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AN EXAMPLE OF AN EXPLICITLY CALCULABLE SUPERSYMMETRIC
LOW ENERGY EFFECTIVE LAGRANGIAN: THE HEISENBERG-EULER LAGRANGIAN
OF SUPERSYMMETRIC QED

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A B S T R A C T

We generalize Schwinger's operator technique, which allows the calculation of low energy effective actions in the presence of constant background fields to supersymmetric theories. The superfield formalism is employed. The generalization is performed within the framework of supersymmetric QED. The low energy effective action of supersymmetric QED is calculated. This action is naturally expressed as a D-term. It turns out that besides the supersymmetric field strengths W and $\bar{W}$, all those covariant derivatives thereof must appear, which cannot be reduced to ordinary derivatives. The explicit form of the lowest terms of an asymptotic expansion of the effective action, valid for large electron mass, is worked out in detail.

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1. INTRODUCTION

There is quite a large number of models of elementary particles, which are tailored along the basic pattern of QCD. The underlying dynamics is determined by a gauge theory coupled to some fundamental matter fields. Yet there are no physical states corresponding to the fields appearing in the underlying theory. Rather the fundamental particles are confined. The physical states at low energy are represented by bound states composed of the fundamental fields.

A convenient way of describing the low energy phenomenology of such theories consists in writing down an effective Lagrangian, which contains the spectrum and the interactions of the composite states. In general such a Lagrangian must respect all the symmetries of the underlying theory. Effective Lagrangians of such a kind, the so-called chiral models, have been very successful in describing the low energy phenomenology of QCD [1].

Over the last few years the various opportunities which supersymmetry offers to the construction of models of the fundamental particles have been recognized largely. There is in particular the attractive possibility that the known particles are bound states of some underlying supersymmetric preon theory. For a recent review of such models see e.g. [2]. The features of such models are commonly studied by writing down an effective Lagrangian for the low energy degrees of freedom, which satisfies all the symmetries of the underlying theory.

So far essentially two different approaches have been pursued to construct such effective Lagrangians. In the first one the formation of a certain condensate is assumed, which breaks some of the global invariances of the underlying theory. The effective theory is then given by a supersymmetric σ -model, which describes the interactions of the corresponding Goldstone supermultiplets, see e.g. [3]. This approach essentially copies the chiral models of QCD, however not incorporating the effect of the chiral anomaly. In the second approach one writes down an effective theory, which respects all the symmetries of the underlying theory, including those which are anomalous. Effective Lagrangians of this kind have been constructed for pure $N = 1$ Super-Yang-Mills (SYM) [4] and $N = 1$ supersymmetric QCD (SQCD) [5]. The results obtained from these effective Lagrangians have essentially been confirmed by recent instanton calculations [6]. Some partial results are known for the $N = 2$ Yang-Mills theory [7]. For a review see [8].

Both approaches have their problems. In particular in the first approach the relation with the underlying theory remains relatively obscure. For example it cannot be excluded that there are further zero mass composite states beside the Goldstone multiplets. Though the second approach assures that all the symmetries are taken into account in an adequate manner, it is not clear in terms of which composite fields the effective Lagrangian should be formulated. This unsatisfactory situation is related to the inherent limitations of the general approach which is employed to construct these effective Lagrangians. As there is no definite information about the low-energy composite fields of the effective theory, several supermultiplets of composite fields are just postulated.

In view of this situation it is important to gain a clear understanding of the relation between the underlying and the effective theory in those models, where this relation can be made explicit. In particular it is important to understand into which composite supermultiplets the fundamental fields arrange.

Some of these questions have been studied in the context of the two dimensional supersymmetric CP^{n-1} model [9]. The results obtained indicate that the field content which has been used so far to construct effective Lagrangians has to be enlarged. This conclusion is supported by results obtained in the context of the supersymmetric Jona-Lasinio model [10] and by super-positronium calculations [11].

Unfortunately the interpretation of the results obtained for the supersymmetric CP^{n-1} model is not straightforward. In [9] Schwinger's operator method [12, 13] has been used on the component level to calculate the effective action. This procedure requires that some of the components of the gauge supermultiplet have to be set to zero. Therefore manifest supersymmetry is lost. The result obtained in this way is then again rewritten in a manifestly supersymmetric manner. However, such a procedure is not unique. In order to obtain a unique result a generalization of Schwinger's method is required, which allows for a manifestly supersymmetric calculation.

In the present paper we shall generalize Schwinger's method in such a way that it becomes possible to calculate a functional determinant in the presence of a slowly varying background superfield in a manifestly supersymmetric way. Our method is based on heat kernel techniques in superspace. Such methods have previously been considered in [14–18]. To remain as close as possible to Schwinger's original work, we have decided to perform this generalization in the framework of supersymmetric QED (SQED).

With the method developed in this paper it is immediately possible to calculate the effective low-energy Lagrangian of SQED, which is the generalization of the well-known Heisenberg–Euler [19] Lagrangian of ordinary QED. Though the dynamics of SQED is quite distinct from the supposed dynamics of those models, which are of interest for the construction of supersymmetric preon models, we nevertheless believe that it is instructive to have an example of an effective supersymmetric Lagrangian, where the relation between the underlying theory and the effective theory is completely under control. In this context we would like to stress that our method is non-perturbative. It is valid for arbitrary strength of the background superfields. To solve the problems outlined above, one should use similar methods in the context of other models, notably the CP^{n-1} model in two dimensions.

Though the primary motivation of our investigation were the above-mentioned problems, our results may have a wider range of interest. Technically we offer a new procedure to calculate superspace heat kernels for constant background superfields. From these kernels one can calculate manifestly supersymmetric effective actions. In order to illustrate the full power of our method, we shall keep more details than only the derivation of the effective action of SQED would have required.

Effective potentials in superspace have previously been considered in [20] and [21]. An effective action for SQED and the two dimensional CP^{n-1} model has already been given in [22]. Unfortunately, the method which is employed in [22] does not lead to a manifestly supersymmetric effective action. Therefore the results of [22] do not allow to draw any conclusion concerning the problems outlined above.

Apart from the technical aspects, the supersymmetric Heisenberg–Euler Lagrangian may have interesting applications if taken for itself. It may serve as the point of departure for the formulation of classical models of supersymmetric confining theories, as they have been proposed for non-supersymmetric theories, e.g. in [23]. Furthermore as our method yields results which are valid for arbitrary field strength, they may be of interest in cosmology.

The rest of the paper is organized as follows. In the next section we shall clarify some definitions and conventions for later reference. Then we shall show how to functionally integrate out the matter superfields of SQED using superfield methods. We shall represent the resulting effective action as the coincidence limit of a chiral heat kernel defined in superspace. In the third section we shall develop the necessary machinery to calculate the heat kernel in the presence of some slowly varying superfields. Special attention is paid not to violate the chirality of the heat kernel by performing an appropriate

approximation. In section 4 we shall employ the results of section 3 to calculate the low-energy effective action of SQED. Section 5 is reserved for some concluding remarks. Some technical details will be presented in two appendices.

2. PRELIMINARIES

In this section we shall mainly clarify our notation and give some intermediate steps in performing the functional integral over the matter fields.

2.1 Gauge covariant formulation

The effective action which we are going to calculate is gauge invariant. Therefore it is useful to formulate everything in terms of gauge covariant quantities, just like in Schwinger's original approach [12, 13]. The formalism, which we shall develop in the next section, will almost exclusively make use of the commutators of the gauge covariant derivatives. Therefore we shall list the relevant definitions and relations in the present section.

Apart from those changes, which are stated explicitly, we shall follow the conventions of [24]. Deviating from [24], we shall always convert Lorentz-indices into a pair of spinor indices via contraction with the generalized Pauli matrices σ . Therefore small Latin indices from the beginning of the alphabet always stand for a pair of a dotted and an undotted Greek index, e.g. $x_a = x_{\alpha\dot{\alpha}}$.

To write both D_α and $\bar{D}_{\dot{\alpha}}$ in a gauge covariant manner, it is necessary to introduce two vector superfields V and U . Covariant derivatives $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_{\dot{\alpha}}$ are then defined by

$$\begin{aligned} \mathcal{D}_\alpha &= D_\alpha + \phi_\alpha \quad , \quad \phi_\alpha = D_\alpha V \quad ; \\ \bar{\mathcal{D}}_{\dot{\alpha}} &= \bar{D}_{\dot{\alpha}} + \bar{\phi}_{\dot{\alpha}} \quad , \quad \bar{\phi}_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}} U \quad . \end{aligned} \tag{2.1.1}$$

The anticommutators among the covariant derivatives $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_{\dot{\alpha}}$ are given by

$$\{ \mathcal{D}_\alpha, \mathcal{D}_\beta \} = \{ \bar{\mathcal{D}}_{\dot{\alpha}}, \bar{\mathcal{D}}_{\dot{\beta}} \} = 0 \quad , \tag{2.1.2}$$

$$\{ \mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}} \} = -2i \mathcal{D}_\alpha \quad , \tag{2.1.3}$$

where

$$\mathcal{D}_a = \partial_a + \phi_a \quad , \quad \phi_a = \frac{i}{2} (\bar{D}_{\dot{\alpha}} \phi_\alpha + D_\alpha \bar{\phi}_{\dot{\alpha}}) \quad . \tag{2.1.4}$$

The commutators with $\mathcal{D}_a$ are

$$[\mathcal{D}_a, \bar{\mathcal{D}}_{\dot{\beta}}] = i \epsilon_{\dot{\alpha}\dot{\beta}} W_\alpha \quad , \tag{2.1.5}$$

$$[\mathcal{D}_\alpha, \mathcal{D}_\beta] = i \epsilon_{\alpha\beta} \bar{W}_i \quad , \quad (2.1.6)$$

$$[\mathcal{D}_a, \mathcal{D}_b] = R_{ab} \quad , \quad (2.1.7)$$

where the field strengths W_α , $\bar{W}_i$ and R_{ab} are defined by

$$W_\alpha = \frac{1}{4} \bar{D}^2 D_\alpha (U - V) \quad , \quad (2.1.8)$$

$$\bar{W}_i = \frac{1}{4} D^2 \bar{D}_i (U - V) \quad , \quad (2.1.9)$$

and

$$R_a{}^b = \frac{1}{2} (D_a W^\beta \delta_{\dot{\alpha}}^{\dot{\beta}} + \bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\beta}} \delta_a^\beta) \quad . \quad (2.1.10)$$

Due to the definitions (2.1.8, 9) we have

$$\bar{D}_{\dot{\alpha}} W_\beta = D_\alpha \bar{W}_{\dot{\beta}} = 0 \quad (2.1.11)$$

and

$$D W = \bar{D} \bar{W} \quad (2.1.12)$$

2.2 The functional integral over the matter superfields

As it is well-known how to handle functional integrals over superfields (e.g., [25]), we shall only sketch the derivation of the general form of the effective action briefly.

