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THE ROLE OF FLUCTUATIONS IN THE MECHANISM OF DECONFINEMENT

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ABSTRACT

The fixed-point structure of the effective theory of Polyakov loops in $SU(N)$ lattice gauge theories is presented in the Migdal-Kadanoff approximation. Emphasis is put on the possibility of a description in terms of $Z(N)$ spin degrees of freedom. As a by-product of the analysis, we argue that the excess free energy of an adjoint source can show a characteristic deconfinement-like crossover behaviour at a lower temperature than that of the deconfinement phase transition itself.

1. - INTRODUCTION

Deconfinement in gauge theories at finite temperature has been carefully studied from several points of view in the last few years. The second-order nature of the phase transition in the SU(2) gauge theory¹⁾ as well as the first-order transition in the SU(3) gauge theory²⁾ are now clearly established in four dimensions. Recent Monte Carlo investigations are now mostly concerned with the difficult and controversial issue of the effect of dynamical fermions on the phase transitions³⁾.

To further the understanding of the transitions in the gauge sector alone, Monte Carlo studies have also moved into the area of SU(N > 4). Contrary to some naive theoretical expectations, these studies have shown rather conclusively that the deconfinement phase transition is of first order in SU(4)⁴⁾, and in fact appears to be discontinuous for all gauge groups SU(N > 4) in four dimensions⁵⁾.

Analytic understanding of these phenomena dates back to the original papers of Polyakov and Susskind⁶⁾. There it was shown that the deconfinement phase transition is associated with the breaking of the global centre symmetry of the gauge group, hence for SU(N) the breaking of global Z(N) symmetry. This was pursued further in the papers of Svetitsky and Yaffe⁷⁾, where it was argued that the deconfinement transitions in (D+1)-dimensional SU(N) gauge theories could be understood in terms of spontaneous symmetry breakdown in D-dimensional Z(N) spin systems with only short-range spin-spin interactions. Since in three dimensions no such system with a Z(3) symmetry is known to have a second-order phase transition, this led to the prediction of a first-order deconfinement phase transition in QCD without quarks. This prediction was, in fact, soon beautifully confirmed by Monte Carlo simulations. For continuous phase transitions, the same Z(N) spin system analogy could give predictions for the critical exponents on the basis of universality. Several of these predictions have subsequently been confirmed by direct Monte Carlo studies⁸⁾.

The closest one has come to an explicit realization of this correspondence between (D+1)-dimensional SU(N) gauge theories at finite temperature and D-dimensional Z(N) spin systems is an effective action for the Polyakov line $W(x)$, the order parameter of the theory. This effective action is of the form^{9),10)}

$$S = \frac{1}{2} \int \sum_{x,j} \{ W(x) W^\dagger(x+j) + W^\dagger(x) W(x+j) \} \quad (1.1)$$

with $J = J(g^2, T)^*$ and $W(x) = \text{Tr} \prod_{i=1}^{\beta} U_0(\vec{x}, \tau+i)$. This action, originally derived in the strong coupling region, also follows from a Migdal-Kadanoff-inspired bond-moving scheme¹¹⁾. The effective action (1.1), whether treated by mean field techniques or by Monte Carlo methods, gives an impressively accurate description of the deconfinement phase transitions - usually to an accuracy of a few per cent^{10),12)}.

It is, however, important to keep in mind that the effective action (1.1), although obviously invariant under global $Z(N)$ transformations $W(x) \rightarrow zW(x)$, $z \in Z(N)$, is not of the form of a $Z(N)$ -valued spin theory. Rather, it is a $Z(N)$ -invariant matrix field theory integrated over the full $SU(N)$ Haar measure for $W(x)$. In fact, if at strong coupling one integrates out all degrees of freedom from this effective action except $Z(N)$ spin degrees of freedom, one obtains the N -state clock model⁹⁾, which gives wrong predictions for the order of the phase transitions for the gauge groups $SU(N > 4)$ in four dimensions. This indicates that $Z(N)$ singlet degrees of freedom may play a role in triggering the phase transition associated with the $Z(N)$ breaking.

In this paper, we address the question to what extent the phase transition associated with deconfinement in (1.1) can be described in terms of just $Z(N)$ -valued spin degrees of freedom. Our framework will be that of approximate real-space renormalization group transformations of the Migdal-Kadanoff form^{13),14)}. For lower-dimensional spin systems, this scheme is known to provide the correct fixed-point structure, and it gives, with some improvements, quite accurate critical indices¹⁵⁾. The renormalization group transformations allow us gradually to probe longer and longer wavelengths of the theory, and thereby identify the relevant degrees of freedom close to the fixed-point Hamiltonian. Our emphasis will be on the role played by $Z(N)$ invariant degrees of freedom.

Another question which we wish to address simultaneously is the importance of fluctuations. Although a lowest-order mean field study, which fixes the theory at the saddle point and hence ignores fluctuations, gives a very good description of the behaviour of the Polyakov line $W(x)$ itself, it simultaneously induces similar behaviour even into $Z(N)$ -invariant quantities, such as for instance $W(x)W^\dagger(x)$. Interestingly, we find that in the Migdal-Kadanoff approximation the

*) At strong coupling $J = 2(1/Ng^2)^{n_\tau}$ and at weak coupling $J = 2/Ng^2 n_\tau$ for $N > 2$. Here n_τ is the periodicity in the time direction, and $T = 1/a_\tau n_\tau$ (a_τ = lattice spacing in the τ -direction) is the temperature.

quantity $\langle W(x)W^\dagger(x) \rangle$ starts to grow in a direct neighbourhood $J_1 < J < J_c$ of the phase transition point J_c , where $W(x)$ itself gets an expectation value. If this reflects the true nature of the transition, it means that sources in the adjoint representation of the gauge group should show deconfinement-like crossover behaviour at a scale slightly different from that of sources in the fundamental representation. This phenomenon, which, in the Migdal-Kadanoff approximation, is strongest for the gauge group $SU(2)$, will be discussed in Section 3.

In Section 2, we give a short description of the real-space renormalization group transformations within the Migdal-Kadanoff scheme, and set up the appropriate formalism. Using a double-character expansion for $SU(2)$, we show analytically the existence of an unstable fixed point of the renormalization group flow in dimensions $D > 1$, when projected down on the J -axis. This demonstrates the existence of a phase transition at some intermediate value of J , the quantitative study of which is the subject of the following sections.

