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Combined Use of the Calkul Method and $N=1$ Supersymmetry to Calculate QCD Six Parton Processes

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Abstract

Concise expressions are presented for the two independent helicity amplitudes of the subprocess $gggqq$. The result has been derived using the improved CALKUL method and is given in terms of spinor inner products in manifestly covariant and crossing symmetric forms. Changing the color factors of the quarks from the fundamental to the adjoint representation we obtain the helicity amplitudes of the four gluon two gluino subprocess. Simple $N=1$ supersymmetric relations have been found which express the helicity amplitudes of the six gluon parton process in terms of the helicity amplitudes of the $4g2\tilde{g}$ process. In this way we have avoided the direct calculation of 220 Feynman diagrams. Gauge invariance and the validity of the supersymmetric relations have been tested with an independent numerical calculation.

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1. INTRODUCTION

Four jet events can be copiously produced at present and future hadron colliders [1,2]. While four jet processes may provide us with a new testing ground for perturbative QCD, they are more significant as a serious background to many interesting new physics signatures. Production of gauge bosons, Higgs bosons, technipions etc. all may give rise to four jet events [3–6]. Therefore a quantitative description of four jet events may become crucial for event selection of interesting new physics signatures at future hadron colliders. At high energies four jet events are expected to occur also due to double parton scattering. To investigate this question, we need have a quantitative description of the standard QCD mechanism.

The full set of QCD subprocesses involving six partons is given in terms of 545 Feynman diagrams. Therefore the size of the calculation requires non-trivial computational techniques. A straightforward numerical method has been attempted previously [7] to get an estimate for the production rate of four jet events by calculating numerically the amplitudes of the $q_1 q_1 q_2 q_2 q_3 q_3$ and $gg q_1 q_1 q_2 q_2$ subprocesses. It has been found that the numerical program is too slow, in particular if the $4g2q$, $6g$ subprocesses are also included. Due to massless kinematics and the high symmetry in the external legs, however, it is conjectured that very concise expressions may exist for the cross-sections of the $2 \rightarrow 4$ subprocesses similarly to the case of the $2 \rightarrow 3$ parton processes.

Recently – using the improved [8] CALKUL method [9] – simple formulae have been found for the two subprocesses involving six quarks [10] and two gluons plus four quarks [11], respectively. In a recent interesting paper [12], $N=2$ supersymmetry has been applied as a calculational trick to simplify the calculation of the helicity amplitudes of the subprocess involving six gluons.

In this paper I present concise expressions for the helicity amplitudes of the subprocess involving four gluons and two quarks. Furthermore I give the helicity amplitudes of the $6g$ subprocess in terms of the amplitudes of the subprocess involving four gluons and two massless gluinos (the supersymmetric partners of the gluons) by employing $N=1$ supersymmetric relations. This new way of using $N=1$ supersymmetry is very powerful: it makes the calculation of 220 Feynman diagrams redundant!

2. THE CALCULATION

The $4g2q$ amplitude is described in terms of 126 Feynman diagrams, which can be divided into two types: 105 diagrams containing no four gluon vertex and the remaining 25 diagrams each containing one four gluon vertex. It is easy to prove that the amplitudes described by the diagrams having a four gluon vertex can be uniquely decomposed into terms having exactly the same colour factors and denominator factors as one of the amplitudes described by the 105 Feynman diagrams having no four gluon vertex. Therefore if the terms which have the same colour factor and denominator are combined together we end up with 105 effective amplitudes. It is remarkable that in the case of the $6g$ subprocess we can proceed similarly. Again there are 105 amplitudes having no four gluon vertices. The 115 amplitudes having one or two four gluon vertices can be decomposed according to their colour and denominator factors again into one of the amplitudes given by the Feynman diagrams containing no four gluon vertex. Therefore the number of the effective diagrams is the same for the two subprocesses. If we change the colour representations of the quarks to the adjoint representation (quark becomes gluino) the colour factors and the denominator factors of the 105 effective amplitudes of the $4g2\tilde{g}$ and $6g$ subprocesses are the same. This is the first very clear indication of the existence of some relations between the amplitudes of the two subprocesses.

