CERN-TH.4316/85

HIGH-TEMPERATURE EXPANSIONS IN CONTINUUM QUANTUM FIELD THEORY

K. Dietz^{*)}

CERN - Geneva

and

Th. Millack

Physikalisches Institut der Universität Bonn
D-5300 Bonn, Germany

ABSTRACT

High-temperature or, equivalently, strong coupling expansions are established by applying heat-kernel techniques to the Green's functional of the Schrödinger picture of quantum field theory. The trace of the Green's functional is considered and a high-temperature expansion for the partition function is derived. It is shown that a simple, physically motivated rescaling of parameters leads to well-defined expressions. A generalized Thomas-Fermi expansion for the density of states finally obtains.

^{*)}Permanent address: Physikalisches Institut der Universität Bonn, Univ. Bonn, 5300-Bonn, Germany.

1. Introduction

The Schrödinger picture of quantum field theory (QFT) introduces concepts which might be useful for conceiving approximation methods for the description of complex physical systems. In particular, the concept of physical states represented by wave-functionals allows for the possibility of transferring intuitions developed in many-body theory to similar problems in the theory of relativistic systems; for instance, ansätze for variational calculations can be formulated in this framework.

A particularly transparent situation arises if we use the "coordinate representation" of the Schrödinger picture in which fields act as multiplicative operators - coordinates - in the space of wave-functionals, the momentum operator as a functional derivative versus these fields. Historically, such a picture (which should be adequately named as "quantum continuum mechanics" and which is doubtlessly the most direct and naive formulation of a quantum field theory) appeared to be suspicious and fell into a certain disrepute until it was shown by Symanzik¹⁾ that the Schrödinger initial value problem can be given a well-defined meaning in the framework of renormalised QFT.

In this paper we are going to describe approximation methods for the Green's functionals of the Schrödinger equation formulated in this picture. As a model we simply take a φ^4 -theory, it should become clear, however, that this theory serves only as a vehicle for a concise description unencumbered by degrees of freedom which are not to the point and that our methods can be applied to more complex and physically realistic situations (for an example see ref. (2)).

For the sake of a self-contained representation a perturbative expansion of the Green's functional for small couplings is shortly described. We then establish, for the euclidean version of the theory, a high-temperature expansion by deriving an asymptotic series for the corresponding functional heat-kernel. This technique is finally used to derive an asymptotic series for the density of states which, for fermionic systems, closely resembles a Thomas-Fermi expansion appropriatively translated to QFT notions.

2. Green's functionals in the Schrödinger picture

In this section we collect a few ingredients of the Schrödinger picture of QFT which are necessary for a coherent presentation. The latter will be deliberately formal in order to emphasize the quantum continuum-mechanical stand-point, the necessity of redefining operator products in a physically adequate manner will appear automatically and will have to be performed (as usual) a posteriori.

As coordinates we shall take a real scalar field $\varphi(\vec{x})$ which, in the Schrödinger picture, is identified with the classical field $\varphi(\vec{x}, t_0)$ at an arbitrary but fixed time t_0 . The momentum is then

$$\pi(\vec{x}) = -i \frac{\delta}{\delta \varphi(\vec{x})} . \quad (2.1)$$

The Hamiltonian we assume to read as

$$H = \frac{1}{2} \int d^3x \left(\pi^2(\vec{x}) + \tilde{V}(\varphi) \right) , \quad (2.2)$$

physical states are represented as time-dependent complex valued functionals $\Psi[\varphi, t]$ the time-dependence of which is governed by the Schrödinger equation

$$i \partial_t \Psi[\varphi, t] = \int d^3x \frac{1}{2} \left(- \frac{\delta^2}{\delta \varphi(\vec{x}) \delta \varphi(\vec{x})} + \tilde{V}(\varphi) \right) \Psi[\varphi, t] \quad (2.3)$$

$$=: H \Psi[\varphi, t] .$$

Clearly, this expression is ill-defined as it stands and has to be treated¹⁾ with appropriate care. We shall encounter certain repercussions of this problem the physical significance of which will be discussed.