The Migdal-Kadanoff recursion relations are exact in one dimension, where the system (1.1) corresponds to the $(1+1)$ -dimensional gauge theory at finite temperature. We start Section 3 with this exact numerical solution for the $SU(2)$ gauge group in order to explore in detail the disordered phase of the effective spin model (besides demonstrating explicitly the absence of phase transitions in this case). In any higher number of dimensions, a clear phase transition is seen. We localize this phase transition by two methods. The first method is a fixed-point effective potential scheme. Here we consider the most general nearest neighbour Hamiltonian, and display explicitly the evolution of the fixed-point effective potential as well as the fixed-point correlation between asymptotically larger and larger domains. The second method is based on a more conventional technique in which the phase transition is located by a peak in the specific heat. These two methods yield consistent results.

In Section 4 we use the same framework to study $SU(2)$, $SU(3)$ and $SU(4)$ in a discrete subgroup approximation. The qualitative features as a function of the gauge group $SU(N)$ and the number of dimensions D are outlined.

Section 5 contains our conclusions and some suggestions for further study along these lines.

2. - THE MIGDAL-KADANOFF FORMALISM

In this section we shall briefly review the Migdal-Kadanoff scheme^{13),14)} for an action of the type (1.1). The resulting recursion relations provide us with a set of approximate real space renormalization group transformations which permits the study of the relevant degrees of freedom at larger and larger wavelengths. The recursion relations map one theory of lattice spacing a into an effectively "block-spinned" action of lattice spacing λa . In the Migdal-Kadanoff formalism, this mapping is accomplished by a combination of decimation, in which a set of intermediate $W(x)$'s are integrated out, and bond moving, in which the couplings J of the theory are redistributed on the lattice.

To begin, we first derive the recursion relations in terms of a double character expansion for the action (1.1). Later, a general framework without reference to this expansion will be discussed. We start with the nearest neighbour Boltzmann factor

$$F[W_1, W_2] = e^{\frac{1}{2} \text{Tr} W_1 \text{Tr} W_2^\dagger + \text{h.c.}} \quad (2.1)$$

of the action (1.1), and expand it on a (double) character basis:

$$F[W_1, W_2] = \sum_{i,j} F_{ij} d_i d_j \chi_i(W_1) \chi_j^*(W_2) \quad (2.2)$$

Here a character $\chi_i(W)$ represents the Polyakov line $W(x)$ in the i 'th irreducible representation of dimension $d_i = \chi_i(1)$.

By the orthonormality condition

$$\int dW \chi_i(W) \chi_j^*(W) = \delta_{ij} \quad (2.3)$$

it follows that

$$F_{ij} = \frac{1}{d_i d_j} \int dW_1 dW_2 F[W_1, W_2] \chi_i^*(W_1) \chi_j(W_2) \quad (2.4)$$

We now begin the Migdal-Kadanoff transformation of the scale factor λ by decimation: an integration over $(\lambda-1)$ $W(x)$'s in one direction (see Fig. 1). This gives an intermediate $F[W_1, W_2, \lambda a]$:

$$F_{\text{int}}[W, W', \lambda a] = \int dW_1 \cdots dW_{\lambda-1} F[W, W_1, a] F[W_1, W_2, a] \cdots F[W_{\lambda-1}, W', a] \quad (2.5)$$

Using the double character expansion for each $F[W_i, W_j, a]$, this can be written

$$F_{\text{int}}[W, W', \lambda a] = \sum_{i_{11}} \sum_{i_{12}} \cdots \sum_{i_{\lambda-1,j}} F_{i_{11}} F_{i_{12}} \cdots F_{i_{\lambda-1,j}} d_i d_{i_1} d_{i_1} \cdots d_{i_{\lambda-1}} d_j \times \\ \int dW_1 \cdots dW_{\lambda-1} \chi_i(W) \chi_{i_1}^*(W_1) \chi_{i_1}(W_1) \cdots \chi_j^*(W') \quad (2.6)$$

Defining

$$\tilde{F}_{ij} \equiv F_{ij} d_i d_j \quad (2.7)$$

and making use of the orthonormality condition (2.3), we get

$$F_{\text{int}}[W, W', \lambda a] = \sum_{i_{11}} \sum_{i_{12}} \cdots \sum_{i_{\lambda-1,j}} \tilde{F}_{i_{11}}(a) \tilde{F}_{i_{12}}(a) \cdots \tilde{F}_{i_{\lambda-1,j}}(a) \chi_i(W) \chi_j^*(W') \quad (2.8)$$

or simply

$$F_{\text{int}}[W, W', \lambda a] = \sum_{i,j} (\tilde{F}^\lambda(a))_{ij} \chi_i(W) \chi_j^*(W') \quad (2.9)$$

The second step in the Migdal transformation^{*)} consists of bond moving in the remaining (D-1) directions (see Fig. 2). This gives the new nearest-neighbour Boltzmann factor F with

$$F[W, W', \lambda a] = \left[\sum_{i,j} (\tilde{F}^\lambda(a))_{ij} \chi_i(W) \chi_j^*(W') \right]^{\lambda^{D-1}} \quad (2.10)$$

The full renormalization group transformation consists of iterating the procedure outlined above. After N steps, one reaches a system of lattice spacing $\lambda^N a$.

*) In Kadanoff's scheme¹⁴⁾, the steps of decimation and bond moving are interchanged. As $\lambda \rightarrow 1$, the two methods become identical.

Let us briefly summarize the use of the recursion relations in locating (second-order) phase transition points (first-order transitions can also be identified in this scheme, but this requires a little more care). In general, the renormalization group transformations will induce a flow in the infinite dimensional space of coupling constants (next-nearest-neighbour interactions, etc., etc.). For a system with just two stable phases, the renormalization group flow will be attracted by two "sinks", one in the small-J region, the other in the large-J region. These two fixed points, which correspond to the disordered and frozen phases respectively, are both stable. The two phases will be separated by a hypersurface, the critical surface (see Fig. 3). In general, one more fixed point lies on this surface. This fixed point, although attractive on the surface itself, is unstable. The correlation length on the critical surface is infinite, and where this surface crosses the J-axis we will see a second-order phase transition.

It follows from this discussion that in order to identify a phase transition along the J-axis, it is sufficient to show that starting with sufficiently large J in the action (1.1), the coupling renormalization^{*)} flows towards $J = \infty$, and that when starting at small J it flows towards $J = 0$.

