merely indicate the breakdown of the semi-classical approximation, it also gives us some suggestions about the behaviour of the wave function and the penetration of classical Lorentzian paths into the Euclidean region.

3.2. - The Lorentzian region

For large b the wave function near the line $V = 0$ can be obtained by the analytic continuation of the wave function of the Euclidean region given by Eq. (22) to the Lorentzian region. We get

$$\Psi(a, b) = B \exp\left(\frac{1}{3W(b)}\right) \cos\left[(a^2 W(b) - 1)^{3/2} / (3W(b)) - \frac{\pi}{4}\right], \quad (23)$$

where B is a slow varying amplitude.

In the deep Lorentzian region, i.e., distant from the line $V = 0$, the wave function in Eq. (23) is no longer valid. However, in this region we can apply the WKB approximation to estimate the behaviour of the wave function, interpreting it as a superposition of quantum states peaked around a certain class of classical paths. This class of classical paths gives the rapidly oscillating phase factor in the semi-classical approximation. The initial conditions for these paths can be deduced from the behaviour of the wave function as follows.

As we have seen from the analysis of the classical Euclidean solutions, the wave function will have an exponential behaviour in the Euclidean region, the region under the curve $V = 0$ in Fig. 2. This result gives us the boundary conditions for the class of Lorentzian paths which contribute to the wave function. Namely, such Lorentzian paths must have a minimum radius or "bounce" at or near $V = 0$.

The initial conditions for bounce solutions of the Lorentzian paths in the case of large b are given by

$$a = 1/\sqrt{W(b_0)}, \quad \frac{da}{dt} = 0, \quad b = b_0, \quad \frac{db}{dt} = 0 \quad (24)$$

at bounce point, which we take here the origin of the time, i.e., $t = 0$. b_0 denotes the value of b at this point. Here, for simplicity, we assume that the velocity b vanishes at the bounce point. We shall discuss this assumption in more detail later.

The Lorentzian solutions start off at the line of the turning points as shown in Fig. 6. We can divide these solutions into four types, characterized as follows.

Those solutions which have an initial value b_0 lying in the region between the local minimum and the local maximum of $W(b)$ (i.e., between the values $1/\sqrt{6\lambda}$ and $1/\sqrt{2\lambda}$) wind up along the potential valley of the local minimum of $W(b)$, and after a long duration of expansion, the paths roll down again. It is worth to remark that the nearer to the local maximum of $W(b)$ the classical solutions start, the higher they go up, i.e., the larger the maximal size of the corresponding Universe reaches.

The numerical examples show that the paths of classical Universes, which can reach a size big enough to be interesting, have to start at initial values within a width of at least 10^{-2} around the local maximum of $W(b)$. One such example is shown in Fig. 6b, where the upper bounce point of the path lies so far up that it is not reached by the numerical computation¹⁴⁾.

This means that sufficient inflation occurs generically in this toy model. To see this, one has to recall the interpretation of the wave function. We shall come back to this point in our discussion.

For initial values b_0 between $1/\sqrt{6\lambda}$ and $1/\sqrt{2\lambda}$, more distant from the local maximum of the potential $W(b)$, the solutions go slightly up and soon roll down inside the potential valley and then run towards the axis $a = 0$, $b = \infty$ (Fig. 6a).

For the initial value b_0 less than $1/\sqrt{6\lambda}$, the solutions immediately collapse after a little expansion, i.e., the paths just run down from the turning point, towards and asymptotically along the axis $a = 0$, $b = \infty$ (this type of paths is not shown in Fig. 6).

Beyond the value of b which corresponds to the local maximum of $W(b)$, i.e., $b_0 > 1/\sqrt{2\lambda}$, the solutions show a continuous expansion of both scales, a and b . No contraction of the internal space will occur (Fig. 6c).

We have already observed that the gathering of the Euclidean paths suggests a non-vanishing velocity of the scale b of the internal space at the bounce points. This affects the above result in such a way that those Universes which correspond to classical Lorentzian paths with non-negligible velocity of b , have

a stronger exponential expansion while climbing up to the potential maximum, as was observed in the investigation of the double-well potential case¹³⁾.

According to the interpretation of the wave function as a superposition of quantum states peaked around a certain class of classical paths, we can approximately describe its behaviour at a semi-classical level by knowledge of this class of classical paths. The Lorentzian bounce solutions described above are a part of this class. To describe the behaviour of the wave function of our model Universe, we need the whole class of classical paths and hence the boundary conditions on the full light cone. However, we face the fact that one half of the light cone of the origin lies in the Lorentzian region. This can be seen from the structure of the potential in (x,y) variables (see Fig. 3): the line $a = 0, b = \infty$ lies in the Euclidean region, whereas the line $a = 0, b = 0$ is located in the Lorentzian region. In addition to this fact, the potential $V(a,b)$ is singular on the latter axis. Therefore, we have to find an appropriate criterion to deal with the specification of the boundary conditions along this axis.