Introducing stationary states

$$\Psi[\varphi, t] = e^{-i E_n t} \Psi_n[\varphi] \quad (2.4)$$

we arrive at the "eigenvalue equation"

$$\mathcal{H} \Psi_n[\varphi] = E_n \Psi_n[\varphi]. \quad (2.5)$$

The Green's functional $\mathcal{G}[\varphi, \varphi'; t, t']$ obeying

$$(i \partial_t - \mathcal{H}) \mathcal{G}[\varphi, \varphi'; t, t'] = \delta(\varphi - \varphi') \delta(t - t') \quad (2.6)$$

where the functional δ -distribution is (formally) defined as usual

$$\delta(\varphi - \varphi') = \prod_{\vec{x}} \delta(\varphi(\vec{x}) - \varphi'(\vec{x}))$$

can now be written as

$$\mathcal{G}[\varphi, \varphi'; t, t'] = \sum_n \Psi_n[\varphi] \Psi_n^*[\varphi'] e^{-i E_n(t-t')} \Theta(t-t') \quad (2.7)$$

It is well known³⁾ that $\mathcal{G}[\varphi, \varphi'; t, t']$ has a representation as a conditional functional integral

$$\mathcal{G}[\varphi_f, \varphi_i; t_f, t_i] = \int_{(\varphi_f, \varphi_i)} D\phi e^{i S[\phi]} \quad (2.8)$$

where the action $S[\phi]$ is given by

$$S[\phi] = \frac{1}{2} \int_{t_i}^{t_f} dt \int d^3x \left\{ \dot{\phi}^2 - \tilde{V}(\phi) \right\}$$

and the integration runs along paths with initial and final configurations $\varphi_i(\vec{x})$ at t_i and $\varphi_f(\vec{x})$ at t_f .

Some precisions are in place. A path is understood to be a map of the time-interval $[t_i, t_f]$ into a set of functions

$$\text{path: } t \rightarrow \varphi_t(\vec{x})$$

the measure $D\phi$ in (2.8) is then constructed as follows: let $t_0 = t_i < t_1 \dots < t_{N+1} = t_f$ be a subdivision of the interval $[t_i, t_f]$ and

$$\begin{aligned} \Delta t_k &= t_{k+1} - t_k \\ \phi_k(\vec{x}) &:= \phi_{t_k}(\vec{x}) \end{aligned}$$

the measure $D\phi$ is then understood as the limit of (formally written)

$$D\phi = \prod_{k=1}^N \frac{D\phi_k}{(2\pi \Delta t_k)^{\gamma/2}} \quad ; \quad \gamma = \delta^3(0) \cdot V \quad (2.9)$$

for arbitrarily small subdivision, where V is the volume in which the system is enclosed, i.e. γ is simply the number of degrees of freedom (in a lattice approximation of the position-space continuum we have $\gamma = (\text{Volume of the system})/(\text{Volume of the lattice cell})$).

For our further considerations we specify the potential

$$\tilde{V}(\phi) = (\phi A^2 \phi)(\vec{x}) + V(\phi) \quad (2.10)$$

where $V(\phi) = \frac{\lambda}{2} \phi^4(\vec{x})$; the term quadratic in ϕ describes free particles with a dispersion given by the differential operator A , the spectrum of the latter being bounded or unbounded from below distinguishes the non-relativistic and the relativistic case (these boundedness properties are the only ones relevant for our purposes).

The Green's functional $\mathcal{G}[\varphi, \varphi'; t, t']$ is now written as

$$\mathcal{G}[\varphi_f, \varphi_i; t_f, t_i] = e^{-\frac{i}{2} \int d\vec{x} V(i \frac{\delta}{\delta \vec{\varphi}(\vec{x})})} \hat{\mathcal{G}}[\varphi_f, \varphi_i; t_f, t_i; j] \quad (2.11)$$

with

$$\hat{\mathcal{G}}[\varphi_f, \varphi_i; t_f, t_i; j] = \int_{(\varphi_f, \varphi_i)} D\phi e^{i S_0[\phi] - i \int d\vec{x} \phi \hat{j}(\vec{x})} \quad (2.12)$$

and

$$S_0[\phi] = \frac{1}{2} \int (\dot{\phi}^2 - \phi A^2 \phi); \quad (2.13)$$

$\hat{\mathcal{G}}$ can be explicitly calculated ($T:=t_f-t_i$)

$$\begin{aligned} \hat{\mathcal{G}}[\varphi_f, \varphi_i; t_f, t_i; j] &= \det\left(\frac{A}{2\pi i \sin AT}\right)^{1/2} \exp i \left[\frac{1}{2} \int \right. \\ &\quad \left\{ \varphi_f (A \cotg AT) \varphi_f + \varphi_i (A \cotg AT) \varphi_i - 2 \varphi_f \frac{A}{\sin AT} \varphi_i \right\} - \\ &\quad \int_{t_i}^{t_f} dt \int \left\{ \varphi_f \frac{\sin(A(t-t_i))}{\sin AT} j + \varphi_i \frac{\sin(A(t_f-t))}{\sin AT} j \right\} - \\ &\quad \left. \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \iint j \frac{\sin(A(t_f-t)) \sin(A(t'-t_i))}{A \sin AT} j \right]. \end{aligned} \quad (2.14)$$

Equation (2.11) now yields a perturbative expansion for $\hat{\mathcal{G}}[\varphi_f, \varphi_i; t_f, t_i]$ in the usual manner, divergences appear and have to be renormalised following the procedures outlined in ref. (1).