In the case of the gauge group $SU(2)$, all this can be made more precise, since in that case all integrals involved can be performed analytically. The characters and the Haar measure can here be characterized by a single parameter θ :

$$\chi_j(W) = \frac{\sin\{\frac{1}{2}(2j+1)\theta\}}{\sin\{\frac{1}{2}\theta\}} \quad (2.11a)$$

$$dW = \frac{d\theta}{2\pi} \sin^2(\frac{1}{2}\theta), \quad 0 \leq \theta \leq 4\pi \quad (2.11b)$$

The dimension of the j 'th representation is now simply $d_j = 2j + 1$, where j is the isospin (i.e., $j = 0, \frac{1}{2}, 1, 3/2, \dots$). Furthermore, in terms of angles θ and θ' , our nearest-neighbour Boltzmann factor takes the form

*) This is defined by the projection of the full renormalization group flow down on the J axis.

$$F(\theta, \theta') = e^{4\beta \cos(\theta/2) \cos(\theta'/2)} \quad (2.12)$$

Note that this Boltzmann factor can be written in a factorized form,

$$F(\theta, \theta') = e^{2\beta \cos[\frac{1}{2}(\theta - \theta')]} e^{2\beta \cos[\frac{1}{2}(\theta + \theta')]} \quad (2.13)$$

Then, using

$$e^{A \cos \varphi} = \sum_{n=-\infty}^{\infty} I_n(A) e^{in\varphi} \quad (2.14)$$

where $I_n(x)$ is the modified Bessel function of order n , we get for the coefficients F_{ij} in the double character expansion (2.2):

$$F_{ij} = \sum_{m,n} I_n(2J) I_m(2J) \frac{4}{d_i d_j} \int_0^{2\pi} \int_0^{2\pi} \frac{d\theta d\theta'}{(2\pi)^2} \sin[\theta] \sin[\theta'] \times \\ \sin[d_i \theta] \sin[d_j \theta'] e^{im(\theta - \theta')} e^{in(\theta + \theta')} \quad (2.15)$$

The θ -integrals in (2.15) are trivial, and we are left with

$$F_{ij} = \frac{1}{(2i+1)(2j+1)} \left\{ I_{j-i}(2J) [I_{i+j+2}(2J) + I_{i+j}(2J)] \right. \\ \left. - I_{i+j+1}(2J) [I_{i-j+1}(2J) + I_{j-i+1}(2J)] \right\} \quad (2.16)$$

with the restriction that either both i and j are half-integers or both i and j are integers. In all other cases $F_{ij} = 0$.

Let us now go to the very weak coupling region ($J \ll 1$). For the first few coefficients F_{ij} , we have

$$F_{0,0} = I_0(2J) [I_0(2J) + I_2(2J)] - 2 I_1(2J)^2 \\ F_{1/2, 1/2} = \frac{1}{4} [I_0(2J) [I_1(2J) + I_3(2J)] - 2 I_1(2J) I_2(2J)]$$

Using power series expansions of the Bessel functions, this yields

$$\tilde{F}_{0,0} = F_{0,0} = 1 + \frac{1}{2}J^2 + O(J^3) \quad (2.17a)$$

$$\tilde{F}_{1/2,1/2} = J + O(J^3) \quad (2.17b)$$

and we now have a truncated $\tilde{F}_{ij}$ matrix which is diagonal. We then get immediately^{*)}

$$\begin{aligned} F(\theta, \theta', \lambda a) &= [\tilde{F}_{0,0}^\lambda + \tilde{F}_{1/2,1/2}^\lambda \chi_{1/2}(\theta) \chi_{1/2}(\theta')]^{\lambda^{D-1}} \\ &= (1 + \frac{1}{2}J^2)^\lambda \left[1 + \frac{J^\lambda}{(1 + \frac{1}{2}J^2)^\lambda} \chi_{1/2}(\theta) \chi_{1/2}(\theta') \right]^{\lambda^{D-1}} \\ &= \text{const.} \cdot \exp \{ J^\lambda \lambda^{D-1} \chi_{1/2}(\theta) \chi_{1/2}(\theta') \} \end{aligned} \quad (2.18)$$

The system has therefore to this order evolved into a system of a form identical to the original system, but at a different (renormalized) value of the coupling constant:

$$J' = \lambda^{D-1} J^\lambda < J \quad \text{for } J \ll 1 \quad (2.19)$$

The renormalization flow is hence attracted towards $J = 0$ in this region ($J \ll 1$).

How is the flow in the opposite region ($J \gg 1$)? From the action (1.1), it follows that as $J \rightarrow \infty$, the $SU(2)$ -valued Polyakov lines $W(x)$ will freeze into $W(x) = \pm 1$, i.e., into $Z(2)$. At $J = \infty$, the theory is therefore Ising-like. For dimensions $D > 1$, such Ising-like systems have a large- J flow towards $J = \infty$. The coupling renormalization flow in the $J \gg 1$ region is therefore opposite that of the $J \ll 1$ region in dimensions $D > 1+\epsilon$. This demonstrates the existence of at least one unstable fixed point along the J -axis in those dimensions, and hence the existence of a second-order phase transition at some intermediate J_c . Some

^{*)} In one includes higher values of the isospin, one must first diagonalize the matrix $\tilde{F}_{ij}$. (This is always possible since the matrix is real and symmetric.)

details of this analysis will be published in a separate note¹⁶⁾.

We can analyze the $J \gg 1$ region in a little more detail by making use of the asymptotic expansions for Bessel functions. This is quite similar to the earlier analysis by Nauenberg and Toussaint¹⁷⁾ for SU(2) gauge systems^{*}). We will restrict ourselves to $D = 1$, and then show that in this case the coupling J becomes marginal as $J \rightarrow \infty$.

Choosing a $\lambda = 2$ scale transformation, we have for the renormalized Boltzmann factor

$$F_{ij}^R = \sum_k d_k^2 F_{ik}(J) F_{kj}(J) \quad (2.20)$$

up to an irrelevant normalization factor which we can choose such that we have a J' with

$$F_{ij}^R(J) = F_{ij}(J'), \quad J \gg 1 \quad (2.21)$$

Inserting the asymptotic expression

$$I_n(2J) = \frac{e^{2J} e^{-n^2/4J}}{\sqrt{4\pi J}}, \quad J \gg 1 \quad (2.22)$$

for the Bessel functions, we find that for $J \gg 1$, the coefficients F_{ij} factorize in the form

$$F_{ij} = \frac{1}{4\pi J d_i d_j} e^{4J} a_i a_j \quad (2.23)$$

with

$$a_k = e^{-\frac{(k+1)^2}{2J}} - e^{-\frac{k^2}{2J}} \quad (2.24)$$

This means that

^{*}) See also Ref. 18).

$$F_{ij}^R(J) = \frac{1}{(4\pi J)^2 d_i d_j} e^{8J} a_i a_j \sum_k a_k^2 \quad (2.25)$$

i.e.,

$$F_{ij}^R(J) = F_{ij}(J') \quad (2.26)$$

with

$$J = J'$$

and the normalization factor N_F given by

$$N_F = \frac{1}{4\pi J} e^{4J} \sum_k a_k^2 \quad (2.27)$$

This indicates that the coupling J becomes marginal in the region $J \gg 1$ in the $D = 1$ dimension. However, a little subtlety is involved in reaching this conclusion, because the k -summation runs either over integers or half-integers, depending on whether (i,j) are integers or half-integers. One can show¹⁶⁾ that the two normalization factors are equal to leading order in J , so the conclusion is correct.