Our starting point is the following matter part of the action of SQED

$$S' = \int d^4x \left\{ \int d^2\theta (\bar{\phi} e^{+V'} \phi + \chi e^{-V'} \bar{\chi}) + \frac{1}{2} m \int d^2\theta \phi \chi + \frac{1}{2} m \int d^2\bar{\theta} \bar{\phi} \bar{\chi} \right\} \quad , \quad (2.2.1)$$

where ϕ and χ are both chiral superfields

$$\bar{D}_{\dot{\alpha}} \phi = \bar{D}_{\dot{\alpha}} \chi = 0 \quad , \quad (2.2.2)$$

and

$$V' = V - U \quad . \quad (2.2.3)$$

In order to perform the functional integral over ϕ and χ it is useful to rewrite the action as

$$S' = \int d^4x_1 d^4x_2 \int d^2\hat{\Theta}_1 d^2\hat{\Theta}_2 \bar{\psi}(z_1) M(z_1, z_2) \psi(z_2) \quad , \quad (2.2.4)$$

where

$$\psi = \begin{pmatrix} \phi \\ \bar{\chi} \end{pmatrix} \quad , \quad \bar{\psi} = (\chi, \bar{\phi}) \quad , \quad (2.2.5)$$

and

$$M(z_1, z_2) = \begin{pmatrix} -\frac{1}{8} m \bar{D}^2(z_2) & \frac{1}{16} \bar{D}^2(z_1) e^{-V(z_1)} D^2(z_2) \\ \frac{1}{16} D^2(z_1) e^{V(z_1)} \bar{D}^2(z_2) & -\frac{1}{8} m D^2(z_2) \end{pmatrix} \delta^8(z_1 - z_2) \quad (2.2.6)$$

z denotes a point of superspace. The hat symbol above the Grassmann variables in (2.2.4) indicates that for chiral (antichiral) superfields the integration measure $d^2\theta(d^2\bar{\theta})$ has to be taken. The covariant derivatives in (2.2.6) act on everything appearing to their right hand side; δ^8 represents the δ -function on the full superspace

$$\delta^8(z_1 - z_2) = \delta^4(x_1 - x_2) (\Theta_1 - \Theta_2)^2 (\bar{\Theta}_1 - \bar{\Theta}_2)^2 \quad . \quad (2.2.7)$$

Performing the functional integral over ψ and $\bar{\psi}$ we get the one loop effective action

$$\Gamma = i T \tau \log M \quad . \quad (2.2.8)$$

In the massless case M can be brought into diagonal form by the 'doubling trick' [25]. In the massive case, this trick does not work successfully. Therefore we shall use instead the generalization of the relation

$$\det \begin{pmatrix} 1 & B \\ A & 1 \end{pmatrix} = \det (1 - AB) \quad (2.2.9)$$

valid for arbitrary finite dimensional matrices A and B. Using this trick and performing a gauge transformation in (2.2.8) M has to be substituted by M', where

$$M' = \left(-\frac{1}{4} m^2 + \frac{1}{16} \bar{D}^2(z_1) D^2(z_1) \right) \bar{\delta}^6(z_1 - z_2) . \quad (2.2.10)$$

$\bar{\delta}^6$ is the δ -function on the gauge chiral subspace

$$\bar{\delta}^6(z_1 - z_2) = -\frac{1}{4} \bar{D}^2 \delta^8(z_1 - z_2) . \quad (2.2.11)$$

Alternatively we can substitute M''

$$M'' = \left(-\frac{1}{4} m^2 + \frac{1}{16} D^2(z_1) \bar{D}^2(z_1) \right) \delta^6(z_1 - z_2) \quad (2.2.12)$$

with

$$\delta^6(z_1 - z_2) = -\frac{1}{4} D^2 \delta^8(z_1 - z_2) \quad (2.2.13)$$

into (2.2.8). To make Γ manifestly Hermitian, we shall define the effective action by

$$\Gamma = \frac{1}{2} i \left\{ \text{Tr} \log M' + \text{Tr} \log M'' \right\} . \quad (2.2.14)$$

It is convenient to represent the logarithm of M' and M'' in (2.2.14) as a proper time integral. For the operator M' we have

$$\text{Tr} \log M' = -\text{Tr} \left\{ \int_0^\infty \frac{d\tau}{\tau} e^{\frac{1}{4} i \tau m^2} h(z, z; \tau) \right\} , \quad (2.2.15)$$

and the analogous relation holds for M''. It is understood that the Feynman $i\epsilon$ prescription has been adapted, i.e. $m^2 \rightarrow m^2 + i\epsilon$. $h(z, z; \tau)$ is the coincidence limit of the heat kernel^{*)} $h(z_1, z_2; \tau)$. It is defined to be the solution of

$$\left(i \frac{d}{d\tau} - \sigma \right) h(z_1, z_2; \tau) = 0 , \quad (2.2.16)$$

which satisfies the initial condition

$$h(z_1, z_2; 0) = \bar{\delta}^6(z_1 - z_2) . \quad (2.2.17)$$

^{*)} We would like to point out that h is not a heat kernel in the strict sense. Nevertheless we shall call h a heat kernel in conformity with the common nomenclature.

The operator O is given by

$$\bar{O} = \frac{1}{16} \bar{D}^2(z_1) D^2(z_1) \quad . \quad (2.2.18)$$

As a consequence of (2.2.16) and (2.2.17) h is chiral, i.e.

$$\bar{D}_i(z_1) h(z_1, z_2; \tau) = 0 \quad , \quad (2.2.19)$$

and

$$\bar{D}'_i(z_2) h(z_1, z_2; \tau) = 0 \quad , \quad (2.2.20)$$

where

$$\bar{D}'_i(z_2) = \bar{D}_i(z_2) - \bar{\phi}_i(z_2) \quad . \quad (2.2.21)$$

Consequently we can drop all those contributions to (2.2.18) which annihilate the chiral subspace:

$$\bar{O} \hat{=} \square = -\frac{1}{2} D_a(z_1) D^a(z_1) - \frac{1}{2} W(z_1) D(z_1) - \frac{1}{4} DW(z_1) \quad (2.2.22)$$

Note that the $\square$ operator commutes with $\bar{D}_a(z_1)$

$$[\square, \bar{D}_i(z_1)] = 0 \quad . \quad (2.2.23)$$

A similar representation can be written down for $\log M''$ in terms of $\bar{h}$, where

$$(i \frac{d}{d\tau} - \bar{O}) \bar{h}(z_1, z_2; \tau) = 0 \quad , \quad (2.2.24)$$

$$\bar{h}(z_1, z_2; 0) = \delta^6(z_1 - z_2) \quad , \quad (2.2.25)$$

$$\bar{O} = \frac{1}{16} D^2(z_1) \bar{D}^2(z_1) \quad . \quad (2.2.26)$$

As $\bar{h}$ is constructed in complete analogy to h , we shall concentrate on the evaluation of h in the following.

3. GENERAL FORMALISM

In this section we shall demonstrate how the heat kernel h defined by the equations (2.2.16) and (2.2.17) can be calculated for slowly varying background fields W and $\bar{W}$.

In non-supersymmetric theories there are various techniques known to calculate heat kernels systematically. The most powerful one is probably the Schwinger–De Witt expansion [26]. This expansion

can be generalized to supersymmetric theories without any problem [14, 15, 17]. However as it is an expansion in powers of the proper time parameter τ and as the coefficient functions are defined recursively, it is very difficult to go beyond the first few powers of τ . Fortunately as we are interested in the case of a slowly varying background field, we do not need the full power of the Schwinger–De Witt expansion.

It is well known that for non-supersymmetric theories it is possible to write down a closed expression for the heat kernel in the presence of a constant background field. The necessary machinery has been pioneered by Schwinger [12, 13]. Alternative techniques have been presented in [27, 28].

Schwinger’s approach is essentially based on interpreting the operator O appearing in equation (2.2.16) as a Hamiltonian of a dynamical system. The heat kernel is then interpreted as the transition amplitude for the system to evolve from the state z_2 at time $\tau = 0$ into the state z_1 at time τ . In contrast to the methods used in [27, 28], Schwinger’s method has the advantage to deal almost exclusively with gauge covariant quantities. Therefore we find that it is a suitable starting point for the formulation of a procedure which allows to calculate h in a manifestly supersymmetric way.

3.1 Manifestly supersymmetric ‘constant field’ approximation

The essential technical ingredient, which makes it possible to write down a solution for h in a closed form is the linearization of a certain differential equation [see equation (3.2.2)]. In the non-supersymmetric theory this linearization is only possible if the field strength of the background gauge field is independent of the position in space–time. One could try to carry this procedure over to the supersymmetric case, i.e. set $\partial_a R_{bc}$ and the fermionic components of the multiplets W and $\bar{W}$ to zero. This is essentially the method adopted in [9]. However, this procedure leads to an effective action, which is supersymmetric only if we put those constraints, already employed in the approximation, on the fields appearing finally in the effective action. In particular the effective action will not be supersymmetric for arbitrary configuration of the fields.

In order to obtain a manifestly supersymmetric effective action, the approximation, which makes h calculable, has to satisfy two requirements. Firstly the approximation has to be formulated exclusively in terms of superfields and covariant derivatives thereof. In other words we have to set some of the covariant derivatives of the supersymmetric field strength to zero. This requirement alone, however, does not guarantee that the approximated form of h remains chiral. To guarantee that h remains chiral for arbitrary configurations of the fields, it depends on, we have to take care, not to set any $\bar{D}_\alpha$ derivative of those fields, which do not vanish, to zero.