It is interesting to note and typical for the Schrödinger picture that the necessity of renormalisation already appear in zeroth order. To see this, we compute the partition functions of bosonic and fermionic excitations

$$Z = \text{tr} \hat{\mathcal{G}} \Big|_{\substack{j=0 \\ T=-i\beta}} = \det^{-1} \left(e^{A \frac{\beta}{2}} (1 - e^{-A\beta}) \right) \quad (2.15)$$

for bosons, i.e. for periodic boundary conditions $\varphi_f = \varphi_i$ and

$$Z = \text{tr} \hat{\mathcal{G}} \Big|_{\substack{j=0 \\ T=-i\beta}} = \det^{-1} \left(e^{A \frac{\beta}{2}} (1 + e^{-A\beta}) \right) \quad (2.16)$$

for fermions, i.e. involving antiperiodic boundary conditions

Since we employed ordinary functional integration not involving Grassmannian fields we have to correct for an overall sign in $\ln Z$. The point we should like to make is that $\text{tr} \hat{\mathcal{G}}$ gives the correct result without any

regularisation procedure. Let us consider the latter case and write

$$\begin{aligned} \ln Z &= \ln \operatorname{tr} \hat{\mathcal{G}} \Big|_{\substack{\hat{x}=0 \\ T=-i\beta}} \\ &= -\frac{\beta}{2} \sum_{\alpha} \varepsilon_{\alpha} \operatorname{sign} \varepsilon_{\alpha} - \sum_{\alpha} \ln(1 + e^{-\beta \varepsilon_{\alpha} \operatorname{sign} \varepsilon_{\alpha}}) \end{aligned} \quad (2.17)$$

where we introduced the spectrum $\{\varepsilon_{\alpha}\}$ of A and the Fermi energy of our system

$$\operatorname{sign} \varepsilon_{\alpha} = \begin{cases} +1 & \text{for } \varepsilon_{\alpha} > \varepsilon_F \\ -1 & \text{for } \varepsilon_{\alpha} < \varepsilon_F \end{cases} . \quad (2.18)$$

Clearly, the first term is divergent and requires regularisation. That it leads to physical consequences can be seen by calculating e.g. the fermion number N. Assume that A contains a background gauge field A_{μ} and compute N

$$N = - \int d^3x \, \beta^{-1} \frac{\delta \operatorname{tr} \hat{\mathcal{G}}}{\delta A_0(\vec{x})} \Big|_{\substack{\hat{x}=0 \\ T=-i\beta}} . \quad (2.19)$$

Using ζ -function regularisation one finds

$$N = \frac{1}{2} \sum_{\alpha} \operatorname{sign} \varepsilon_{\alpha} |\varepsilon_{\alpha}|^{-s} \Big|_{s \rightarrow 0} \quad (2.20)$$

which is the well-known⁴⁾ connection between the fermion number and the open space generalisation⁵⁾ of the Atiyah-Patodi-Singer spectral asymmetry. Interesting physical consequences resulting from boundary conditions and/or global topological properties of the gauge fields have been discussed in the literature⁶⁾⁴⁾.

This concludes our attempt to assemble the ingredients necessary for formulating our main point, a task to which we now turn.

3. High-temperature expansion for the euclidean Green's functional

It is clear that the Hamiltonian $\mathcal{H}$ defined in (2.3) has a continuous elliptic structure, i.e. any finite discretisation leads to an elliptic differential operator. For the latter an asymptotic expansion for small β - high temperatures - of $\text{tr} e^{-\beta \mathcal{H}}$ - the partition function - is well known⁷⁾. In this section we shall investigate the behaviour of this expansion in the continuum limit and show that this limit admits an unambiguous physical interpretation.