The 105 effective amplitudes can be described in terms of two diagrams shown in fig.1. The diagram of topology A leads to 90 Feynman diagrams after permuting the labels of the external legs, while the diagram of topology B gives 15 Feynman diagrams via the allowed label permutations. In the case of the subprocess involving four gluon plus two quarks (or two gluinos) the external legs having e.g. labels 5 and 6 may be considered as fermion lines with an arrow pointing e.g. from 5 to 6. The label convention of the 105 diagrams is given in Table 1. The entries of Table 1 give the permuted labels of diagrams of topology A (without asterisk) and diagrams of topology B (with asterisk).¹ The labels of Table 1 uniquely define the 105 effective diagrams. Each of the diagrams can be decomposed into the form

$$M(\ell) = T\{A\}(\ell) \cdot N(\ell)/D(\ell) \quad (1)$$

where $D(\ell)$ is the product of the inverse scalar propagators defined by the diagram of label ℓ of Table 1, $T\{A\}(\ell)$ is the colour factor with some set of external colour indices $\{A\}$ and $N(\ell)$ is the effective numerator function. Only the $N(\ell)$'s depend on the helicities of the external partons. For example diagram 82 for the 4g2q subprocess is defined as

$$\begin{aligned} N(82) &= \bar{u}(k_6)_1(\kappa_1 + \kappa_6)\not{\epsilon}_2(\kappa_1 + \kappa_2 + \kappa_6)\not{\epsilon}_3 \cdot \\ &\quad (-\kappa_5 - \kappa_4)\not{\epsilon}_4 u(k_5) \\ D(82) &= (k_1 + k_2)^2 (k_1 + k_2 + k_6)^2 (k_5 + k_4)^2 \\ T(82)\{A\} &= (\lambda^{a_1} \lambda^{a_2} \lambda^{a_3} \lambda^{a_4})_{\alpha_6 \alpha_5} / 2^4 \end{aligned}$$

where $k_1, k_2, \dots, k_6$ denote the four momenta of the external legs with the convention

$$\sum_{i=1}^6 k_i = 0 \quad , \quad (2)$$

$\not{\epsilon}$ is the gluon polarization vector contracted with the Dirac γ_μ (with helicity indices suppressed), κ denotes the four momenta contracted with γ_μ , λ^a denotes the Gell-Mann matrix of SU(3) colour and $a_1, \dots, a_4, \alpha_5, \alpha_6$ are the colour components of the gluons and quarks, respectively. In the case of the 4g2g subprocess $i(\lambda^a/2)_{\beta\gamma}$ is replaced by the antisymmetric matrix $(F^a)_{bc} = -i f_{abc}$.

The colour factors of the 105 Feynman diagrams can be decomposed as linear combinations of the 24 colour structures given by the diagrams of label 82-105. This decomposition is the same for the 4g2q and 4g2g process

$$T(\ell)\{A\} = \sum_{i=1}^{24} R(i, \ell) S(i)\{A\}, \quad \ell = 1, 2, \dots, 105 \quad (3)$$

¹ The peculiar convention which mixes the order of the labels of the topology A and B has no significance. Notice that both diagram A and B are symmetric under the interchange $i_1 \leftrightarrow i_2$, $i_5 \leftrightarrow i_6$ and $i_1 i_2 i_3 i_4 i_5 i_6 \leftrightarrow i_6 i_5 i_4 i_3 i_2 i_1$ while diagram B is also symmetric under the interchange of the pair of indices $i_1 i_2 \leftrightarrow i_3 i_4$ and cyclic permutations.