There are several levels of approximation which can be formulated in accordance with these rules, and which allow for the construction of h in a closed form. Let us list them in turn

$$(A) \quad W_\alpha \neq 0 \quad D_\alpha W_\beta = \partial_\alpha W_\beta = \bar{W}_\beta = 0$$

$$(B) \quad W_\alpha \neq 0 \quad D_\alpha W_\beta \neq 0 \quad \partial_\alpha W_\beta \neq 0 \\ \bar{W}_\alpha = \partial_\alpha D_\beta W_\gamma = \partial_\alpha \partial_\beta W_\gamma = 0$$

$$(C) \quad W_\alpha \neq 0 \quad D_\alpha W_\beta \neq 0 \quad \partial_\alpha W_\beta \neq 0 \quad \bar{W}_{\dot{\alpha}} \neq 0 \quad \bar{D}_{\dot{\alpha}} \bar{W}_{\dot{\beta}} \neq 0$$

$$\partial_\alpha D_\beta W_\gamma = \partial_\alpha \partial_\beta W_\gamma = \partial_\alpha \bar{W}_{\dot{\beta}} = 0$$

Note that due to (2.1.12) $\partial_\alpha W_\beta \neq 0$ ($\partial_\alpha \bar{W}_{\dot{\beta}} = 0$) implies $\bar{D}^2 \bar{W}_{\dot{\alpha}} \neq 0$ ($D^2 W_\alpha = 0$). To find a suitable approximation for the effective action Γ , we must keep in mind that according to our definition (2.2.14) Γ consists of two parts, which are the Hermitian conjugate of each other. Therefore the approximations (A) and (B) do not lead to a consistent approximation for Γ , though both give a well-defined heat kernel. We are therefore forced to work with approximation (C).

If we require in (A)-(C) a certain superfield not to vanish, it is meant to be so for all its components. Therefore, rewriting the various approximations in component form, they seem to be inconsistent. However this inconsistency is only apparent. As long as we write everything in terms of superfields, there is obviously no inconsistency. Therefore we have a perfectly well-defined approximation scheme. As we do not claim to improve the approximation by keeping some component fields at some places, whilst discarding the same component fields at other places, this approximation is also well-defined at the component level. We would like to stress that the only purpose of keeping some of the small contributions, is to obtain a manifestly supersymmetric result.

3.2 Construction of the heat kernel

We shall now turn to the solution of the heat equation (2.2.16) with the Laplacian (2.2.22). Our procedure will essentially consist of two steps. First we shall introduce the 'Heisenberg representation' of the operators $\mathcal{D}_\alpha$, $\mathcal{D}_{\dot{\alpha}}$ and W_α . Then we shall employ this representation in the heat equation (2.2.16), which simplifies it to such an extent that it can immediately be integrated.

3.2.1 The 'Heisenberg representation'

Let us start with the definition of the Heisenberg representation of the operators $\mathcal{D}_\alpha$, $\mathcal{D}_{\dot{\alpha}}$ and W_α , the 'Hamiltonian' $\square$ is composed of. It is convenient to combine these operators into a common vector O_A

$$\mathcal{O}_A(z_1) = \begin{pmatrix} \mathcal{D}_\alpha(z_1) \\ \mathcal{D}_{\dot{\alpha}}(z_1) \\ W_\alpha(z_1) \end{pmatrix} . \quad (3.2.1)$$

The Heisenberg representation $O_A(\tau)$ is defined as usually by the equation of motion

$$i \frac{d}{d\tau} \mathcal{O}_A(\tau) = [\square, \mathcal{O}_A(\tau)] \quad , \quad \mathcal{O}_A(\tau=0) = \mathcal{O}_A(z_1) . \quad (3.2.2)$$

The transformed operator $W(\tau)$ acts on $h(z_1, z_2; \tau)$ like

$$W_\alpha(\tau) h(z_1, z_2; \tau) = W_\alpha(z_2) h(z_1, z_2; \tau) . \quad (3.2.3)$$

To deduce the action of $\mathfrak{D}_a(\tau)$ on $h(z_1, z_2; \tau)$ one has to take into account that $\mathfrak{D}_a$ does not commute with $\bar{\mathfrak{D}}_{\dot{a}}$. It is therefore natural to add first a compensating piece to $\mathfrak{D}_a$, which makes it commute with $\bar{\mathfrak{D}}_{\dot{a}}$. One possible choice is

$$\tilde{\mathfrak{D}}_a(\tau) = \mathfrak{D}_a(\tau) - i \bar{\Theta}_{1,\dot{a}} W_a(\tau), \quad (3.2.4)$$

$$[\tilde{\mathfrak{D}}_a(\tau), \bar{\mathfrak{D}}_{\dot{a}}] = 0.$$

$\tilde{\mathfrak{D}}_a(\tau)$ acts now in a simple manner on $h(z_1, z_2; \tau)$

$$\tilde{\mathfrak{D}}_a(\tau) h(z_1, z_2; \tau) = [-\mathfrak{D}'_a(z_2) - i \bar{\Theta}_{2,\dot{a}} W_a(z_2)] h(z_1, z_2; \tau), \quad (3.2.5)$$

where

$$\mathfrak{D}'_a(z_2) = \partial_a - \phi_a(z_2). \quad (3.2.6)$$

The action of $\mathfrak{D}_a(\tau)$ on $h(z_1, z_2; \tau)$ can be deduced in a similar fashion. However, we shall not need it in this paper.

Formally (3.2.2) is solved by

$$\sigma_A(\tau) = e^{-i\tau\Box} \sigma_A(z_1) e^{+i\tau\Box}. \quad (3.2.7)$$

In general the evaluation of (3.2.7) requires the knowledge of all the higher (anti)-commutators of $\Box$ with σ_A . Using the commutation relations presented in section 2.1 the lowest (anti)-commutator is readily evaluated to be

$$[\Box, \sigma_A(z_1)] = \tilde{\Box}_A{}^B \sigma_B(z_1), \quad (3.2.8)$$

where

$$\tilde{\Box}_A{}^B = \begin{pmatrix} \frac{1}{2} \mathcal{G}_A{}^\beta & -i \delta_A^\beta \bar{W}^\beta & 0 \\ \frac{1}{2} \mathcal{H}_A{}^\beta & \mathcal{R}_A{}^b & -\frac{i}{2} \delta_A^\beta \bar{W}_\beta \\ 0 & -\mathcal{H}^b{}_A & \frac{1}{2} \tilde{\mathcal{G}}_A{}^\beta \end{pmatrix}, \quad (3.2.9)$$

and

$$\mathcal{G}_a^\beta = D_a W^\beta, \quad \tilde{\mathcal{G}}_a^\beta = \mathcal{G}_a^\beta, \quad \mathcal{H}_a^\beta = \partial_a W^\beta. \quad (3.2.10)$$

We have already dropped all those terms, which vanish in the approximation (C). Looking at the higher commutators one immediately recognizes the virtue of the constant field approximation. In each approximation presented in section 3.1 all the higher commutators of $\square$ with $\bar{\square}_A^B$ vanish. Consequently we can solve the equation of motion (3.2.2) in a closed form

$$\mathcal{O}_A(\tau) = S_A^\beta(\tau) \mathcal{O}_B(z_1), \quad (3.2.11)$$

where

$$S_A^\beta(\tau) = (e^{-i\tau \tilde{\mathcal{O}}})_A^\beta. \quad (3.2.12)$$

One may cast (3.2.12) into a form where the dependence on the fields of $\square$ becomes explicit, e.g. by expanding the exponential in powers of $\bar{W}$. However, as will be shown in section 4, we only require the knowledge of the lowest term of such an expansion. Equation (3.2.12) is evaluated for $\bar{W} = 0$ in appendix B.

As has been stressed in the foregoing section, an important property of our approximation is the preservation of the chirality of h . Let us therefore demonstrate in detail how this is achieved. The fact that $\square$ commutes with $\bar{\mathcal{D}}_a(z_1)$, together with the initial condition (2.2.17) ensure that $h(z_1, z_2; \tau)$ is chiral before performing any approximation. This becomes obvious by rewriting the heat equation (2.2.16) as an integral equation. Though $\bar{\mathcal{D}}_a(z)$ commutes with $\square$, it does not commute with the operators O_A , $\square$ consists of, individually. Rather $\bar{\mathcal{D}}_a(z_1)$ acts as a kind of step operator. Introducing the matrix Δ_{AB}^C

$$\Delta_{AB}^C = \begin{pmatrix} 0 & -2i \delta_A^i \delta_B^j & 0 \\ 0 & 0 & i \epsilon_{AB} \delta_C^D \\ 0 & 0 & 0 \end{pmatrix} \quad (3.2.13)$$

the (anti)-commutator of O_A with $\bar{\mathcal{D}}_a(z_1)$ is written in an economical way as

$$[\bar{\mathcal{D}}_a(z_1), \mathcal{O}_B(z_1)] = \Delta_{AB}^C \mathcal{O}_C(z_1). \quad (3.2.14)$$

This commutator is responsible for the vanishing of the commutator of $\square$ with $\bar{\mathcal{D}}_a(z_1)$ and of the chirality of $h(z_1, z_2; \tau)$ in turn. Consequently we have to take care that our approximation respects these relations.

Going into the Heisenberg picture does not affect the relation (3.2.14). Let us show that this remains true even after having dropped those terms, which vanish in the approximation (C), i.e. for (3.2.11). We shall first prove a relation which holds for arbitrary powers of the matrix $\tilde{\square}$. Then we shall use this relation to demonstrate that (3.2.14) is indeed true after having performed the approximation (C).

The matrix $\tilde{\square}$ satisfies the important relation

$$\bar{D}_i(z_1) (\tilde{\square}^n)_B^C + (-)^{(b+d)} (\tilde{\square}^n)_B^D \Delta_{iD}^C = \Delta_{iB}^D (\tilde{\square}^n)_D^C, \quad (3.2.15)$$

where $b, d = 0(1)$ if b, d is bosonic (fermionic). This relation can immediately be proved by induction. Taking into account that

$$\bar{D}_i R_b^c = -i \delta_i^j \mathcal{H}_{b,j}^c + i \epsilon_{\beta i} \mathcal{H}^c_{\beta}, \quad (3.2.16)$$

and

$$\bar{D}_i \mathcal{H}_{\beta}^j = -2i \mathcal{H}_{i\beta}^j, \quad (3.2.17)$$

it is readily checked that (3.2.15) is true for $n = 1$. Taking the $\bar{D}_i(z_1)$ derivative of $(\tilde{\square}^{n+1})_B^C$ we get

$$\begin{aligned} \bar{D}_i(z_1) (\tilde{\square}^{n+1})_B^D &= (-)^{(b+c)} \tilde{\square}_B^C (\bar{D}_i \tilde{\square}^n)_C^D \\ &\quad + (\bar{D}_i(z_1) \tilde{\square}_B^C) (\tilde{\square}^n)_C^D. \end{aligned} \quad (3.2.18)$$

Inserting (3.2.15) into the first term on the RHS of (3.2.18) and the same relation with $n = 1$ into the second term, we reobtain (3.2.15) with $n \rightarrow (n+1)$, which proves (3.2.15).