We write equation (2.3) as a heat-transport equation by Wick-rotating $t \rightarrow -i\tau$

$$(-\partial_\tau - \mathcal{H}) \bar{\mathcal{G}} [\varphi, \varphi'; \tau, \tau'] = 0 \quad (3.1)$$

and impose the usual boundary condition

$$\bar{\mathcal{G}} [\varphi, \varphi'; \tau, \tau] = \delta(\varphi - \varphi'). \quad (3.2)$$

An asymptotic expansion for small β is now obtained by splitting off the most singular part determined by the kinetic term in (3.1)

$$\bar{\mathcal{G}} [\varphi, \varphi'; \tau, \tau'] = \bar{\mathcal{G}}_0 [\varphi, \varphi'; \tau, \tau'] \sum_{n=0}^{\infty} a_n[\varphi, \varphi'] \beta^n \quad (3.3)$$

$$\beta = \tau - \tau'$$

with $\bar{\mathcal{G}}_0$ obeying

$$(-\partial_\tau - \mathcal{H}_0) \bar{\mathcal{G}}_0 [\varphi, \varphi'; \tau, \tau'] = 0$$

$$\bar{\mathcal{G}}_0 [\varphi, \varphi'; \tau, \tau] = \delta(\varphi - \varphi')$$

where

$$H_0 = \frac{1}{2} \int d^3x \left(- \frac{\delta^2}{\delta\varphi(\vec{x}) \delta\varphi(\vec{x})} \right) \quad (3.4)$$

The functionals $a_n[\varphi, \varphi']$ are seen to fulfil functional differential equations which are straightforwardly determined by inserting (3.3) into (3.1). We obtain

$$\int d^3x (\varphi - \varphi')(\vec{x}) \frac{\delta a_n[\varphi, \varphi']}{\delta\varphi(\vec{x})} = -\kappa a_n[\varphi, \varphi'] - H[\varphi] a_{n-1}[\varphi, \varphi'] \quad (3.5)$$

Solving these equations we get for the diagonal coefficients

$$\begin{aligned} a_0[\varphi, \varphi] &= 1 \\ a_1[\varphi, \varphi] &= -\frac{1}{2} \int d^3x \tilde{V}(\varphi) \\ a_2[\varphi, \varphi] &= \frac{1}{2} a_1[\varphi, \varphi]^2 - \frac{1}{6} H_0[\varphi] a_1[\varphi, \varphi] \\ a_3[\varphi, \varphi] &= -\frac{1}{12} a_1[\varphi, \varphi] H_0 a_1[\varphi, \varphi] - \\ &\quad \frac{1}{24} H_0[\varphi] (a_1[\varphi, \varphi])^2 + \frac{1}{6} a_1[\varphi, \varphi]^3 \end{aligned} \quad (3.6)$$

etc..

Our goal is to derive an asymptotic expansion for the partition function

$$Z = \text{tr } \bar{\mathcal{G}} = \int D\varphi \bar{\mathcal{G}}[\varphi, \varphi; \beta, 0] \quad (3.7)$$

and for the density of states

$$dE_S(E) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} d\tau e^{E\tau} \text{tr} \bar{g}[\varphi, \varphi'; \tau, 0] dE \quad (3.8)$$

Our main concern will be to show that finite expressions appear if certain parameters are rescaled in a physically motivated manner. From (2.14) we read off

$$\bar{g}_0[\varphi, \varphi'; \beta, 0] = \left(\frac{1}{2\pi\beta}\right)^{3/2} e^{-\frac{1}{2\beta} \int d^3x (\varphi - \varphi')^2(\vec{x})} \quad (3.9)$$

and obtain

$$Z = \left(\frac{1}{2\pi\beta}\right)^{3/2} \int D\varphi \sum_{n=0}^{\infty} a_n[\varphi, \varphi] \beta^n. \quad (3.10)$$

Our problem now is to compute the integrals $\int D\varphi a_n[\varphi, \varphi]$, functional integrals of polynomials in φ . We shall show that a physically well-defined sense can be given to such integrals if infinities corresponding to familiar concepts like the number of degrees of freedom, the volume of configuration space, i.e. the "volume" of the space of field coordinates $\varphi(\vec{x})$, can be unambiguously identified.