With the label convention defined by Table 1 and with the definition

$$S(i)\{A\} = T\{A\}(81+i), \quad i=1,2,\dots,24 \quad (4)$$

we find that the matrix $R(i,\ell)$ has only elements with values 1, -1 or 0. The non-vanishing matrix elements are

$$\begin{aligned} R(i_a(k), \ell_a) &= 1, & R(j_a(k), \ell_a) &= -1 \\ R(i_b(k), \ell_b) &= 1, & R(j_b(k), \ell_b) &= -1 \\ R(i_c(i), i+45) &= 1, & R(j_c(i), i+45) &= -1 \\ R(r, r+81) &= 1 \end{aligned}$$

$$k=1,2,\dots,60, \quad i=1,2,\dots,36, \quad r=1,2,\dots,24 \quad (5a)$$

where $i_a(k)$, $i_b(k)$, $j_a(k)$, $j_b(k)$ are integer valued vectors of length 60, $i_c(i)$, $j_c(i)$ are integer valued vectors of length 36. They are given in Table 2. The indices ℓ_a and ℓ_b are defined as

$$\ell_a = k \bmod(15) \quad (5b)$$

$$\begin{aligned} \ell_b &= k+3, & \text{if } 13 \leq k \leq 36 \\ \ell_b &= k-9, & \text{if } 37 \leq k < 54 \\ \ell_b &= k-15, & \text{if } 55 \leq k \leq 60 \end{aligned} \quad (5c)$$

The absolute square of the amplitude can be written in the form

$$|A|^2 = \left| \sum_{i=1}^{105} M(i) \right|^2 = \sum_{k=1}^{24} C_v(k, \ell) \cdot U(k) \cdot U(\ell)^*, \quad (6a)$$

where the vector $U(\ell)$ is defined by the equation

$$U(k) = \sum_{\ell=1}^{105} R(\ell, k) N(\ell)/D(\ell) \quad , \quad k=1,2,\dots,24 \quad (6b)$$

the asterisk * denotes complex conjugation and $v=q$ or $v=g$. With the convention given above the matrices C_q, C_g have only nine independent matrix elements which are given as elements of the nine components vectors $Q_q(i), Q_g(i)$, $i=1,\dots,9$. In the general case of $SU(N)$ colour the $Q_q(i), Q_g(i)$ vectors are given in Table 3. The matrices $C_v(k, \ell)$ are constructed from the $Q_v(i)$ vectors as

$$\begin{aligned} C_v(i, i) &= Q_v(i), & i &= 1,\dots,12 \\ C_v(k, \ell) &= Q_v[m(r)], & k &= 1,\dots,12; \quad \ell = k+1, \dots, 25-k \\ r &= (k-1)(24-k) + \ell + 1 \end{aligned} \quad (7a)$$

where $m(r)$ is an integer valued vector of length 144. Its components are given in Table 4. The remaining matrix elements of $C_v(i,j)$ are defined by the symmetry properties

$$\begin{aligned} C_v(j,i) &= C_v(i,j) \\ C_v(j,25-i) &= C_v(i,25-j), \quad \text{if } i=1,\dots,11; j=i+1,\dots,12 \\ C_v(25-i,j) &= C_v(i,25-j), \quad \text{if } i=1,\dots,12; j=1,\dots,24. \end{aligned} \quad (7b)$$

The numerator functions of the $4g2q$ and $4g2\tilde{g}$ processes are the same for all external helicity values of the quarks and gluinos.