A possible choice, within the scope of this paper, is to demand that the wave function vanishes on the axis $b = 0$. This might considerably alleviate the singularity in the Wheeler-De Witt equation which would otherwise occur. Of course, it is not excluded that quantum effects may intervene to change the singular behaviour at $b = 0$. Since the singularity at $b = 0$ has a topological origin, this change in the singular behaviour is likely to involve a compactification process itself, a problem outside the scope of this paper.

4. - DISCUSSION

We have investigated a quantum cosmological model of the Kaluza-Klein theory with the topology $R \times S^3 \times S^n$. An ATF is introduced so that a stable compactification solution may exist at the classical level. The boundary conditions for the wave function of the Kaluza-Klein model are formulated, applying the idea of "no boundary". We specify the domain C of the path integral as all regular compact Euclidean $(D+1)$ geometries, the boundary of which is given by $S^3 \times S^n$. The wave function on the minisuperspace is analyzed by using the WKB approximation.

The model examined here is one of the simplest of this type. However, even in this case, the attempt to write the wave function on its minisuperspace encounters problems, namely, in order to define the initial conditions, one has

to take the point $b = 0$, $a = 0$ into considerations. This point, which corresponds to a line $x = y$ (one side of the light cone) in (x,y) co-ordinates, lies in the Lorentzian region. Furthermore, the potential is singular along this line.

The singularity appearing on the axis of the light cone $a = 0$, $b = 0$ may have several origins. One possible interpretation is to regard it as a remnant of the change in topology which may occur in the beginning of the spontaneous compactification, i.e., a change from S^{n+3} into $S^n \times S^3$, a process which is not dealt with in this paper.

The present analysis of the wave function relies on the WKB approximation. We have interpreted the class of the Lorentzian Universes which satisfy the initial condition suggested by the analysis as a subclass of the complete set of the classical paths which contribute to the wave function.

An interesting result is that only classical paths with initial values of b close to the local maximum of the potential lead to satisfactory large expansion of the scale of the four-dimensional space-time, together with a compactified internal space of Planck size.

On the other hand, following the interpretation of the wave function as a superposition of quantum states peaked around a class of classical paths, we conclude that in this toy model there always exist Universes which become sufficiently large. Therefore, no fine tuning of the shape of the potential is necessary.

Finally, in our theory the ATF plays an important role. The presence of the ATF guarantees, as already mentioned, the stable classical solution for compactification. On the other hand, due to the Bianchi identity of the ATF, the vacuum expectation value cannot depend on the time but has to be a constant. It has no dynamical degree of freedom in the minisuperspace (this fact is used in our ansatz). The reason for this can be understood as that the ATF has a topological origin. However, the mechanism of developing this vacuum expectation value is not yet clear. A possible interpretation of the appearance of a non-vanishing vacuum expectation value of the ATF may be that it develops in connection with a change in topology of the Universe.

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FIGURE CAPTIONS

Fig. 1 The lines $b = \text{const.}$ of the total potential $V(a,b)$ in units of Planck lengths.

Fig. 2 The line $V = 0$. We show the zero line of the total potential as the function $a = 1/W(b)$.

Fig. 3 The zero line of the total potential in (x,y) co-ordinates. The light cone of the origin is represented by the line $y = |x|$.

Fig. 4 Torus with the two radii representing the two scales a and b .

Fig. 5 The Euclidean paths near the local maximum of the potential $W(b)$. In this numerical example we choose $f^2 = 1/36$ and $\lambda = 5.0$. The maximum is located at $b \cong 1/0.32$. The initial values of the paths are:

a) $b_0 = 1/2.0 \cong 0.5$;

b) $b_0 = 1/3.0 \cong 0.33$;

c) $b_0 = 1/3.2 \cong 0.312$;

d) $b_0 = 1/5.2 \cong 0.19$;

e) $b_0 = 1/5.5 \cong 0.18$.

Fig. 6 Graphs of three typical Lorentzian solutions. In this example, we also choose $f^2 = 1/36$ and $\lambda = 5.0$. We show solutions with initial values of

a) $b_0 = 1/3.0$;

b) $b_0 = 1/3.2$;

c) $b_0 = 1/3.5$.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6