For the actual calculation we follow the usual procedure to define functional integrals by a discretisation procedure and introduce a finite lattice $\mathcal{L}$ with volume V in the position space $\mathbb{R}_3$. The number of lattice points is denoted by $\bar{\gamma}$, the volume of lattice cells is ΔV . Of course,

$$\bar{\gamma} = \frac{V}{\Delta V}$$

and, as already remarked,

$$\bar{\gamma} \rightarrow \gamma$$

in the continuum limit (see eq.(2.9)). Operators $B(\vec{x}, \vec{x}')$ operating in function spaces defining our field coordinate manifolds are denoted with "matrix-indices" $(\vec{x}, \vec{x}')$ and are represented in the discretisation proce-

duced by finite matrices

$$B(\vec{x}, \vec{x}') \rightarrow B_{ij}$$

We then have

$$\text{tr } B = \int d^3x B(\vec{x}, \vec{x}) \rightarrow \sum_i B_{ii} \Delta V$$

The discretized integrals

$$\int D\varphi F[\varphi] \rightarrow \int \prod_{i=1}^{\bar{r}} d\varphi_i F(\varphi_i) \quad (3.11)$$

are computed using polar coordinates

$$\varphi_i := \varphi(\vec{x}_i), \quad \vec{x}_i \in \mathcal{L}$$

$$\varphi_1 = r \cos \theta_1$$

$$\vdots$$

$$\varphi_k = r \cos \theta_k \prod_{j=1}^{k-1} \sin \theta_j \quad (3.12)$$

$$\vdots$$

$$\varphi_{\bar{r}} = r \prod_{j=1}^{\bar{r}-1} \sin \theta_j$$

$$0 \leq \theta_j \leq \pi, \quad 1 \leq j \leq \bar{r}-2$$

$$0 \leq \theta_{\bar{r}-1} \leq 2\pi$$

$$r^2 := \sum_{i=1}^{\bar{r}} \varphi_i^2.$$

In the calculation the normalisation of the discretized version of $\bar{\mathcal{G}}$ given in (3.9)

$$\bar{\mathcal{G}}_0[\varphi, \varphi'; \beta, 0] = \left(\frac{\Delta V}{2\pi\beta} \right)^{\bar{r}/2} e^{-\frac{1}{2\beta} \sum_{k=1}^{\bar{r}} (\varphi_k - \varphi'_k)^2 \Delta V} \quad (3.13)$$

plays an important role. From (2.10) and (3.6) we obtain after a lengthy but straightforward computation

$$\int D\varphi \alpha_1[\varphi, \varphi] \rightarrow -\frac{1}{2} \Delta V \Omega_\varphi \left(\text{tr} A^2 \frac{R^2}{\bar{\gamma}+2} + \frac{3g}{2} \frac{\bar{\gamma} R^4}{(\bar{\gamma}+2)(\bar{\gamma}+4)} \right) \quad (3.14)$$

and

$$\begin{aligned} \int D\varphi \alpha_2[\varphi, \varphi] \rightarrow & \frac{1}{8} (\Delta V)^2 \Omega_\varphi \left(\frac{R^4}{(\bar{\gamma}+2)(\bar{\gamma}+4)} \left((\text{tr} A^2)^2 + 2 \text{tr} A^4 \right) + \right. \\ & 3g \text{tr} A^2 \frac{R^6}{(\bar{\gamma}+2)(\bar{\gamma}+6)} + \frac{3g^2 R^8 \bar{\gamma} (3\bar{\gamma}+32)}{4(\bar{\gamma}+2)(\bar{\gamma}+4)(\bar{\gamma}+6)(\bar{\gamma}+8)} \\ & \left. - \frac{2}{3} \frac{\text{tr} A^4}{(\Delta V)^2} - 18 \frac{\bar{\gamma}}{\bar{\gamma}+2} \frac{R^2}{(\Delta V)^2} \right) ; \end{aligned} \quad (3.15)$$

in this calculation we restricted the integration over the lattice variables to a sphere with radius R

$$R^2 \geq \sum_{i=1}^{\bar{\gamma}} \varphi_i^2$$

the volume of which we denoted by Ω_φ

$$\begin{aligned} \Omega_\varphi &:= \int \prod_{i=1}^{\bar{\gamma}} d\varphi_i \\ &= \frac{2 \pi^{\bar{\gamma}/2}}{\Gamma(\bar{\gamma}/2)} \frac{R^{\bar{\gamma}}}{\bar{\gamma}} =: \Omega^{(\bar{\gamma})} \frac{R^{\bar{\gamma}}}{\bar{\gamma}} . \end{aligned} \quad (3.16)$$

Performing the Laplace transformation (3.8) we obtain the following (asymptotic) expansion for the density of states

$$\begin{aligned}
 S(E) dE &= \Omega_{\varphi} \left(\frac{\Delta V}{2\pi^2} \right)^{\bar{\gamma}/2} \frac{1}{2} \Omega^{(\bar{\gamma})} E^{\bar{\gamma}/2-1} dE \\
 &\left(1 - \frac{R^2 \Delta V}{4E} \left(\text{tr} A^2 + \frac{3g}{2} R^2 \right) + \right. \\
 &\frac{R^2 (\Delta V)^2}{32 E^2} \left((\text{tr} A^4)^2 + 2 \text{tr} A^4 + 3g R^2 \text{tr} A^2 + \frac{9(gR^2)^2}{4} \right) \\
 &\left. - \frac{\bar{\gamma}^2}{16 E^2} \left(\frac{1}{3} \text{tr} A^2 + g R^2 \right) + \dots \right). \quad (3.17)
 \end{aligned}$$