For the calculation of the numerator functions I have used the improved CALKUL method as formulated in ref.[8]. The gluon helicity vectors have the form

$$\epsilon_\mu^\lambda(k_i, k_6) = (\lambda/\sqrt{2}) \langle i, \lambda | \gamma_\mu | 6, -\lambda \rangle / \langle i, \lambda | 6, -\lambda \rangle \quad (8)$$

where $\lambda = \mp$ and $\langle i, -|j, + \rangle$ or $\langle i, +|j, - \rangle$ denote the spinor inner products (see ref.[7]) i.e. scalar products of massless spinors of given helicity. They can be expressed in terms of momentum components as follows

$$\begin{aligned} \langle ij \rangle &\equiv \langle i, -|j, + \rangle = \sqrt{(k_{i-} k_{j-})} \exp\{i\phi_i\} - (i \leftrightarrow j) \\ \langle ij \rangle^+ &\equiv \langle j, +|i, - \rangle^* = \sqrt{(k_{i-} k_{j-})} \exp\{i\phi_i\} - (i \leftrightarrow j) \end{aligned} \quad k_{0i}, k_j > 0$$

where

$$k_{i\mp} = k_{0i} \mp k_{iz}, \quad \cos(\phi_i) = k_{ix} / \sqrt{(k_{ix}^2 + k_{iy}^2)} \quad (9a)$$

denote the light cone variables. To maintain manifest crossing symmetry we also define inner products of negative energy spinors.

For $p_0, q_0 > 0$ we have

$$\begin{aligned} \langle -p, \lambda | q, -\lambda \rangle &= i \langle p, \lambda | q, -\lambda \rangle, \quad \langle p, \lambda | -q, -\lambda \rangle = i \langle p, \lambda | q, -\lambda \rangle \\ \langle -p, \lambda | -q, -\lambda \rangle &= - \langle p, \lambda | q, -\lambda \rangle. \end{aligned} \quad (9b)$$

Using the properties of the inner products

$$\langle i, \lambda | i, \lambda' \rangle = 0, \quad \langle i, \lambda | j, \lambda \rangle = 0, \quad \langle i, -|j, + \rangle \langle j, +|i, - \rangle = 2k_j \cdot k_i \quad (9c)$$

we can easily see that when all four external gluons have the same helicities the amplitude vanishes. Since along the quark line helicity is conserved, up to interchanges of momentum labels and parity there are only two independent helicity amplitudes. We have chosen to calculate

and

$$\begin{aligned}
 & A(k_1, +; k_2, +; k_3, +; k_4, -; k_5, -; k_6, -) \\
 & A(k_1, +; k_2, +; k_3, -; k_4, -; k_5, -; k_6, -)
 \end{aligned}
 \tag{10}$$

The corresponding numerator functions are denoted as $N_+(\ell)$, $N_-(\ell)$, respectively, where their dependence on the external parton four momenta k_i has been suppressed. The helicity amplitudes of positive quark helicity can be obtained from the amplitudes of negative quark helicity with the help of the identity

$$\langle k_i, - | \gamma_{\mu_1} \dots \gamma_{\mu_1} | k_j, - \rangle \equiv (-1)^{\ell+1} \langle k_j, + | \gamma_{\mu_1} \gamma_{\mu_1-1} \dots \gamma_{\mu_1} | k_i, + \rangle
 \tag{11}$$

Flipping the quark helicities maps the numerator functions into themselves according to the label interchange $5 \leftrightarrow 6$ (see Table 1). Since the momentum flow along the quark(gluino) line is also changed by the application of eq.(11) certain sign changes are also generated. Therefore we obtain

$$\begin{aligned}
 & A(k_1, +; k_2, +; k_3, \lambda; k_4, -; k_5, +; k_6, +) = \\
 & \sum_{\ell=1}^{105} T(\ell) \{A\} \cdot \Theta(\ell) \cdot N_{\lambda}(k_5 \leftrightarrow k_6) \cdot IM(\ell) / D(\ell)
 \end{aligned}
 \tag{12}$$

where the label map $IM(\ell)$ and the sign changes $\Theta(\ell)$ are given in Table 5. Since the map generated by flipping the helicity (see eq.(10)) is reflexive ($IM(IM(\ell)) = \ell$, $\Theta(\Theta(\ell)) = 1$) in Table 5 only the independent and non trivial elements of the maps are indicated. The map is trivial for $\ell = 1, \dots, 15$ ($IM(\ell) = \ell$ and $\Theta(\ell) = 1$).