Equation (3.2.15) immediately implies

$$\bar{D}_i(z_1) S_B^D = \Delta_{iB}^C S_C^D - (-)^{(b+c)} S_B^C \Delta_{iC}^D. \quad (3.2.19)$$

Taking the (anti)-commutator of $O_B(\tau)$ with $\bar{D}_i(z_1)$ we see that

$$[\bar{D}_i(z_1), O_B(\tau)] = (\bar{D}_i S_B^C) O_C(\tau) + (-)^{(b+c)} S_B^C \bar{D}_i O_C(z_1). \quad (3.2.20)$$

If we now insert the relations (3.2.14) and (3.2.19) into the RHS of (3.2.20) we get

$$[\bar{D}_i(z_1), O_B(\tau)] = \Delta_{iB}^C O_C(\tau), \quad (3.2.21)$$

which proves that the approximated operators have the same algebraic relations with $\bar{D}_i(z_1)$, as the full operators have.

The relation (3.2.19) is not only important to ensure (3.2.21), it may also be used to facilitate the explicit computation of the matrix S . Relation (3.2.19) determines all the elements of the matrix S solely from either S_{1k} or S_{n3} , $k = 1, 2, 3$.

Let us now turn to the Heisenberg representation of the operators y_1 , θ_1 and $\bar{\theta}_1$, where

$$y_{1,a} = x_{1,a} - 2i \theta_{1,a} \bar{\theta}_{1,i} \quad (3.2.22)$$

The Heisenberg representation of these operators is defined in the same manner as previously. As $\bar{\theta}_{1,i}$ commutes with $\square$, it is not affected by going into the Heisenberg representation. To solve the equations of motion of $y_1(\tau)$ and $\theta_{1,\alpha}(\tau)$ we need to know the following commutators

$$[\square, y_{1,a}(\tau)] = 2 (\mathcal{D}_a(\tau) - i \bar{\theta}_{1,i} W_a(\tau)) \quad (3.2.23)$$

and

$$[\square, \theta_{1,a}(\tau)] = -\frac{1}{2} W_a(\tau) \quad (3.2.24)$$

which hold in general. With this piece of information we can immediately express $y_a(\tau)$ and $\theta_{1,\alpha}(\tau)$ in terms of $\mathcal{D}_a(\tau)$ and $W_a(\tau)$

$$y_a(\tau) - y_a(0) = -2i \int_0^\tau d\tau' (\mathcal{D}_a(\tau') - i \bar{\theta}_{1,i} W_a(\tau')) \quad (3.2.25)$$

$$-2i \Delta_a(\tau) = \int_0^\tau d\tau' W_a(\tau') \quad (3.2.26)$$

where

$$\Delta_a(\tau) = \theta_a(0) - \theta_a(\tau) \quad (3.2.27)$$

As we shall see later it is convenient to replace the operator $y_a(\tau) - y_a(0)$ in favour of

$$\begin{aligned} \eta_a(\tau) &= \frac{i}{2} (y_a(\tau) - y_a(0) - 2 \bar{\theta}_{1,i} \Delta_a(\tau)) \\ &= \int_0^\tau d\tau' \mathcal{D}_a(\tau') \end{aligned} \quad (3.2.28)$$

The relations (3.2.23)–(3.2.28) are completely general. In the approximation (C), $\eta(\tau)$ and $\Delta(\tau)$ can be represented as linear combinations of the operators of $O_A(z_1)$ via (3.2.11). Defining the matrix

$$T = i \tilde{\square}^{-1} (e^{-i\tau \tilde{\mathcal{D}}} - 1) \quad (3.2.29)$$

$\eta(\tau)$ and $\Delta(\tau)$ turn out to be

$$\eta_a(\tau) = T_{21 a}{}^\beta \mathcal{D}_\beta(z_1) + T_{22 a}{}^b \mathcal{D}_b(z_1) + T_{23 a}{}^\beta W_\beta(z_1), \quad (3.2.30a)$$

$$2i \Delta_a(\tau) = T_{31} a^\beta \mathcal{D}_\beta(z_1) + T_{32} a^b \mathcal{D}_b(z_1) + T_{33} a^\beta W_\beta(z_1). \quad (3.2.30b)$$

As T is given by a series in $\tilde{\square}$, it satisfies relation (3.2.19) with S replaced by T . Due to their definitions $\eta(\tau)$ and $\Delta(\tau)$ act on $h(z_1, z_2; \tau)$ simply by multiplication with the functions η and Δ

$$\eta(\tau) h(z_1, z_2; \tau) = \eta h(z_1, z_2; \tau), \quad \Delta(\tau) h(z_1, z_2; \tau) = \Delta h(z_1, z_2; \tau), \quad (3.2.31a, b)$$

$$\eta_a = \frac{i}{2} (\gamma_{2\bar{a}} \gamma_{1a}) - 2 \bar{\Theta}_{1,\bar{a}} \Delta_a, \quad \Delta = \Theta_1 - \Theta_2. \quad (3.2.32)$$

Note that η commutes with $\mathcal{D}_\alpha(z_1)$

$$[\mathcal{D}_a(z_1), \eta] = 0 \quad (3.2.33)$$

3.2.2 Solution of the heat equation

Now we are ready to solve the heat equation. In the non-supersymmetric case the strategy is the following. In this case we have only to deal with the counterparts of $\mathcal{D}_a$ and η_a , whilst all the other operators are not present. Therefore we can express the non-supersymmetric counterpart of $\mathcal{D}_a$ and $\square$ in turn with the help of (3.2.30a) in terms of $\eta_a(\tau)$. As $\eta_a(\tau)$ acts multiplicatively on $h(z_1, z_2; \tau)$ the operator equation (2.2.16) is turned into an ordinary differential equation, where the action of the Laplacian has factorized. Therefore this equation can be integrated easily. Though this procedure is straightforward in the non-supersymmetric case, in the present case we first have to overcome some obstacles.

Turning to the supersymmetric case, one is tempted to express $\mathcal{D}_\alpha(z_1)$ and $\mathcal{D}_a(z_1)$ in terms of W_α , $\eta_a(\tau)$, and $\Delta_a(\tau)$ by solving (3.2.30), and achieve in this way the factorization of the Laplacian in (2.2.16). Though it is possible to completely factorize the action of $\square$ on $h(z_1, z_2; \tau)$ this is not very useful. The reason is the following. If we express the action of $\square$ on $h(z_1, z_2; \tau)$ along the proposed procedure, the function which replaces the $\mathcal{D}_a \mathcal{D}^a$ part of $\square$ [see (2.2.22)] behaves for small τ like τ^{-3} . Exponentiating the integral of this function, we get a formal expression for h . However, the expression obtained in this way does not satisfy the boundary condition (2.2.17). Therefore supersymmetry prevents us from making use of the completely factorized version of (2.2.16).

We shall instead proceed as follows. The relation (3.2.11) allows to eliminate $W_\alpha(z_1)$ in favour of $W_\alpha(\tau)$

$$W_a(z_1) = (S_{33}^{-1})_a^\beta (W_\beta(\tau) - S_{31\beta}^b \mathcal{D}_b - S_{32\beta}^c \mathcal{D}_c) \quad (3.2.34)$$

With the help of this relation it follows from (3.2.30a) that

$$\mathcal{D}_a(z_1) = (\tilde{T}_{22}^{-1})_a^b \left\{ \eta_b(\tau) - \tilde{T}_{23b}^c W_c(\tau) - \tilde{T}_{21b}^c \mathcal{D}_c \right\}, \quad (3.2.35)$$

where

$$\check{T}_{21 a}^{\beta} = T_{21 a}^{\beta} - T_{23 a}^{\gamma} (S_{33}^{-1})_{\gamma}^{\delta} S_{31 \delta}^{\beta}, \quad (3.2.36a)$$

$$\check{T}_{22 a}^b = T_{22 a}^b - T_{23 a}^{\gamma} (S_{33}^{-1})_{\gamma}^{\delta} S_{32 \delta}^b, \quad (3.2.36b)$$

$$\check{T}_{23 a}^{\beta} = T_{23 a}^{\gamma} (S_{33}^{-1})_{\gamma}^{\beta}. \quad (3.2.36c)$$

We can now insert (3.2.35) into $\square$. Taking into account that

$$\begin{aligned} & [\check{T}_{22 a}^b \mathcal{D}_b(z_1), \eta_c - \check{T}_{21 c}^{\delta} \mathcal{D}_{\delta}(z_1) - \check{T}_{23 c}^{\delta} W_{\delta}(z_2)] \\ &= i \check{T}_{22 a c} - i \check{T}_{22 a}^b \check{T}_{21 c}^{\delta} \epsilon_{\beta \delta} \bar{W}_{\beta}(z_1), \end{aligned} \quad (3.2.37)$$

the heat equation for h becomes

$$(i \frac{d}{d\tau} - \hat{\square}) h(z_1, z_2; \tau) = 0, \quad (3.2.38)$$

where

$$\begin{aligned} \hat{\square} = & -\frac{1}{2} \check{T}_{22 a}^{-1 b} \check{T}_{22}^{-1 d} [\eta_b - \check{T}_{21 b}^{\gamma} \mathcal{D}_{\gamma}(z_1) - \check{T}_{23 b}^{\gamma} W_{\gamma}(z_2)] \\ & \times [\eta_d - \check{T}_{21 d}^{\epsilon} \mathcal{D}_{\epsilon}(z_1) - \check{T}_{23 d}^{\epsilon} W_{\epsilon}(z_2)] \\ & - \frac{1}{2} i \check{T}_{22 a}^{-1 c} + \frac{1}{2} i \check{T}_{22}^{-1 a b} \check{T}_{21 b}^{\gamma} \epsilon_{\gamma \delta} \bar{W}_{\delta} \\ & - \frac{1}{2} W^{\delta}(z_1) \mathcal{D}_{\delta}(z_1) - \frac{1}{4} (DW). \end{aligned} \quad (3.2.39)$$

Our procedure may look superficially complicated; however, we necessarily have to take this route in order to get the correct behaviour of $\hat{\square}$ for small τ . Note that $\hat{\square}$ is not a c-number, it still contains the differential operator $\mathcal{D}_{\alpha}(z_1)$. Nevertheless the form of $\hat{\square}$ is simple enough to allow for an explicit solution of the heat equation.