Proceeding to $D = 1+\epsilon$ dimensions, the large- J representation of F_{ij} enables one to find the actual value of the critical coupling $J_c = (8\pi \epsilon \ln 2)^{-1}$ and the corresponding "thermal" exponent $\nu_J = \epsilon^{-1}$ ¹⁶⁾. This result, when compared with the Ising results¹³⁾, explicitly confirms that the $SU(2)$ effective action belongs to the Ising universality class around $D = 1$.

In later sections, we shall use the $\lambda = 2$ Migdal-Kadanoff procedure without reference to the character expansion, phrasing it simply in terms of the block-spin Hamiltonians themselves. This will be described next.

After n iterations of our starting action (1.1), the most general effective action can be written in the form

$$Z^{(n)} = \int d[W] e^{\sum_{ij} \left\{ \frac{1}{2D} [v_n(W_i) + v_n(W_j)] + u_n(W_i, W_j) \right\}} \quad (2.28)$$

where the sum on i and j runs over nearest neighbours. Here $v_n(W)$ is defined by the local part of the action density, and $u_n(W_i, W_j)$ represents the correlation between nearest neighbours. Then, by definition, $u_n(0, W) = u_n(W, 0) = 0$.

The bond-moving step now simply results in multiplying the u_n -term by λ^{D-1} . For the local part v_n , we have the well-known ambiguity that the integration over variables to be decimated can be performed without even shifting the potential at all. As a rule, we consider here recursions where the v_n -terms belonging to intermediate $W(x)$'s are distributed symmetrically between the neighbouring block spins^{*)}.

With these remarks, the complete Migdal-Kadanoff recursion formula takes the following form:

$$K_n \exp\{\Delta v_n(W_i) + \Delta v_n(W_j) + u_n(W_i, W_j)\} \\ = \int dW_k \exp\{a_D v_{n-1}(W_k) + \lambda^{D-1} [u_{n-1}(W_i, W_k) + u_{n-1}(W_k, W_j)]\} \quad (2.29)$$

with

$$v_n(W_i) = v_{n-1}(W_i) + 2D \Delta v_n(W_i) \quad (2.30)$$

The normalization factor K_n is arbitrary. In the disordered phase it is convenient to choose^{**)}

$$v_n(W=0) = 0 \quad (2.31)$$

which uniquely fixes K_n at each iteration.

The coefficients a_D in Eq. (2.29) arise from the distribution of the local

*) We have checked in two dimensions that the results will not change significantly if the local v_n -term is left untouched in the bond-moving step.

**) In the ordered phase, a different normalization must be chosen.

action density part in the bond-shifting step (in Fig. 4 associated with the site in the centre). The values for $D = 1, 2, 3$ are

$$a_1 = 1, \quad a_2 = 3/2, \quad a_3 = 7/3 \quad (2.32)$$

This completes our discussion of the general formalism. For each value of J , one starts the iteration at $v_0(W) = 0$ and $u_0(W_i, W_j) = \frac{1}{2}J \{ \text{Tr} W_i \text{Tr} W_j^\dagger + \text{Tr} W_i^\dagger \text{Tr} W_j \}$ in accordance with (1.1). The iterations then proceed until the fixed-point Hamiltonian has been reached within the required accuracy.

3. - THE SU(2) GAUGE GROUP

The main objective of this section is to investigate to what extent a representation of the action (1.1) for SU(2) in terms of Z(2) spin variables is justifiable. For SU(2) the effective theory can be rewritten^{*})

$$Z = \int_x \prod_{x,e} [d\theta(x) \sin^2 \frac{\theta(x)}{2}] \exp \left\{ J \sum_{x,e} \cos \left(\frac{\theta(x)}{2} \right) \cos \left(\frac{\theta(x+e)}{2} \right) \right\} \quad (3.1)$$

Exploiting the evenness of the integrand and introducing the scalar $W(x) \equiv \cos(\theta(x)/2)$ as a new dynamical variable, we can write

$$Z = \int_x \prod_{x,e} [dW(x) \sqrt{1-W(x)^2}] \exp \left\{ J \sum_{x,e} W(x) W(x+e) \right\} \quad (3.2)$$

up to an irrelevant normalization factor.

The effective potential

$$V_{\text{classical}}(W) = -J D W^2 - \frac{1}{2} \ln[1-W^2] \quad (3.3)$$

clearly favours the neighbourhood of the Z(2) singlet point $W(x) = 0$ for sufficiently small J . From this simple argument one would not expect a Z(2) spin large

^{*}) Note that for convenience we have rescaled the coupling J from Eq. (1.1) by a factor of four.

distance description to be justified close to $J=0$, since here the $W(x)$'s will cluster around zero. The question is whether this feature will persist in the fixed-point Hamiltonian all the way up to higher values of J .

In order to perform the renormalization group transformations (2.37) on the system (3.2), it is convenient to write [see, e.g., Eq. (25.5.40) of Ref. 19)]

$$\int_{-1}^1 dx \sqrt{1-x^2} f(x) = \sum_{i=1}^n w_i f(x_i) + R_n \quad (3.4)$$

with

$$w_i = \frac{\pi}{n+1} \sin^2\left(\frac{i\pi}{n+1}\right) \quad (3.5a)$$

$$x_i = \cos\left(\frac{i\pi}{n+1}\right) \quad (3.5b)$$

for $i = 1, 2, \dots, n$. This amounts to representing the class functions in $SU(2)$ by n selected values which can be made denser at will. Since we might expect the role of the invariant class to be important, we choose n to be odd; this ensures that $W(x) = 0$ is always included.