For the helicity combination of the numerator function N_+ we obtain $3!$ -fold symmetry in the four momenta of the gluons corresponding to the permutations of the gluons of positive helicity. Therefore it is sufficient to calculate N_+ for one fixed permutation of the gluon momenta k_1, k_2, k_3 . The result can be conveniently expressed in terms of four simple combinations

$$\begin{aligned}
 V_1(1) &= xc \cdot gc \cdot (f \cdot wc + d \cdot vc) \cdot B \\
 V_2(1) &= xc \cdot gc \cdot (f \cdot uc + d \cdot sc) \cdot B \\
 V_3(1) &= -xc \cdot rc \cdot (f \cdot qc + d \cdot pc) \cdot B \\
 V_4(1) &= -uc \cdot oc \cdot (f \cdot xc - b \cdot sc) \cdot B
 \end{aligned}$$

where $B = 4 \cdot b^3 / f \cdot d \cdot c$. and we use the shorthand notation of the inner products² defined in Table 6. We generate six components for these expressions $V_i(\ell)$, $i=1,2,3,4,5,6$ corresponding to the permutations of the gluon momenta $\{1,2,3; 1,3,2; 2,3,1; 2,1,3; 3,1,2; 3,2,1\}$. With the gluon polarisation vectors of eq.(8) and the first helicity choice of eq.(10) it is easy to see that 57 out of the 105 numerator functions are vanishing. The 48 non-vanishing numerator functions can be written as

² The adjoint inner products $\langle ij \rangle^+$ are denoted by the same letter as $\langle ij \rangle$ but with an extra 'c' appended. For example $a = \langle 56 \rangle$, therefore $ac = \langle 56 \rangle^+$.

$$\begin{aligned} n_+(j) &= n_+(j+9) = V_1(j) \\ n_+(j+3) &= n_+(j+12) = -n_+(j+21) = V_2(j) \\ n_+(j+6) &= -n_+(j+15) = n_+(j+18) = V_3(j) \end{aligned}$$

for $j = 1, 2, 3$;

and

$$n_+(j+24) = n_+(j+30) = n_+(j+36) = -n_+(j+42) = V_4(j)$$

for $j = 1, 2, 3, 4, 5, 6$.

(13)

The non-vanishing functions $n_+(j)$ are related to our standard labels by the equation

$$N_+(\text{pmap}(i)) = \eta_i \cdot n_+(i), \quad i = 1, 2, \dots, 48,$$

(14)

where the integer array $\text{pmap}(i)$ is given in Table 7. The appearance of the sign factor η_i is due to an unfortunate sign convention. $\eta_i = 1$ for all $i = 1, 2, \dots, 48$ but $\eta_i = -1$ when $i = 9$ and $i = 11$.

The calculation of the numerator functions for the second helicity choice is more complicated, with only 33 numerator functions vanishing in this case. Since now the numerator functions are symmetric under the independent interchange of the gluon momenta $k_1 \leftrightarrow k_2$ and $k_3 \leftrightarrow k_4$, we have only 20 independent numerator functions modulo this symmetry. The 20 entries of Table 8 define 20 auxiliary functions. Their relation to the 72 non-vanishing numerator functions is uniquely given by the labels j_1 and j_2 , where j_1 has values 1, 2, 3, 4 while j_2 can have values only 1 and 2. The values of j_1 and j_2 correspond to the permutation of the gluon momentum labels as

$$1 = \{1, 2, 3, 4\}; 2 = \{2, 1, 3, 4\}; 3 = \{1, 2, 4, 3\}; 4 = \{2, 1, 4, 3\}; \text{ for } j_1$$

$$1 = \{1, 2, 3, 4\}; 2 = \{2, 1, 3, 4\}; \quad \text{for } \ell_1$$

and

$$1 = \{1, 2, 3, 4\}; 2 = \{1, 2, 4, 3\}; \quad \text{for } j_2$$

(15a)