In the approximation in which we are working, $\mathcal{D}_{\alpha}(z_1)$ commutes with all the elements of the matrix T . Therefore in solving the heat equation (3.2.38) we have only to take care of the fact that $\mathcal{D}_{\alpha}(z_1)$ does not commute with $W_{\alpha}(z_1)$. Taking this into account, the heat equation has the following solution

$$h(z_1, z_2; \tau) = U(\tau) e^{\Phi(\tau) + \Xi(\tau)} G(z_1, z_2), \quad (3.2.40)$$

where

$$U(\tau) = \exp \left\{ \frac{1}{2} i \tau W^a(z_1) \mathcal{D}_a(z_1) \right\}, \quad (3.2.41)$$

$$\begin{aligned} \Phi(\tau) = & -i \int_0^\tau d\tau' \left[-\frac{1}{2} \check{T}_{22}^{-1}{}^a{}_b \check{T}_{22}^{-1}{}^{ad} \left(\eta_b - \check{T}_{23}{}_b{}^c W_f(z_2) \right) \right. \\ & \times \left(\eta_d - \check{T}_{23}{}_d{}^e W_g(z_2) \right) \\ & \left. - \frac{1}{2} i \check{T}_{22}^{-1}{}^a{}_a + \frac{1}{2} i \check{T}_{22}^{-1}{}^{ad} \check{T}_{21}{}_d{}^e \epsilon_{ae} \bar{W}_a - \frac{1}{4} DW \right], \end{aligned} \quad (3.2.42)$$

and

$$\begin{aligned} \Xi(\tau) = & -i \int_0^\tau d\tau' U^{-1}(\tau') \left[\check{T}_{22}^{-1}{}^a{}_b \check{T}_{22}^{-1}{}^{ad} \times \right. \\ & \left(\eta_b - \check{T}_{23}{}_b{}^c W_f(z_2) \right) \check{T}_{21}{}_d{}^e \mathcal{D}_e(z_1) \\ & \left. + \frac{1}{4} \check{T}_{21}{}_b{}^c \check{T}_{21}{}_d{}^e \mathcal{D}^2(z_1) \right] U(\tau'). \end{aligned} \quad (3.2.43)$$

The lower limit of the range of integration in (3.2.42) is fixed by

$$\lim_{\tau \rightarrow 0} \Phi(\tau) = -\frac{i}{2} \frac{1}{\tau} \eta_a \eta^a - 2 \log \tau. \quad (3.2.44)$$

The function $C(z_1, z_2)$ parametrizes all the solutions of (3.2.38). Note that a solution of (3.2.38) is a solution of the initial heat equation (2.2.16) as well, only if it satisfies the relations (3.2.2), (3.2.4) and (3.2.31). We shall use this property in appendix A to determine the function $C(z_1, z_2)$. In order to satisfy the initial condition (2.2.17) we must have $C(z_1, z_2) \sim \Delta^2$. Remembering that

$$\lim_{\tau \rightarrow 0} \frac{1}{\tau^2} \exp \left[-\frac{i}{2\tau} \eta_a \eta^a \right] \Delta^2 = i (4\pi)^2 \delta^4(y_1 - y_2) \Delta^2, \quad (3.2.45)$$

the initial condition (2.2.17) is satisfied, if

$$C(z_1, z_2) \Big|_{y_1=y_2} = \frac{-i}{(4\pi)^2} e^{-U(z_1) + U(z_2)} \Delta^2 \Big|_{y_1=y_2}. \quad (3.2.46)$$

Though the explicit form of $C(z_1, z_2)$ is not required for the calculation of the effective action in the next section, it is important to check that there indeed exists such a function and we deal with a solution of (2.2.16).

4. THE HEISENBERG-EULER EFFECTIVE LAGRANGIAN

So far we have worked out the machinery necessary in order to calculate the supersymmetric heat kernel $h(z_1, z_2; \tau)$ in a manifestly supersymmetric manner. In the present section we shall deduce the

effective Lagrangian by taking the coincidence limit of the general expression (3.2.40) for the heat kernel obtained in the foregoing section.

Let us recall that the effective action Γ is obtained from the heat kernel $h(z_1, z_2; \tau)$ by letting the points z_1 and z_2 coincide [see (2.2.14)]. In the last section we obtained a representation of $h(z_1, z_2; \tau)$ essentially in terms of the differential operator $\mathcal{D}_\alpha(z_1)$ and the function $C(z_1, z_2)$. As $C(z_1, z_2) \sim \Delta^2$, letting z_1 and z_2 coincide, all those contributions to the coincidence limit of $h(z_1, z_2; \tau)$ vanish, where Δ^2 is not annihilated by the action of $\mathcal{D}^2(z_1)$. In other words, we need only to keep the highest term of an expansion of $U(\tau) e^{\Xi(\tau)}$ in powers of the operator $\mathcal{D}_\alpha(z_1)$. Expanding

$$\Xi(\tau) = \Sigma(\tau) \mathcal{D}_\alpha(z_1) + \Omega(\tau) \mathcal{D}^2(z_1) \quad (4.1)$$

we have

$$e^{\Xi(\tau)} = 1 + \Sigma(\tau) \mathcal{D}(z_1) + \left(\Omega(\tau) - \frac{1}{4} \Sigma^2(\tau) \right) \mathcal{D}^2(z_1), \quad (4.2)$$

as $\mathcal{D}_\alpha(z_1)$ does not act on $\Sigma(\tau)$ and $\Omega(\tau)$. Expanding $U(\tau)$ requires more care. Taking into account the non-commutativity of $\mathcal{D}_\alpha(z_1)$ and $W_\alpha(z_1)$ one may verify that in the present approximation

$$U(\tau) = 1 + W \mathcal{D}^{-1} (e^{\frac{1}{2} i \tau \mathcal{D}} - 1) \mathcal{D}(z_1) - \frac{1}{4} [W \mathcal{D}^{-1} (e^{\frac{1}{2} i \tau \mathcal{D}} - 1)]^2 \mathcal{D}^2(z_1). \quad (4.3)$$

With these results the coincidence limit of $h(z_1, z_2; \tau)$ turns out to be

$$h| = \frac{-i}{(4\pi)^2} e^{\Phi|} \left\{ [W \mathcal{D}^{-1} (e^{\frac{1}{2} i \tau \mathcal{D}} - 1)]^2 + 2 W \mathcal{D}^{-1} (e^{\frac{1}{2} i \tau \mathcal{D}} - 1) \Sigma| + \Sigma^2| - 4 \Omega| \right\}. \quad (4.4)$$

The symbol $|$ indicates that the coincidence limit $z_1 = z_2 = z$ has been taken.

Note that $h|$ is chiral

$$\bar{D}_\alpha(z) h| = 0. \quad (4.5)$$

This property is an immediate consequence of (2.2.19) and (2.2.20).

The ultimate form of the effective action should be formulated in terms of composite superfields. Though working this form out from (4.4) is straightforward, it requires an awful amount of computation. Fortunately we may simplify matters at this point by exploiting the fact that $h|$ is chiral. As $\Sigma| \sim W$ the following expansion of $h|$ in powers of the superfields W and $\bar{W}$ can be read off (4.4)

$$h| = W^2 a + b, \quad (4.6)$$

$$a = a' + a''_i \bar{W}^i + a''' \bar{W}^2, \quad (4.7)$$

and similarly for b . We can now see (4.5) to calculate the coefficient functions a'' and a''' (b'' and b''') from a' (b') recursively.

$$a''_i = \bar{\mathcal{G}}^{-1}_{ij} \dot{\beta}^j \bar{D}_\beta a', \quad (4.8)$$

$$a''' = \frac{1}{4} \bar{\mathcal{G}}^{-1}_{ij} \dot{\beta}^j \bar{D}_\beta a''_i, \quad (4.9)$$

where

$$\bar{\mathcal{G}}_{ij} \dot{\beta}^j = \bar{D}_i \bar{W} \dot{\beta} \quad (4.10)$$

The coefficients b'' and b''' satisfy analogous relations. An important role in deriving (4.8) and (4.9) is played by the fact that

$$\{ \bar{\mathcal{G}}^{-1}_{ij} \dot{\beta}^j \bar{D}_i, \bar{\mathcal{G}}^{-1}_{jk} \dot{\beta}^k \bar{D}_j \} = 0, \quad (4.11)$$

which is non-trivial because

$$[\bar{D}_i, \bar{\mathcal{G}}^{-1}_{jk} \dot{\beta}^k] \neq 0 \quad (4.12)$$

As is argued in appendix B, if $\bar{W}$ is set to zero, T_{23} vanishes. Therefore, in order to obtain the coefficient functions a' and b' , we can drop all the hat symbols from the T-matrix elements, and set $\Sigma|$ to zero in (4.4). Denoting the $\bar{W} = 0$ part of $\Phi|(\Omega)$ by $\Phi_0(\Omega_0)$, we get

$$a' = \frac{i}{2(4\pi)^2} e^{\Phi_0} \text{tr}' [(\mathcal{G} \tilde{\mathcal{G}})^{-1} (e^{\frac{1}{2}i\tau \mathcal{G}} - 1) (e^{-\frac{1}{2}i\tau \tilde{\mathcal{G}}} - 1)], \quad (4.13)$$

and

$$b' = \frac{4i}{(4\pi)^2} e^{\Phi_0} \Omega_0. \quad (4.14)$$

In (4.13) $\tilde{\mathcal{G}}$ denotes the transpose of $\mathcal{G}$, considered as a matrix and tr' refers to a sum over either dotted or undotted indices only. Φ_0 is given by

$$\Phi_0 = i \int d\tau' \left[\frac{1}{2} i \text{tr} T_{22,0}^{-1} + \frac{1}{4} DW \right], \quad (4.15)$$

where tr denotes the sum over both, dotted and undotted indices. $T_{22,0}$ is that part of T_{22} which does not depend on $\bar{W}$. For Ω_0 we get

$$\Omega_0 = -\frac{i}{4} \int_0^{\tau} d\tau' \left[T_{22,0}^{-1}{}^{ab} T_{22,0}^{-1}{}^{ad} T_{21,0}{}^{b\gamma} T_{21,0}{}^{d\gamma} e^{-\frac{1}{2}i\tau' DW} \right]. \quad (4.16)$$

Decomposing $G_{\alpha}{}^{\beta}$ into irreducible components

$$\mathcal{G}_{\alpha}{}^{\beta} = -\frac{1}{2} \delta_{\alpha}{}^{\beta} D + G_{\alpha}{}^{\beta} \quad (4.17)$$

where

$$D = DW, \quad G_{\alpha\beta} = G_{\beta\alpha} \quad (4.18a,b)$$

we have

$$\tilde{\mathcal{G}}_{\alpha}{}^{\beta} = \frac{1}{2} \delta_{\alpha}{}^{\beta} D + G_{\alpha}{}^{\beta}. \quad (4.19)$$