As described in the previous section, we now split the most general nearest-neighbour interaction $V(\phi, \psi)$ into a local piece $v(\phi)$ associated with sites, and a correlating piece $u(\phi, \psi)$ associated with bonds between nearest neighbours:

$$V(i, j) = \frac{1}{2D} [v(i) + v(j)] + u(i, j) \quad (3.6)$$

where

$$u(i, j) \equiv u(\phi(i), \psi(i)) = \sum_{p, q=1}^{\infty} a_{pq} \phi(i)^p \psi(j)^q \quad (3.7)$$

Then

$$u(i, \frac{n+1}{2}) = u(\frac{n+1}{2}, j) = 0 \quad (3.8)$$

automatically, corresponding to $W_i = 0$ or $W_j = 0$.

Similarly, the normalization condition (2.39) now reads

$$v_\ell(\frac{n+1}{2}) = 0 \quad (3.9)$$

and at the start we have

$$v_0(i) = 0, \quad u_0(i, j) = \mathcal{F} \cos\left\{\frac{i}{n+1}\pi\right\} \cos\left\{\frac{j}{n+1}\pi\right\} \quad (3.10)$$

The recursion relation (2.37) now takes the form

$$\begin{aligned} & K_\ell \exp\{\Delta v_\ell(i) + \Delta v_\ell(j) + u_\ell(i, j)\} \\ &= \sum_{k=1}^n \omega_k \exp\{a_D v_{\ell-1}(k) + \lambda^{D-1} [u_{\ell-1}(i, k) + u_{\ell-1}(k, j)]\} \end{aligned} \quad (3.11)$$

where K_ℓ is determined by Eq.(3.9). Keeping this normalization, $\Delta v_\ell((n+1)/2) = 0$, implies

$$K_\ell = \sum_{k=1}^n \omega_k \exp\{a_D v_{\ell-1}(k)\} \quad (3.12)$$

At the ℓ 'th step this factor contributes an amount

$$f_\ell = (\ln K_\ell) / 2^{2D} \quad (3.13)$$

to the free energy density $\mathcal{F} = \sum_\ell f_\ell$.

The ℓ 'th step of the iteration starts by computing K_ℓ from (3.12). Then one finds [on account of (3.8) and (3.9)]:

$$\exp\{\Delta v_\ell(i)\} = K_\ell^{-1} \sum_{k=1}^n \omega_k \exp\{a_D v_{\ell-1}(k) + \lambda^{D-1} u_{\ell-1}(i,k)\} \quad (3.14)$$

and finally one determines $u_\ell(i,j)$ from the general matrix element (3.11).

Typically, 15-20 iterations are needed to obtain the fixed-point Hamiltonian to ~ 6 digit accuracy in this restricted (nearest-neighbour) coupling space. This corresponds to going to a lattice of spacing 10^4 - 10^6 in original lattice units.

At the fixed point, we can compute the effective fixed-point potential

$$u^*(i) \equiv -V^*(i,i) = -v^*(i) - D u^*(i,i) - \ln \omega_i \quad (3.15)$$

while at the same time monitoring the domain-domain coupling $u^*(i,j)$ for nearest-neighbour blocks.

It is instructive to consider this procedure first in the case $D = 1$. Of course, in this case we can solve for the partition function exactly. Using, for instance, the character expansion of Section 2, we obtain

$$\begin{aligned} Z &= \int d[W] \exp\left\{J \sum_x \text{Tr}[W(x)] \text{Tr}[W(x+1)]\right\} \\ &= \int d[W] \prod_x \sum_{i,j} F_{ij}(J) \chi_i(W(x)) \chi_j(W(x+1)) d_i d_j \\ &= \text{Tr}\left\{[F^{(1)}(J)]^N\right\} \end{aligned} \quad (3.16)$$

with F_{ij} defined in Eq. (2.16) and $F_{ij}^{(1)}(J) \equiv F_{ij} d_i^2$ (no sum on i). N indicates the number of points x . This is a completely analytic function of J , and of course does not have a phase transition. Since in $D = 1$ the Migdal-Kadanoff scheme becomes exact, we should be able to reproduce this result by studying directly the effective fixed-point potential $u^*(i)$ and the coupling $u^*(i,j)$.

For this one-dimensional case, Fig. 5 shows the evolution of the fixed potential $u^*(\cos(\theta/2))$ as the coupling J is varied. (The curves shown are smooth interpolations between the values computed in the discretized version of the model; here with $n = 13$.) The important feature is the following: at an intermediate coupling $J_1 \sim 2.0$, the fixed-point potential develops a double-well

structure. For $J < J_1$, the potential is centred around $J = 0$ with just one minimum.

However, the switch to a double-well structure at $J = J_1$ does not imply the breakdown of the $Z(2)$ symmetry. In fact, if one computes the correlation between neighbouring domains, $u^*(i,j)$, one finds $u^*(i,j) = 0$ identically for all the values of J , including $J > J_1$. What happens at $J = J_1$ is that here the system starts forming (approximately) $Z(2)$ -valued domains. Since these domains are uncorrelated, one will always find $\langle W(x) \rangle = 0$ when averaging over many (asymptotically large) domains. As J increases, the rate of convergence towards $u^*(i,j) = 0$ slows down, apparently leading to $u^*(i,j) \neq 0$ just at $J = \infty$. This is simply the analogue of the $T = 0$ "transition" in the one-dimensional Ising model.

We conclude from this exercise that only for $J > J_1$ will the action (3.2) be well represented by $Z(2)$ -valued "spins". For $J < J_1$, the neighbourhood of $Z(2)$ singlet degrees of freedom dominates the partition function. However, the $Z(2)$ symmetry is never spontaneously broken in one dimension, except possibly at $J = \infty$. Then the two wells are infinitely deep, so at this point one deals with an ideal Ising situation.

Let us now turn to the more interesting case of $D = 2$. The fixed-point potential now evolves in the following manner with increasing J . At a coupling $J_1 \approx 0.605$, the effective potential switches from having a single minimum centred on $W = 0$ to a double-well structure (see Fig. 6). As in the one-dimensional case, this does not signal the breakdown of the $Z(2)$ symmetry of the system, because we find $u^*(i,j) = 0$ identically for all J , $0 < J < J_c \approx 0.855$. This means that for all J in this region, $\langle W(x) \rangle = 0$ when we average over the block spins. The appearance of a double-well structure therefore only tells us that the various domains will tend to orient themselves along $Z(2)$ values, but when we average over these domains we will obtain zero magnetization. In this way, the effective fixed-point potential shows that only for $J \gtrsim J_1$ can we expect the system to be well represented by $Z(2)$ spin variables.