Expressing the energy E in terms of the properly discretized momenta

$$\Pi(\vec{x}) = -i \frac{\delta}{\delta \varphi(\vec{x})} \longrightarrow \left\{ \frac{1}{\Delta V} p_i \right\}$$

i.e. writing

$$E = \frac{1}{2\Delta V} \sum_i p_i^2 \quad (3.18)$$

we observe that

$$\prod_{k=1}^{\bar{\gamma}} \frac{dp_i}{2\pi} = \frac{1}{2} \left(\frac{\Delta V}{2\pi^2} \right)^{\bar{\gamma}/2} d\Omega^{(\bar{\gamma})} E^{\bar{\gamma}/2-1} dE \quad (3.19)$$

appears as a common factor trivially integrated over the $\bar{\gamma}$ -sphere for $S(E) dE$.

We then introduce a mass-scale M and a rescaled coupling constant g'

$$\begin{aligned}
 R^2 \Delta V &=: \frac{\bar{\gamma}}{M} \\
 g' &= g \frac{\bar{\gamma}}{\Delta V M^3} \quad \text{or} \quad g' M^2 = g R^2 \quad (3.20)
 \end{aligned}$$

in order to reexpress the density of states

$$\rho(E) dE \longrightarrow \tilde{\rho}(p_i) \prod_{i=1}^{\bar{\gamma}} \frac{dp_i}{2\pi}$$

with

$$\begin{aligned} \tilde{\rho}(p_i) \prod_{i=1}^{\bar{\gamma}} \frac{dp_i}{2\pi} &= \Omega_{\varphi} \prod_{i=1}^{\bar{\gamma}} \frac{dp_i}{2\pi} \left(1 - \frac{\bar{\gamma}}{4ME} \left\{ \text{tr} A^2 + \frac{3g' M^2}{2} \right\} \right. \\ &\quad + \frac{\bar{\gamma}^2}{32(ME)^2} \left\{ (\text{tr} A^2)^2 + 2 \text{tr} A^4 + 3g' M^2 \text{tr} A^2 + 9 \frac{g'^2 M^4}{4} \right\} \\ &\quad \left. - \frac{\bar{\gamma}^2}{16 E^2} \left\{ \frac{1}{3} \text{tr} A^2 + g' M^2 \right\} + \dots \right) \end{aligned} \quad (3.21)$$

The continuum limit $\Delta V \rightarrow 0$ is now performed such that M and g' remain fixed. Since

$$\frac{\bar{\gamma}}{M} = R^2 \Delta V \xrightarrow{\Delta V \rightarrow 0} \int \varphi^2(\vec{x}) d^3x \xrightarrow{M \text{ fixed}} \infty$$

i.e. $R^2 \Delta V$ approaches the L_2 norm of φ which itself goes to infinity for fixed M since the number of degrees of freedom becomes unlimited for $\Delta V \rightarrow 0$.

The density of states becomes a well-defined functional

$$\tilde{\rho}(p_i) \prod_{i=1}^{\bar{\gamma}} \frac{dp_i}{2\pi} \longrightarrow \rho(\pi) D\pi \quad (3.22)$$

of the momentum $\pi(\vec{x})$ (which is the Fourier conjugate of $\varphi(\vec{x})$) if we show that the energy per degree of freedom $E/\bar{\gamma}$ is a finite quantity in the frame of our anticipated approximation scheme. We have

$$\begin{aligned} \rho(\pi) D\pi &= \Omega_{\varphi} D\pi \left[1 - \frac{\bar{\gamma}}{4ME} \left\{ \text{tr} A^2 + \frac{3g' M^2}{2} \right\} + \right. \\ &\quad \frac{\bar{\gamma}^2}{32 M^2 E^2} \left\{ (\text{tr} A^2)^2 + 2 \text{tr} A^4 + 3g' M^2 \text{tr} A^2 + 9 \frac{g'^2 M^4}{4} \right\} - \\ &\quad \left. \frac{\bar{\gamma}^2}{16 E^2} \left\{ \frac{1}{3} \text{tr} A^2 + g' M^2 \right\} + O\left(\left(\frac{\bar{\gamma}}{E}\right)^3\right) \right] \end{aligned} \quad (3.23)$$