The calculation of e^{Φ_0} is most conveniently performed along the lines proposed in Schwinger's original paper [12]. $S_{22,0a}{}^b$ is calculated in appendix B. Using this result and taking into account that

$$\text{tr } R = 0, \quad (4.20)$$

we get

$$\text{tr } T_{22,0}^{-1} = \frac{1}{2} i \text{tr} [R \coth(\frac{1}{2} i \tau R)]. \quad (4.21)$$

The $d\tau$ -integration in Φ_0 can now be carried out, leading to

$$\begin{aligned} \Phi_0 &= -\frac{1}{2} \text{tr} \log \left(-\frac{2i}{\tau R} \text{sh}(\frac{1}{2} i \tau R) \right) \\ &\quad - 2 \log \tau + \frac{i}{4} \tau D. \end{aligned} \quad (4.22)$$

In order to exponentiate Φ_0 to get a' and b' we have to bring R_{ab} into diagonal form. Here we can take advantage of the fact that $R_a{}^b$ is diagonalized by diagonalizing $G_{\alpha}{}^{\beta}$ and $\bar{G}_{\alpha}{}^{\beta}$, which constitute $R_a{}^b$ [see (2.1.10)], in the corresponding subspace individually. $\bar{G}_{\alpha}{}^{\beta}$ is defined by

$$\bar{D}_{\alpha} \bar{W}^{\beta} = \bar{\mathcal{G}}_{\alpha}{}^{\beta} = \frac{1}{2} \delta_{\alpha}{}^{\beta} D + \bar{G}_{\alpha}{}^{\beta}, \quad \bar{G}_{\alpha\beta} = \bar{G}_{\beta\alpha}. \quad (4.23)$$

With the help of this observation it is immediately demonstrated that

$$e^{\Phi_0} = \frac{1}{8} e^{\frac{i}{4} \tau D} \frac{\text{tr}' G^2 - \text{tr}' \bar{G}^2}{\text{tr}' (\text{ch } \frac{1}{2} i \tau \bar{G}) - \text{tr}' (\text{ch } \frac{1}{2} i \tau G)}. \quad (4.24)$$

Before examining the coefficient function a in more detail, which will turn out to give the only contribution to the effective action, let us first have a closer look at b . From (4.14) and (4.16) it follows that b is of the same order of magnitude as $T_{21,0a}^\beta$. We have calculated $T_{21,0a}^\beta$ in appendix B. The result shows that b is of the same order of magnitude as $\mathcal{K}^2$. As there are terms of the same order of magnitude contributing to $\bar{h}$, which have been dropped in the present approximation, the contribution of b to the effective Lagrangian may be dropped in a consistent approximation. However, there are no competing terms as long as we restrict our attention to $h|$. Therefore we may make partial use of the information contained in b .

In particular, we may obtain some of those chiral composite superfields, which are of the order of magnitude of $\mathcal{K}^2$. Expanding

$$b = \frac{1}{(4\pi)^2} \left(\frac{i}{2}\right) \sum_{n=0}^{\infty} b_n \frac{1}{(2n+4)!} \left(\frac{i}{2}\right)^{2n} \quad (4.25)$$

we get for the first few coefficients

$$\begin{aligned} b_0 &= \mathcal{K}^2 \\ b_1 &= 2 \mathcal{K}^{a,b} \mathcal{K}_{b,c} R_a^{2c} + \frac{5}{2} \mathcal{K}^2 R_a^{2a} \\ &\quad - 2 \mathcal{K}^{a,b} R_a^{bc} \mathcal{K}_{b,c} \mathcal{G}_{15} - \frac{1}{2} \mathcal{K}^2 + \tau' \mathcal{G}^2 \\ &\quad - 10i \mathcal{K}^2 \mathcal{K}_{a,b} \bar{W}^i + 6i \mathcal{K}^2 \mathcal{K}_{a,b} \bar{W}^i . \end{aligned} \quad (4.26)$$

Though the general expression for b in the (C) approximation, including all the powers of τ , is quite complicated, if we restrict ourselves to the (B) approximation it simplifies enough that we can write down a closed expression without too much pain. This expression gives some hints of how b manages to be chiral to all orders of τ . Let us only state the final result. In the approximation (B) a becomes simply a constant, therefore

$$\begin{aligned} h| &= \frac{i}{(4\pi)^2} \left\{ W^2 + \tau' H^2 + \tau' f(G) \right. \\ &\quad \left. + \tau' G H_i + \tau' G H^i + \tau' q(G) \right\} , \end{aligned} \quad (4.27)$$

with

$$f(G) = (1 - \text{ch}(\frac{1}{2}i\tau G))^{-4} (G^i \text{sh}(\frac{1}{2}i\tau G) - \frac{1}{2}i\tau) \quad (4.28)$$

$$q(G) = -\frac{1}{2} (1 - \text{ch}(\frac{1}{2} i \tau G))^{-1} G^{-3} \cdot \left[\text{sh}(\frac{1}{2} i \tau G) + \frac{1}{2} i \tau G - \frac{1}{4} \tau^2 G^2 \text{sh}(\frac{1}{2} i \tau G) (1 - \text{ch}(\frac{1}{2} i \tau G))^{-1} \right]. \quad (4.29)$$

H is defined to be the symmetric part of $\mathcal{H}$

$$H_{\alpha\beta} = \frac{1}{2} \mathcal{H}_{\alpha\beta} \quad (4.30)$$

Note that the antisymmetric part of $\mathcal{H}_{\alpha\beta}$ vanishes in approximation (B). It is easy to check that $h|$ is chiral, provided

$$+i \tau G^{-1} \frac{\partial}{\partial G} f(G) = 2 + i \tau q(G) \quad (4.31)$$

(4.28) and (4.29) indeed satisfy this relation. (4.27) clearly indicates that $h|$ cannot be formulated in terms of a finite number of chiral composite superfields.

Let us next turn our attention to the coefficient function a . With the help of the decompositions (4.17) and (4.19), the trace in (4.13) can be reduced, such that

$$a' = \frac{i}{(4\pi)^2} e^{\frac{1}{2} \bar{\Phi}_0} - \frac{1}{4} i \tau D \quad + i \tau \left\{ (G^2 - \frac{1}{4} D^2)^{-1} \right. \\ \left. \times (\text{ch}(\frac{1}{4} i \tau D) - \text{ch}(\frac{1}{2} i \tau G)) \right\} \quad (4.32)$$

Inserting (4.24) into (4.32) we get

$$a' = \frac{-i}{16 (4\pi)^2} + i \tau \left\{ \frac{(G^2 - \bar{G}^2)}{(\frac{1}{4} D^2 - G^2)} \frac{\text{ch}(\frac{1}{4} i \tau D) - \text{ch}(\frac{1}{2} i \tau G)}{\text{ch}(\frac{1}{2} i \tau \bar{G}) - \text{ch}(\frac{1}{2} i \tau G)} \right\} \quad (4.33)$$

Note that a' coincides with the expression which has been adopted in [22] for the full effective action.

If we insert (4.33) into (2.2.14) and (2.2.15) to obtain the effective action Γ , we recognize that the $d\tau$ integral over the τ independent part of (4.33) is divergent. This divergence is an ultraviolet divergence, which can be removed by a wave function renormalization of the field W . Adding the classical action Γ_0

$$\Gamma_0 = \int d^4x d^2\theta \frac{1}{4} W^2 + h.c. \quad (4.34)$$

to (2.2.14) we get

$$\Gamma = \Gamma_0 - \frac{1}{2} i \int d^4x d^2\theta \int_0^\infty \frac{d\tau}{\tau} e^{\frac{1}{4} i \tau m^2} W^2 a_{\text{sub}} + h.c., \quad (4.35)$$

where

$$a_{\text{sub}} = a - a(\tau=0) \quad (4.36)$$

The $\bar{W} = 0$ component a'_{SUB} of (4.36) can be expressed as

$$a'_{SUB} = \frac{-i}{82} \left(\frac{1}{4\pi}\right)^2 \left[t\tau' \bar{G}^2 - \frac{1}{2} D^2 \right] \times t\tau \left[(G^2 - \frac{1}{4} D^2)^{-1} \frac{\text{ch}(\frac{1}{2}i\tau G) - \text{ch}(\frac{1}{4}i\tau D)}{\text{ch}(\frac{1}{2}i\tau G) - \text{ch}(\frac{1}{2}i\tau \bar{G})} + (G \leftrightarrow \bar{G}) \right] \quad (4.37)$$

We may now use (4.8) and (4.9) to calculate a'' and a''' .

Though this procedure is straightforward, it is quite laborious. Fortunately there is a way which prevents us from having to make explicit use of (4.8) and (4.9). It is actually sufficient to know that the higher-order terms in (4.7) are determined uniquely, once we know a' . Therefore, if we find an expression for a_{SUB} , which reduces to (4.37) for $\bar{W} = 0$, this expression is unique. Taking into account that

$$\frac{1}{2} \bar{D}^2 \bar{W}^2 = t\tau' \bar{G}^2 - \frac{1}{2} D^2 - 4i \mathcal{H}_a, {}^a \bar{W}^2, \quad (4.38)$$

we see that the first bracket of (4.37) is the $\bar{W}^2 = 0$ part of $\frac{1}{2} \bar{D}^2 \bar{W}^2$. Consequently we have

$$a_{SUB} = \frac{-i}{64} \left(\frac{1}{4\pi}\right)^2 \bar{D}^2 \left\{ \bar{W}^2 + t\tau \left[(G^2 - \frac{1}{4} D^2)^{-1} \times \frac{\text{ch}(\frac{1}{2}i\tau G) - \text{ch}(\frac{1}{4}i\tau D)}{\text{ch}(\frac{1}{2}i\tau G) - \text{ch}(\frac{1}{2}i\tau \bar{G})} + (G \leftrightarrow \bar{G}) \right] \right\} \quad (4.39)$$

Inserting (4.39) into (4.35) and introducing the electric charge by rescaling the fields W and $\bar{W}$, we finally get

$$\Gamma = \Gamma_0 + \frac{e^2}{(16\pi)^2} \int d^4x d^2\theta d^2\bar{\theta} W^2 \bar{W}^2 \times t\tau \left\{ (G^2 - \frac{1}{4} D^2)^{-1} \int_0^\infty \frac{d\tau}{\tau} e^{\frac{1}{4}i\tau m^2} \frac{\text{ch}(\frac{1}{2}i\tau G) - \text{ch}(\frac{1}{4}i\tau D)}{\text{ch}(\frac{1}{2}i\tau G) - \text{ch}(\frac{1}{2}i\tau \bar{G})} + (G \leftrightarrow \bar{G}) \right\} \quad (4.40)$$

To write the effective action as a D-term, we have made use of the relation

$$d^2\bar{\theta} \hat{=} -\frac{1}{4} \bar{D}^2 \quad (4.41)$$

Note that (4.40) is a polynomial in the superfields G , $\bar{G}$ and D , if expanded in powers of τ , i.e. there are no inverse powers of the superfields, G , $\bar{G}$ and D .