For $J > J_c$, the effective potential decreases without bounds with the number of iterations except at the point $W = 0$, where it is fixed by normalization. However, the coupling $u^*(i,j)$ remains finite in this region: $u^*(i,i) = \ln 2$. (This is a numerically established six-digit accurate phenomenon. We have no analytical explanation for this number.) This means that we have a ferromagnetic

coupling between large domains in this region, and the system will now synchronously choose either $W > 0$ or $W < 0$, i.e., $\langle W(x) \rangle \neq 0$. The fact that the effective fixed-point potential decreases without bounds at $J = J_c$ when insisting on a finite potential at $W = 0$ is another indication of a switch to a magnetized (ordered) phase. In this phase the system cannot tunnel through the infinitely high barrier between the two wells. (A different normalization prescription is required in this phase if one wants a finite potential for $W \neq 0$.) We therefore identify J_c as an order-disorder phase transition point. Only here is the global $Z(2)$ symmetry spontaneously broken.

In Fig. 7, we show independent evidence that J_c is a (second-order) phase transition point. We plot here the specific heat

$$C = J^2 \frac{\partial^2 F}{\partial J^2} \quad (3.17)$$

evaluated from Eq. (3.13) as a function of the coupling J . A clear peak is seen in the specific heat at $J \approx J_c^*$. As can be seen from the Figure, the specific heat shows absolutely no structure around $J = J_1$.

The scenario is completely similar in $D = 3$ dimensions. In Fig. 8, we show the fixed-point effective potential for different values of J . There is again a clear (but diminishing) separation between the point $J_1 \approx 0.193$ and $J_c \approx 0.208$, the latter giving the location of the phase transition. Interestingly, we find $u^*(1,1) = \ln 2$ for $J > J_c$ again. The critical coupling J_c is close to the value $1/6$ found in a naive mean field estimation. The deviation is partly due to the effect of fluctuations pushing J_c towards higher values; partly it reflects the approximate nature of the $\lambda = 2$ Migdal-Kadanoff transformation.

The fact that the fixed-point potential develops a two-well structure at a coupling J_1 clearly separated from the phase transition point J_c has an interesting consequence. Although $\langle W(x) \rangle = 0$ in the interval $J_1 < J < J_c$, it is clear that $Z(2)$ invariant quantities, such as $\langle W(x)^2 \rangle$, will be sensitive to the shape of the potential in this region. In particular, at $J \sim J_1$ one would expect $\langle W(x)^2 \rangle$ to show a characteristic rise. Since

*) Even in one dimension the specific heat reaches a rounded maximum at an intermediate $J \approx 4.0$, which is not correlated with J_1 at all. The occurrence of this maximum, which can be thought of as a remnant of the phase transition in two dimensions, is completely analogous to what happens in the Ising model.

$$|\text{Tr}_{\text{fund.}} W(x)|^2 = \text{Tr}_{\text{adj.}} W(x) + 1 \quad (3.18)$$

this rise in $\langle W(x)^2 \rangle$ is associated with a lowering of the free energy of a source in the adjoint representation. [Recall that $\text{Tr}_{\text{adj.}} W(x) \sim \exp\{-(E-E_{\text{vac}})/T\}$, where T is the temperature and E is the excess free energy of the adjoint source.]

Equation (3.18), in conjunction with Figs. 7 and 8, indicates that for $\text{SU}(2)$ adjoint sources may show an almost deconfinement-like lowering of the free energy at $J \gtrsim J_1$, slightly below the true deconfinement temperature. This result seems to be of interest in view of the fact that also at zero temperature an independent "adjoint string tension"^{*}) has been observed at intermediate distances²⁰⁾. The dynamical mechanism of screening may be more complicated than what is expected on an algebraic basis.

We can substantiate these remarks by noting that the $Z(2)$ -invariant quantity $\langle W(x)^2 \rangle$ can be computed directly within our Migdal-Kadanoff framework. Its evaluation consists of performing successive decimation-like integrations in the path integral. When the domain represented by $W(x)$ at the site x grows big enough ($\sim 2^{15}$ in original lattice units), its coupling to neighbouring domains vanishes in the low- J phase (all $J < J_c$). Hence in that region one simply has

$$\langle W(x)^2 \rangle = \frac{\sum_i W^2(i) \exp\{-u^*(i)\}}{\sum_i \exp\{-u^*(i)\}} \quad (3.19)$$

In Fig. 9, we plot $\ln \langle W(x)^2 \rangle$ obtained from Eq. (3.19) as a function of J in the case $D = 3$ (recall that this corresponds to the four-dimensional finite temperature gauge theory). One can clearly see the change of regime around $J = J_1$.

Let us end this section by emphasizing that we are fully aware of the uncertainties related to our prediction of a separate crossover behaviour for adjoint sources. Our analysis would hint at the existence of such a crossover in the one-dimensional model too. This appears to be in contradiction to the exact

^{*}) The adjoint string tension at intermediate distances is actually bigger than the fundamental string tension²⁰⁾. However, since the breaking of the adjoint string involves one more dimensionful scale (the length at which the string breaks), one should be cautious about arguing from this that $J_{\text{adj}} > J_{\text{fund}}$. There does not seem to be any a priori reason against $J_{\text{adj}} < J_{\text{fund}}$.

calculation of the same quantity in the two-dimensional finite temperature gauge model, which for $\langle W(x)^2 \rangle$ gives a J-independent value. This, however, is not a failure of the Migdal-Kadanoff procedure, which is exact in one dimension. Rather it is a shortcoming of the effective action (1.1) in this specific dimensionality. We hope that the effective action provides a more realistic description in higher dimensions.

4. - A DISCRETE SUBGROUP APPROXIMATION

In order to go beyond the gauge group SU(2), it is computationally convenient to apply discrete subgroups. We shall in this section describe in some detail such a calculation for discrete subgroups of SU(2), SU(3) and SU(4). The groups used are known to represent the full groups well in the strong coupling regime, the region in which the effective action (1.1) is most straightforwardly derived. In particular, we shall see that our results for the icosahedral subgroup of SU(2) nicely confirm the numbers obtained in the previous section.