with (see (3.18))

$$E = \frac{1}{2} \int d^3x \pi^2(\vec{x}) ; \quad (3.24)$$

the traces of A^2, A^4 etc. are supposed to be computed in the usual manner by γ -function regularisation⁸⁾, the factor Ω_{φ} , the configuration space volume indicates that the density of states appears as an extensive quantity, as it should, in complete analogy to corresponding notions of statistical mechanics which, nota bene, have to be translated taking into account that in the Schrödinger picture of QFT the fields $\varphi(\vec{x})$ play the role of coordinates. This (infinite) factor drops out if we go, as usual, to intensive quantities. The dimension of position space enters into the regularisation of $\text{tr} A^{2n}$, $n \geq 1$. The high temperature expansion of the partition function can now be written as

$$Z(\beta) = \int D\pi \, \mathcal{S}[\pi] \, e^{-\frac{\beta}{2} \int d^3x \pi^2(\vec{x})} . \quad (3.25)$$

This completes our construction. Averages are determined by

$$\langle F \rangle = \frac{1}{Z} \int D\pi \, \mathcal{S}[\pi] \, F[\pi] \, e^{-\frac{\beta}{2} \int \pi^2} \quad (3.26)$$

in particular we find to leading order in γ

$$\langle E \rangle = \frac{1}{2} \beta^{-1} \gamma \quad (3.27)$$

the generalised equipartition law, and thus corroborate the finiteness of energy per degree of freedom anticipated in writing down equation (3.23) for the density of states.

Clearly, this result can be directly obtained from the partition function (3.10)

$$\langle E \rangle = - \frac{1}{2} \frac{\partial}{\partial \beta} Z$$

$$\begin{aligned}
 &= \frac{\sum_n \int D\varphi \alpha_n[\varphi, \varphi] \left(\frac{\gamma}{2} - n\right) \beta^{-\frac{\gamma}{2} + n - 1}}{\sum_n \int D\varphi \alpha_n[\varphi, \varphi] \beta^{-\frac{\gamma}{2} + n}} \\
 &= \frac{\gamma}{2} \beta^{-1} - \frac{\sum_n n \int D\varphi \alpha_n[\varphi, \varphi] \beta^{n-1}}{\sum_n \int D\varphi \alpha_n[\varphi, \varphi] \beta^n} . \tag{3.28}
 \end{aligned}$$

Inserting the scaled versions of (3.14) and (3.15)

$$\begin{aligned}
 \int D\varphi \alpha_1[\varphi, \varphi] &= -\frac{1}{2} \Omega_\varphi \frac{1}{M} \left(\text{tr} A^2 + \frac{3g' M^2}{2} \right) + O(\gamma^{-1}) \\
 \int D\varphi \alpha_2[\varphi, \varphi] &= \frac{1}{8} \Omega_\varphi \frac{1}{M^2} \left((\text{tr} A^2)^2 + 2 \text{tr} A^4 - 3g' M^2 \text{tr} A^2 + \frac{3g' M^4}{4} \right) \\
 &\quad - \frac{1}{4} \Omega_\varphi \left(\frac{1}{3} \text{tr} A^2 + g' M^2 \right) + O(\gamma^{-1}) \tag{3.29}
 \end{aligned}$$

we see that the coordinate-space volume drops out as it obviously should. We included terms of order γ^{-1} which are straightforwardly read off equations (3.14) and (3.15) to indicate that the large-temperature expansions of $\langle E \rangle$ is order in powers of γ , the number of degrees of freedom (it is important to note that when we treat the infinite quantity γ defined in (2.9) as an ordinary number we tacitly assume a discretisation procedure as the one described above in which the degrees of freedom can be unambiguously counted). We have

$$\begin{aligned}
 \langle E \rangle &= \frac{\gamma}{2} \beta^{-1} + \left\{ \frac{1}{2M} \left(\text{tr} A^2 + \frac{3g' M^2}{2} \right) + \right. \\
 &\quad \left. \frac{\beta}{2} \left(-\frac{\text{tr} A^4}{M^2} + \frac{1}{3} \text{tr} A^2 + g' M^2 \right) + O(\beta^2) \right\} \\
 &\quad + O(\gamma^{-1}) . \tag{3.30}
 \end{aligned}$$

This result sheds new light on the triviality of a pure φ^4 -theory⁹⁾. We see that to leading order in the number γ of degrees of freedom which in continuum QFT is infinite of the type (2.9), the φ^4 -theory describes for large temperatures a non-interacting ideal gas irrespective of the strength of the self-coupling g' and eventual external fields contained in A^2 . It is conceivable that this situation persists if such external fields are promoted to dynamical degrees of freedom. A recent study by Callaway and Petronzio¹⁰⁾ emphasises this point looking at it from a different theoretical perspective.