One might be tempted to argue that it is not essential to have an approximation right from the beginning, which leads to a chiral heat kernel. It appears to be sufficient to calculate $h(z_1, z_2; \tau)$ first in an arbitrary approximation with $\bar{W}_\alpha = 0$, and then to reconstruct the $\bar{W}$ dependence by using the relations (4.8) and (4.9). Unfortunately such a procedure does not work. Setting $\bar{W}_\alpha = 0$ implies $\bar{G} = 0$, which definitely gives a wrong answer for a' . We use the expansion (4.7) only to facilitate the evaluation of h .

Note that the result (4.39) indicates that the effective action is a D-term. In particular, if we had insisted on writing it as an F-term, we would have been forced to introduce infinitely many composite chiral superfields. However, these superfields are nothing else than linear combinations of those superfields appearing in (4.40) and covariant derivatives thereof.

Expanding the integrand of (4.40) in powers of τ , we get an asymptotic expansion of Γ , valid for large m^2

$$\Gamma = \Gamma_0 - \frac{2e^2}{(16\pi)^2} \int d^4x d^2\theta d^2\bar{\theta} W^2 \bar{W}^2 \times \sum_{n=1}^{\infty} \left(\frac{2ie}{m^2} \right)^{2n} \frac{\Gamma(2n)}{(2n+2)!} + \tau a_n \quad (4.42)$$

The first few coefficients a_n are given by

$$\begin{aligned} a_1 &= 1 \\ a_2 &= \frac{3}{2} (G^2 + \bar{G}^2) - \frac{D^2}{4} \\ a_3 &= \frac{10}{3} (G^4 + \bar{G}^4) + \frac{31}{3} G^2 \bar{G}^2 - \frac{11}{12} D^2 (G^2 + \bar{G}^2) + \frac{1}{16} D^4 \end{aligned} \quad (4.43)$$

The effective action (4.40) describes all the low-energy vacuum polarization effects of SQED, caused by the matter superfields χ and ϕ of the fundamental Lagrangian (2.2.1). If we introduce couplings of the fields appearing in the fundamental Lagrangian to some external currents, we can calculate arbitrary S-matrix elements. Interactions between the matter and the gauge fields are introduced through the non-trivial dependence of the matter propagation, the inverse of (2.2.10), on the gauge fields. As all the fields appearing in (4.40) are classical fields we can interpret (4.40) immediately as an effective action for composite fields.

In (4.40) we have represented the effective action in terms of the five composite superfields $W^2 \bar{W}^2$, G^2 , $\bar{G}^2$ and D . The expansion of (4.40) in terms of component fields is straightforward. However, as the resulting formula is quite lengthy, let us only mention some general features. The non-supersymmetric Heisenberg-Euler effective action contains polynomial interactions of the lowest components of the following superfields

$$\tilde{F} = R_{ab} R^{ab}, \quad \tilde{\tilde{F}} = R_{ab} \tilde{R}^{ab} \quad (4.44a,b)$$

In (4.44b) $\tilde{R}^{ab}$ denotes the dual of R_{ab} . In the supersymmetric case we get various additional prices, including complicated four-fermion couplings and derivative couplings. In particular the auxiliary field of the W and $\bar{W}$ multiplets couples via its derivative. Therefore its field equation is no longer a simple algebraic equation, as it was on the fundamental level. However, as there is no kinetic term for the auxiliary field, this does not mean that the auxiliary field becomes a propagating field.

Let us finally demonstrate how (4.40) goes over into the non-supersymmetric Heisenberg–Euler effective action. We have

$$\begin{aligned} (G^2)_\alpha{}^\beta &= -8\delta_\alpha{}^\beta (\tilde{\mathcal{F}} + i\tilde{\tilde{\mathcal{F}}}) , \\ (\bar{G}^2)_{\dot{\alpha}}{}^{\dot{\beta}} &= -8\delta_{\dot{\alpha}}{}^{\dot{\beta}} (\tilde{\mathcal{F}} - i\tilde{\tilde{\mathcal{F}}}) . \end{aligned} \quad (4.45)$$

Therefore keeping the F-components of W^2 and $\bar{W}^2$

$$\begin{aligned} W^2|_{\theta^2} &= \frac{1}{2} (\tilde{\mathcal{F}} + i\tilde{\tilde{\mathcal{F}}})| + \dots , \\ \bar{W}^2|_{\bar{\theta}^2} &= \frac{1}{2} (\tilde{\mathcal{F}} - i\tilde{\tilde{\mathcal{F}}})| + \dots , \end{aligned} \quad (4.46)$$

(4.40) immediately reproduces the well-known non-supersymmetric result. Note that $\tilde{\mathcal{F}}|$ and $\tilde{\tilde{\mathcal{F}}}|$ appear as the lowest as well as the highest component of a composite superfield. A similar structure has been found previously in [9–11] in the context of several other models.

5. CONCLUDING REMARKS

In the present paper we have developed a systematic technique, which allows to calculate supersymmetric effective actions in a manifestly supersymmetric manner. With the help of this technique we were able to deduce the Heisenberg–Euler effective Lagrangian of SQED.

Our work can be generalized in various ways. In addition to the gauge field of SQED, one may introduce another gauge field, coupling to the axial vector current. Or one may include a term, which softly breaks supersymmetry in the underlying Lagrangian and study its manifestation in the effective theory.

The main emphasis in the present investigation has been put upon a clear understanding of how to generalize Schwinger’s operator technique to superspace. We hope that this technique can be applied with only minor modifications to models other than SQED. We have in mind two-dimensional models, or models with an extended supersymmetry.

One may immediately think of extending the construction presented in this paper to non-Abelian gauge theories. However, Schwinger’s technique can only be generalized to a non-Abelian gauge theory at the price of keeping only a subset of the background fields different from zero, which commute among one another.

Unfortunately the vacuum structure of SQED and the supposed vacuum structure of those theories, which are of interest for the construction of supersymmetric subcomponent models, are radically different. It is therefore impossible to draw any immediate conclusions from the present analysis about

those problems, which we have mentioned in the introduction. In order to find answers to those questions our method should be applied to models different from SQED.

Bearing this in mind, we may nevertheless speculate about potential consequences for SQCD. Previous attempts to construct effective actions for SQCD and SYM [4, 5] did only include the composite gauge fields W^2 and $\bar{W}^2$, besides some composite matter superfields. As our analysis has clearly shown, the low-energy effective action of SQED must as well include covariant derivatives of the superfields W and $\bar{W}$. Therefore it seems highly reasonable that the effective actions given in [4, 5] should be extended to include as well covariant derivatives. This does not necessarily imply that the corresponding superfields represent new degrees of freedom. We do not expect that these additional superfields change the vacuum structure as found in [4, 5], which was mainly determined by the anomalous part of the effective action.

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APPENDIX A

The purpose of this appendix is the determination of the function C , appearing in the general expression (3.2.40) of the heat kernel. C will be fixed by solving the equations (3.2.31a), (2.2.19), (3.2.5), and (2.2.20). For this purpose it is useful to rewrite $h(z_1, z_2; \tau)$ as

$$h(z_1, z_2; \tau) = e^{\bar{\Phi}} \bar{V} C(z_1, z_2), \quad (\text{A.1})$$

where

$$V = e^{\Xi'} U(\tau), \quad (\text{A.2})$$

$$\Xi' = U(\tau) \Xi U^{-1}(\tau). \quad (\text{A.3})$$

(A.3) is readily evaluated from (3.2.43) with the help of

$$U(\tau) \mathcal{D}_a(z_1) U^{-1}(\tau) = (e^{-\frac{1}{2}i\tau \mathcal{Q}})_a{}^\beta \mathcal{D}_\beta(z_1). \quad (\text{A.4})$$

We shall start with equation (3.2.31a). Inserting the expansion (3.2.30a) of the operator $\eta(\tau)$ and (A.1) into (3.2.31a) we get

$$\begin{aligned} & V^{-1} (\bar{T}_{21} a^\beta \mathcal{D}_\beta(z_1) + \bar{T}_{22} a^b \mathcal{D}_b(z_1)) \bar{V} C(z_1, z_2) \\ &= \bar{V}^{-1} (\eta_a - \bar{T}_{23} a^\beta W_\beta - \bar{T}_{21} a^\beta (\mathcal{D}_\beta \bar{\Phi}) \\ &\quad - \bar{T}_{22} a^b (\mathcal{D}_b \bar{\Phi})) \bar{V} C(z_1, z_2). \end{aligned} \quad (\text{A.5})$$

As $C(z_1, z_2)$ is independent of τ , (A.5) gives for each power of τ a constraint on $C(z_1, z_2)$. We note that

$$\bar{T}_{22} a^b = \delta_a^b \tau - \frac{1}{2} i \tau^2 R_a^b + \mathcal{O}(\tau^2), \quad (\text{A.6})$$

$$\bar{T}_{21} a^\beta \sim \mathcal{O}(\tau^2), \quad (\text{A.7})$$

$$\bar{T}_{23} a^\beta \sim \mathcal{O}(\tau^2), \quad (\text{A.8})$$

$$\mathcal{D}_a \bar{\Phi} = \frac{1}{\tau} \eta_a + \mathcal{O}(\tau), \quad (\text{A.9})$$

$$V = 1 + O(\tau) \quad . \quad (A.10)$$

Inserting these relations into (A.5) one discovers that to order τ^0 (A.5) is satisfied identically. Taking into account that $C(z_1, z_2) \sim \Delta^2$ we get to order τ

$$\left\{ \partial_a(z_1) + \frac{1}{2} R_a{}^b (\gamma_1 - \gamma_2)_b \right\} C(z_1, z_2) = 0 \quad . \quad (A.11)$$

All the higher orders in τ do not contain any new information. (A.5) is identically satisfied for all τ^n $n > 1$ if $C(z_1, z_2)$ satisfies (A.11) and (A.16).