In the discrete subgroup approach, the partition function associated with (1.1) reads

$$\begin{aligned} Z(G_{\text{disc}} \subset G) \\ = \sum_{\{\delta_x\}} \prod_x n(\delta_x) \exp \left\{ \frac{1}{2} \sum_{x,j} [\chi(\delta_x) \chi^*(\delta_{x+j}) + \text{h.c.}] \right\} \end{aligned} \quad (4.1)$$

where δ_x denotes the conjugacy classes of G_{disc} at the site x , $n(\delta_x)$ is the multiplicity of that class, and $\chi(\delta_x) = \text{Tr}_{\text{fund.}}(g_x \in G_{\text{disc}})$ is the fundamental character of a certain class. The relevant data associated with the classes of the discrete subgroups chosen for the present investigation are listed in Table 1.

The classes δ and $10-\delta$ of the icosahedral subgroup Y_{120} of SU(2) are Z(2) mirrors of each other. As in Section 3, we normalize our effective fixed-point potential $u^*(\chi(\delta_x))$ such that its value is fixed at the Z(2) singlet, here the class $\delta = 5$.

The renormalization group transformations are completely analogous to Eq. (3.11) and we shall therefore not repeat them here, only note that our fixed-point potential now takes the form

$$u^*(\delta) = -v^*(\delta) - Du^*(\delta, \delta) - \ln[n(\delta)] \quad (4.2)$$

with simply $n(\delta_1)$ replacing W_1 .

Figure 10 shows this fixed-point potential in two dimensions for the icosahedral subgroup of $SU(2)$. Beyond a coupling $J_c \approx 0.21$, the potential decreases without bounds with the number of renormalization group transformations. Note that this particular value of this critical coupling agrees nicely with that of the full group obtained in Section 3 (recall the factor of four difference in the normalization of J in that section). At some smaller value of the coupling, $J_1 \approx 0.175$, the potential develops a secondary minimum with lower energy than that of the $\delta = 5$ class. Beyond this point, a large-distance description in terms of just $Z(2)$ valued variables starts to make sense. Note that also the numerical value of J_1 agrees nicely with the number obtained in Section 3.

For the gauge groups $SU(3)$ and $SU(4)$ we use the discrete subgroups $\Delta(3.6^2)$ [one of the dihedral-like subgroups of $SU(3)^{21}$] and $Y_{60} \otimes Y_{60} \otimes Z(4) [\subset SU(2) \otimes SU(2) \subset SU(4)]$ respectively. The relevant data for these subgroups are listed in Table 1.

The evolution of the fixed-point potential of the above subgroup of $SU(3)$ shows an interesting behaviour which is quite different from the case of $SU(2)$. Let us again start in the simple case of $D = 1$. For $J < J_1 \approx 0.52$, the potential has just one single $\Delta(3.6^2)$ -invariant minimum at $\delta = 10$. At J_1 the potential shifts into non-trivial threefold $[Z(3)$ -invariant] degenerate minima at $\delta = (1, 2, 3)$, $|\chi| = 3$. No increase in the number of iterations necessary for reaching the fixed-point Hamiltonian was observed on either side of J_1 , which of course is not a phase transition point. Also, the coupling $u^*(i, j)$ between nearest neighbours was found to be identically zero on either side of J_1 . However, as J increases, the number of iterations needed for convergence does show a slow increase. We interpret this as the appearance of metastable states which will only disappear on large enough lattices. Thus, for example, in Fig. 11 we plot the nearest-neighbour coupling $u(\delta_{\min}, \delta_{\min})$ as a function of the number of iterations. After about 12 iterations, $u(\delta_{\min}, \delta_{\min})$ moves away from the quasi-stable

plateau at $u(\delta, \delta) \approx 1.09$, a number very close to the value $\ln 3$ observed in the ordered phase of the same model in two and three dimensions (see below). Only after ~ 18 iterations has $u(\delta, \delta)$ approached zero.

The values of J_1 and J_c found in $D = 1, 2, 3$ dimensions are listed in Table 2, together with the results of the $SU(4)$ subgroup. The accuracy of the determination of these values was dictated by the demand for a clear separation of J_1 from J_c . As can be seen, the couplings J_1 and J_c are almost indistinguishable from each other for the subgroups of $SU(3)$ and $SU(4)$. This is in clear contrast to the $SU(2)$ case.

The difference $J_c - J_1$ is seen to diminish as the dimension D increases for a given group, or when proceeding to larger N [in $SU(N)$] in a fixed number of dimensions. Both for the $SU(3)$ and $SU(4)$ subgroups J_1 and J_c are so close to each other that no Monte Carlo simulation with the usual statistics can thermalize the system to an accuracy with which one can distinguish them. This indicates that a large-distance description before the phase transition in terms of just $Z(3)$ and $Z(4)$ spin variables is insufficient. This conclusion actually also seems to be indicated by recent accurate population plots of Polyakov loops in the $SU(3)$ pure gauge theory²²⁾. In the direct neighbourhood of the phase transition, a large proportion of the loop values is still scattered around the origin, without having well-defined $Z(3)$ phases. Only exactly at the phase transition point does the adequacy of a $Z(3)$ description become apparent.

The ordered phase is again nicely characterized by a non-vanishing fixed point domain-domain correlation. Interestingly, we find numerically that $u^*(i, i) \approx \ln N$ for $N = 2, 3, 4$ in $SU(N)$.

Let us finally point out that $J_1 \approx J_c$ for $N = 3, 4$ is consistent with a simple large- N argument based on factorization [see also Ref. 23)]. Because of factorization we have, as $N \rightarrow \infty$:

$$\langle \text{Tr}_{\text{adj.}} W(x) \rangle = \langle \text{Tr}_{\text{fund.}} W(x) \text{Tr}_{\text{fund.}} W(x)^\dagger \rangle = |\langle \text{Tr}_{\text{fund.}} W(x) \rangle|^2 \quad (4.3)$$

Equation (4.3) shows that at $N = \infty$ adjoint sources will deconfine at exactly the same temperature as that of sources in the fundamental representation. In fact, factorization implies a complete lack of fluctuations. Since in general $J_1 = J_c$ will imply the existence of fluctuations in the region $J_1 < J < J_c$, it is consistent to assume that $J_1 = J_c$ at large N .