4. Conclusions

We have described a perturbative construction of the Green's functional for the Schrödinger equation formulated in the Schrödinger picture of QFT.

Our main goal was to construct a high-temperature expansion (i.e. an expansion in terms of small imaginary times) of the Green's functional continued to purely imaginary times. The technique used is completely analogous to the one used in deriving an asymptotic expansion of the heat-kernel in the finite dimensional case: factorising the most singular part we obtain functional differential equations for the coefficients of a power-series expansion for small imaginary times assumed to hold as an asymptotic series. Performing a functional trace an asymptotic series for the partition function and the density of states obtains.

In computing this trace functional integrations of polynomials have to be treated: we show that these integrals can be given a well-defined sense if infinities corresponding to quantities which admit a well-understood physical interpretation are identified. Typically the volume of the configuration space and the number of degrees of freedom appear in this context, they are identified by first approximating the theory on a finite lattice and then performing the continuum limit after appropriately re-scaling the parameters of the theory. The (infinite) configuration space volume (formally a "volume" of continuous sets of functions - fields - on position space) drops out if we go to intensive observables in the very same manner as it does in statistical mechanics.

Calculating the energy of our system we arrive at an interesting facet of theorems¹⁰⁾ on the triviality of a φ^4 -theory in space-time dimensions $d \geq 4$: to leading order in the number of degrees of freedom we obtain

at larger temperatures the equipartition law of an ideal gas irrespective of the interaction and of background fields.

The density of states, finally, obtained from the high-temperature expansion by Laplace-transformation appears after rescaling as a perfectly well-defined functional of the momentum density, the Fourier conjugate of the field-variables of the theory. For fermions the resulting expansion corresponds to a Thomas-Fermi approximation generalised to a system with a continuum of degrees of freedom.

References:

- 1) K. Symanzik, Nucl. Phys. B190 (1981) 1.
- 2) K. Dietz, "On the theory of bound states in relativistic quantum field theory", to be published.
- 3) R.P. Feynman and A.R. Hibbs, Quantum Mechanics and Path Integrals, McGraw-Hill, New York 1965;
L.S. Schulman, Techniques and Applications of Path Integration, John Wiley and Sons, New York 1981.
- 4) R. Jackiw and C. Rebbi, Phys. Rev. D13 (1976) 3398;
For a review see A.J. Niemi and G.W. Semenoff, Phys. Rep. C., to be published.
- 5) C. Callias, Comm. Math. Phys. 62 (1978) 213;
J. Lott, Comm. Math. Phys. 93 (1984) 533;
R. Bott and R. Seeley, Comm. Math. Phys. 62 (1978) 235;
M.B. Paranjape and G.W. Semenoff, Phys. Lett. 132 (1983) 369.
- 6) R. Jackiw, in "Quantum Structures of Space and Time", M.J. Duff and C. Isham editors, Cambridge University Press, 1982;
J. Goldstone and F. Wilczek, Phys. Rev. Lett. 47 (1981) 986;
J. Lott, Phys. Lett. B145 (1984) 179.
- 7) R. Seeley, Am. Journ. Math. 91 (1969) 889, 963;
P.B. Gilkey, Index Theorem and Heat Equation, Publish or Perish, Boston 1974;
B. Durhuus, P. Olesen, and J.L. Petersen, Nucl. Phys. B198 (1982) 157;
M. Lüscher, Ann. of Phys. 142 (1982) 359.
- 8) S.W. Hawking, Comm. Math. Phys. 55 (1977) 133;
J.S. Dowker and B. Critchley, Phys. Rev. D13 (1976) 3224;
K. Dietz and Th. Filk, Phys. Rev. D27 (1983) 2944.

- 9) J. Fröhlich, Nucl. Phys. B200 (1982) 281;
C. Aragao de Carvalho, S. Carraciolo and J. Fröhlich, Nucl. Phys. B215
(1983) 209;
M. Aiseman, Comm. Math. Phys. 85 (1982) 1;
G.A. Baker and J.M. Kincaid, Phys. Rev. Lett. 42 (1979) 1431.
- 10) D.J.E. Callaway and R. Petronzio, preprint CERN-TH.4270/85.