Let us next turn to (2.2.19). Inserting (A.1) into (2.2.19) it looks like

$$\bar{V}^{-1} \bar{\partial}_i \bar{V} C(z_1, z_2) = - V^{-1} (\bar{\partial}_i \bar{\Phi}) \bar{V} C(z_1, z_2) \quad . \quad (A.12)$$

As

$$\bar{\partial}_i \bar{\Phi} = - \frac{i}{\tau} \epsilon_{i\dot{\rho}} \Delta_{\dot{\rho}} \eta^{\dot{b}} + O(\tau) \quad , \quad (A.13)$$

(A.12) is trivially satisfied to order τ^{-1} . In the order τ^0 we have to take into account the action of $V^{-1} \dots V$ on Δ_{α} . As

$$\Xi = - \frac{1}{4} i \tau \eta^a \mathcal{K}_a{}^{\beta} \partial_{\beta} + O(\tau^2) \quad , \quad (A.14)$$

we get

$$\bar{V}^{-1} \Delta_a \bar{V} = \Delta_a + \frac{1}{4} i \tau (\eta^{\dot{b}} \mathcal{K}_{\dot{b},a}) - \frac{1}{2} i \tau W_a + O(\tau^2) \quad . \quad (A.15)$$

Inserting this relation into (A.12) we get a second constraint on $C(z_1, z_2)$

$$\begin{aligned} \bar{\partial}_i(z_1) C(z_1, z_2) &= 2i \epsilon_{i\dot{\rho}} \eta^{\dot{b}} \\ &\times \left[\frac{1}{4} \eta^c \mathcal{K}_{c,\dot{\rho}} - \frac{1}{2} i W_{\dot{\rho}} \right] C(z_1, z_2) \quad . \end{aligned} \quad (A.16)$$

Again (A.12) is an identity for all higher powers of τ , if $C(z_1, z_2)$ satisfies the constraints (A.11) and (A.16).

Equation (3.2.5) leads to a third constraint on $C(z_1, z_2)$. Rewriting (3.2.5) with the help of (A.1) we get

$$\begin{aligned}
V^{-1} [- \mathcal{D}'_a(z_2) - (\partial_a(z_2) \bar{\Phi}) - i \bar{\Theta}_{2,i} W_a(z_2)] V \zeta = \\
V^{-1} [(S'_{22}{}^a{}_b - i \bar{\Theta}_{1,i} S'_{32}{}^a{}_b) T_{22}^{-1}{}^c{}_b \eta_c \\
+ \sigma(\tau)] \bar{V} \zeta ,
\end{aligned} \tag{A.17}$$

where we have reduced the action of $\bar{\mathcal{D}}_a(\tau)$ on h via (3.2.11). Noting that

$$S'_{22}{}^a{}_b = S_a{}^b - i \tau R_a{}^b , \tag{A.18}$$

$$S'_{32}{}^a{}_b = i \tau \mathcal{H}^b{}_a , \tag{A.19}$$

we get the third constraint on C

$$\mathcal{D}'_a(z_2) C(z_1, z_2) = [i \bar{\mathcal{D}}_i W_a(z_2) + \frac{1}{2} i R_a{}^b \eta_b] C(z_1, z_2) . \tag{A.20}$$

Let us finally write down the last constraint following from (2.2.20).

$$\bar{\mathcal{D}}'_i(z_2) C(z_1, z_2) = 0 . \tag{A.21}$$

We have checked that the constraints (A.11), (A.16), (A.20) and (A.21) are consistent among each other.

The next step is to solve the four constraints. So far we have dealt exclusively with gauge-covariant quantities. In order to solve the constraints we have to go into a definite gauge. It has turned out that the gauge appropriate for the present problem is the generalization of the Schwinger–Fock gauge to superspace. The generalization has been performed in a previous paper [29]. This gauge has the advantage that the gauge connection appearing as a part of the covariant derivatives can be expressed directly in terms of the field strength and its covariant derivatives, taken at a certain reference point.

The transition into the Schwinger–Fock gauge is accomplished by the following ansatz

$$C(z_1, z_2) = \exp \left\{ - \int_{z_2}^{z_1} d\xi_A \phi^A(\xi) \right\} C'(z_1, z_2) , \tag{A.22}$$

where ξ_A denotes a straight line in superspace running from z_2 to z_1 and the $\phi_A(\xi)$ are the gauge connections, which have been introduced to define the covariant derivatives. The path-dependent phase factor acts like a gauge transformation, which brings, in the approximation (C), $\phi_a(z_1)$ and $\bar{\phi}^{\dot{a}}(z_1)$ into

$$\begin{aligned}
\phi_a(z_1) = & \frac{i}{2} \bar{\Delta}_i W_a(z_2) + \frac{1}{6} (\bar{\Delta}_\beta \mathcal{K}_{a,\beta} + (a \leftrightarrow b)) \eta^b \\
& + \frac{i}{2} (R_{ab}(z_2) - i (\bar{\Delta}_i \mathcal{K}_{b,a} - (a \leftrightarrow b))) \eta^b \\
& + \mathcal{O}(\Delta) \quad ,
\end{aligned} \tag{A.23}$$

$$\bar{\phi}^i(z_1) = -\frac{1}{2} \eta^{i\beta} W_\beta(z_2) + \frac{i}{3} \eta^{i\gamma} \eta^b \mathcal{K}_{b,\gamma} + \mathcal{O}(\Delta) \quad , \tag{A.24}$$

and similarly for $\phi_A(z_2)$, where the roles of z_1 and z_2 are interchanged. As $C' \sim \Delta^2$, we have dropped all those contributions from ϕ_A which are proportional to Δ . Inserting the gauge connections into the constraints (A.11), (A.16), (A.20) and (A.21), C' is determined uniquely to be

$$\begin{aligned}
C'(z_1, z_2) = & \frac{-i}{(4\pi)^2} \exp \left\{ -\frac{1}{2} \bar{\Delta}_i \eta^a W_a(z_2) \right. \\
& \left. + \frac{i}{6} \bar{\Delta}_i \eta^a \eta^b \mathcal{K}_{b,a} \right\} \Delta^2 \quad ,
\end{aligned} \tag{A.25}$$

where we have fixed the normalization in accordance with (3.2.46). Note that owing to the restriction to $y_1 = y_2$ and $\theta_1 = \theta_2$ in (3.2.46), the path-dependent phase factor in (A.22) reproduces exactly the factor $e^{-U(z_1) + U(z_2)}$ of (3.2.46). This follows from the fact that, because of the partial identification of the coordinates of the points z_1 and z_2 , only $\bar{\phi}_\alpha = \bar{D}_\alpha U$ survives in (A.22).

APPENDIX B

In this appendix we shall derive the elements of the matrix S in the limit $\bar{W}_\alpha = 0$. Sending $\bar{W}$ to zero, the $\tilde{\square}$ matrix becomes triangular

$$\tilde{\square}_0 = \begin{pmatrix} \frac{1}{2} \mathcal{G}_a{}^\beta & 0 & 0 \\ \frac{1}{2} \mathcal{H}_a{}^\beta & \mathcal{R}_a{}^b & 0 \\ 0 & -\mathcal{H}^b{}_a & \frac{1}{2} \tilde{\mathcal{G}}_a{}^\beta \end{pmatrix} \quad (\text{B.1})$$

Consequently $S_{A,0}^B$ is triangular too. The diagonal elements of the matrix S are immediately determined to be

$$S'_{11,0}{}_a{}^\beta = (e^{-\frac{1}{2}i\tau \mathcal{G}})_a{}^\beta, \quad (\text{B.2})$$

$$S'_{22,0}{}_a{}^b = (e^{\frac{1}{2}i\tau \mathcal{R}})_a{}^b, \quad (\text{B.3})$$

$$S'_{33,0}{}_a{}^\beta = (e^{-\frac{1}{2}i\tau \tilde{\mathcal{G}}})_a{}^\beta. \quad (\text{B.4})$$

To calculate the off-diagonal elements of S one may be tempted to use the recurrence relations (3.2.19). Though such a procedure is very convenient if $\bar{W}_\alpha \neq 0$, having set $\bar{W}_\alpha = 0$ (3.2.19) does not hold any more. Therefore we determine the off-diagonal elements by solving the following differential equations

$$i \frac{d}{d\tau} S'_{21,0}{}_a{}^\beta = \mathcal{R}_a{}^c S'_{21,0}{}_c{}^\beta + \frac{1}{2} \mathcal{H}_a{}^\gamma (e^{-\frac{1}{2}i\tau \mathcal{G}})_\gamma{}^\beta, \quad (\text{B.5})$$

$$i \frac{d}{d\tau} S'_{32,0}{}_a{}^b = -\mathcal{H}^c{}_a (e^{-i\tau \mathcal{R}})_c{}^b + \frac{1}{2} \tilde{\mathcal{G}}_a{}^\gamma S'_{32,0}{}_b{}^\gamma, \quad (\text{B.6})$$

$$i \frac{d}{d\tau} S'_{31,0}{}_a{}^\beta = -\mathcal{H}^c{}_a S'_{21,0}{}_c{}^\beta + \frac{1}{2} \tilde{\mathcal{G}}_a{}^\gamma S'_{31,0}{}_b{}^\beta, \quad (\text{B.7})$$

with initial conditions

$$S'_{ij,0}(\tau=0) = 0, \quad i \neq j. \quad (\text{B.8})$$

The unique solution of (B.5)–(B.8) is

$$S_{32,0}^b = i \int_0^{\tau} d\tau' (e^{-\frac{1}{2}i\tau' \tilde{q}})_a{}^\sigma \mathcal{H}_{c,\sigma} (e^{-i(\tau-\tau')R})_c{}^b, \quad (\text{B.9})$$

$$S_{21,0}^B = -\frac{1}{2}i \int_0^{\tau} d\tau' (e^{-i\tau'R})_a{}^c \mathcal{H}_{c,\sigma} (e^{-i(\tau-\tau')\frac{1}{2}\tilde{q}})_\sigma{}^B, \quad (\text{B.10})$$

and

$$S_{31,0}^B = (e^{-\frac{1}{2}i\tau \tilde{q}})_a{}^\gamma \left\{ -\mathcal{H}_{c,\gamma} \int_0^{\tau} d\tau' S_{21,0}^B(\tau') \right\}. \quad (\text{B.11})$$

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