5. - CONCLUSION

We believe that analytic understanding of the deconfinement phase transitions in pure gauge theories is not yet at its final stage. As we have shown in this paper, approximate real-space renormalization group transformations of the Migdal-Kadanoff type give important insight into the nature of the relevant block-spin variables near the transition point. In particular, they can show to what extent a $Z(N)$ spin variable description is relevant in the disordered phase. Our conclusion is that within the Migdal-Kadanoff approximation an Ising-like description is quite appropriate close to the critical point of the $SU(2)$ gauge theory at finite T . In this case, the coupling J_1 where the fixed-point effective potential develops a double-well structure is clearly separated from the phase transition point J_c itself. For the gauge groups $SU(3)$ and $SU(4)$ it appears that the degrees of freedom transforming as singlets under $Z(3)$ and $Z(4)$ respectively are important almost all the way up to the phase transition point when starting from the disordered phase. Here the two couplings J_1 and J_c are almost identical: $J_1 \lesssim J_c$. This observation may help to explain the incorrect predictions for the $SU(N)$ ($N > 4$) phase transitions from strong coupling $Z(N)$ -valued spin models. Also, $Z(N)$ singlet degrees of freedom may be partly responsible for triggering the observed first-order phase transitions of these theories.

It should be emphasized that the broken phase is always well described in terms of $Z(N)$ degrees of freedom, since here the singlet class is infinitely suppressed. This means that our findings would not invalidate the possible universality predictions for critical exponents of, e.g., the magnetization in cases where the transition could be second-order. Also, if the discontinuity is weak in the cases of first-order phase transitions, the renormalization flow on the broken side might be similar to the case of continuous transitions.

Our Migdal-Kadanoff investigation started from the effective action for Polyakov lines, itself derivable from the full finite temperature gauge theory by a Kadanoff-type potential-moving scheme. It would clearly be advantageous to start the block-spin procedure already at the level of the original finite temperature gauge theory. This, however, runs into the problem of how to define meaningful real-space renormalization group transformations in the Euclidean formulation of finite temperature gauge theories, a problem which has not yet been resolved²⁴⁾.

We hope that our suggestion of a non-trivial structure associated with the free energy of adjoint sources in, at least, the $SU(2)$ case can be checked by Monte Carlo simulations. It might give us further insight into the question of screening and the nature of the independent "adjoint string" at intermediate distances.

Further work can also be carried out along the lines of this paper. Fermions, which to lowest order in a hopping parameter expansion correspond to the addition of an external magnetic field²⁵⁾ to the action (1.1), can be included. Also, the application of this method to other full $SU(N)$ gauge groups is a straightforward, but interesting extension.

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Group		# Elements					# Classes					# Z(N) rotated classes
Y_{120}		120					9					
Class (δ)	1	2	3	4	5							
Character	2	$(1+\sqrt{5})/2$	1	$(\sqrt{5}-1)/2$	0							4
Multiplicity	1	12	20	12	30							
$\Delta(108)$		108					10					
Class (δ)	1	2	3	4								
Character	3	2	-1	0								$2 \cdot 3 = 6$
Multiplicity	1	6	3	78								
$Y_{60} \otimes Y_{60} \otimes Z(4)$		14400					37					
Class (δ)	1	2	3	4	5	6	7	8	9			
Character	4	$\sqrt{5}+1$	$(3+\sqrt{5})/2$	2	$(\sqrt{5}+1)/2$	$\sqrt{5}-1$	1	$(\sqrt{5}-1)/2$	$(3-\sqrt{5})/2$	0		$3 \cdot 9 = 27$
Multiplicity	1	24	144	40	480	24	688	480	144	6300		

Table 1

Basic group characteristics for the SU(N) subgroups appearing in the present study.

D = 1 D = 2 D = 3

Group	D = 1		D = 2		D = 3	
	J_1	J_c	J_1	J_c	J_1	J_c
Y_{120}	0.56 ± 0.01	∞	0.175 ± 5.10^{-3}	0.215 ± 5.10^{-3}	0.0507 ± 1.10^{-4}	0.0519 ± 1.10^{-4}
$\Delta(108)$	0.52 ± 0.01	∞	0.1459 ± 1.10^{-4}	0.1466 ± 1.10^{-4}	0.0467859 ± 2.10^{-7}	0.0467864 ± 2.10^{-7}
$Y_{60} \quad Y_{60} \quad Z(4)$	0.55 ± 0.01	∞	0.15588 ± 1.10^{-5}	0.15592 ± 1.10^{-5}	> 0.0519190	< 0.0519191

Table 2

Values of the characteristic couplings (J_1 and J_c) for the $SU(N)$ subgroups discussed in the text.

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FIGURE CAPTIONS

- Fig. 1 : A two-dimensional example of decimation with scale factor λ (here $\lambda = 3$).
- Fig. 2 : The bond moving associated with the $\lambda = 3$ decimation of Fig. 1.
- Fig. 3 : A general fixed-point structure in the multidimensional space of couplings (only the original nearest-neighbour coupling J is drawn explicitly). There is a stable fixed point (the "small J sink") in the disordered phase and a stable fixed point (the "large J sink") in the ordered, magnetized phase. The two phases are separated by the critical hypersurface. Where the critical surface crosses the J axis we have a (second-order) phase transition.
- Fig. 4 : Kadanoff-type transformation with scale factor $\lambda = 2$.
- Fig. 5 : The effective fixed-point potential in $D = 1$ dimensions for the gauge group $SU(2)$. A clear double-well (Ising-like) structure is seen for couplings bigger than $J_1 \approx 2.0$. No phase transition is found. (The curves shown are smooth interpolations between the evaluated points, shown as dots.)
- Fig. 6 : The potential $u^*(W)$ for $SU(2)$ in $D = 2$ dimensions. A clear Ising-like structure is seen before the actual phase transition.
- Fig. 7 : A plot of the specific heat for $SU(2)$ in two dimensions. One sees a strong peak in the specific heat at a coupling very close to J_c , confirming the interpretation of J_c as the phase transition point.
- Fig. 8 : The clear distinction between J_1 and J_c persists for $SU(2)$ also up to $D = 3$ dimensions, which corresponds to the full finite temperature gauge theory in four dimensions.
- Fig. 9 : The logarithm of $\langle W^2 \rangle$ versus the coupling J in three dimensions. Note the change of regime at J_1 .

Fig. 10 : The effective fixed-point potential for the Y_{120} subgroup of $SU(2)$. The features are quantitatively similar to those found in Section 3 for the full gauge group.

Fig. 11 : The correlation $u(\delta_{\min}, \delta_{\min})$ at $J = 1.0$ in one dimension for the gauge group $SU(3)$. Only after a large number of iterations does the correlation approach zero. This indicates that metastable domains can be formed between rather large domains, even if the total magnetization will be zero.

Fig. 1

Fig. 2

Fig. 3

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Fig. 8

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Fig. 10

Fig.11