The set of functions $n_-(\ell)$, ($\ell = 1, 2, \dots, 72$) of Table 8 define the numerator functions for the standard label with the help of the equation

$$N_-(\text{nmap}(i)) = \eta_i \cdot n_-(i) \cdot 4 / (f \cdot d \cdot cc \cdot bc), \quad i = 1, 2, \dots, 72$$

(15b)

where $\text{nmap}(i)$ is given in Table 7 and $\eta_i = 1$ for all i except for $\eta_{43} = -1$.

In view of the very simple formula obtained for $N_+(\ell)$ I conjecture that it may also be possible to derive a simple formula for the squared amplitude. The form of the other helicity amplitude does not look too promising.

The amplitude squared summed over all spin states can now be given very easily. First we calculate the sum

$$B1(1,2,3,4,5,6) = |A(1+, 2+, 3+, 4-, 5-, 6-)|^2 + |A(1+, 2+, 3+, 4-, 5+, 6+)|^2$$

$$B2(1,2,3,4,5,6) = |A(1+, 2+, 3-, 4-, 5-, 6-)|^2 + |A(1+, 2+, 3-, 4-, 5+, 6+)|^2$$

Then the sum over all helicity states will give us the expression

$$|M|^2 = 2 \cdot \{B1(1,2,3,4) + B2(1,2,3,4) + B1(1,2,4,3) + B1(1,4,3,2) + B2(1,4,3,2) + B1(4,2,3,1) + B2(4,2,3,1)\}$$
(16)

3. THE SUPERSYMMETRY TRICK

In leading order of perturbation theory the 6g amplitude is given by exactly the same set of Feynman diagrams as in the supersymmetric extension of QCD with R-symmetry. Therefore the helicity amplitudes of the six gluon process are related to the helicity amplitudes of the process involving four gluons and two gluinos.

It is easy to see using for example the CALKUL method that in leading order there are only two independent non-vanishing helicity amplitudes also for the 6g process (modulo parity and Bose-symmetry) since the amplitudes with helicity combination $\{+++++\}$ and $\{++++-\}$ vanish. This result can also be obtained as a consequence of the supersymmetric Ward-identities [13]. Following the method of [13] I have found that

$$A^{6g}(k_1+, k_2+, k_3+, k_4-, k_5-, k_6+) =$$

$$- \langle k_4 k_5 \rangle / \langle k_4 k_6 \rangle \cdot A^{4g2\tilde{g}}(k_1+, k_2+, k_3+, k_4-, k_5+, k_6+)$$
(17)

and

$$A^{6g}(k_1+, k_2+, k_3-, k_4-, k_5-, k_6+) =$$

$$\{- \langle k_3 k_5 \rangle / \langle k_3 k_6 \rangle \cdot A^{4g2\tilde{g}}(k_1+, k_2+, k_3-, k_4-, k_5+, k_6+) + (k_4 \leftrightarrow k_5)\}$$
(18)

where the inner products $\langle ij \rangle$ are defined by eq.(9), the gluinos have momentum labels 5 and 6 and all particles are outgoing (eq.(2)). The relations are manifestly crossing symmetric. If we perform the interchange $4 \leftrightarrow 5$ the color matrix $C_g(i, j)$ will be rotated as follows

$$C_g(i, j) = \sum_{i_1, j_1}^{24} R_c(i, i_1) \cdot C_g(i_1, j_1) \cdot R_c(j, j_1)$$

$$i_1, j_1 = 1$$
(19)

the matrix $R_c(i, j)$ is defined in terms of $R(i, j)$ (see eqs.(5a - 5c))

$$R_c(i, j) = \eta_i \cdot R(j, i_s(i)) \quad i = 1, \dots, 24$$

$$\eta_i = 1 \quad \text{for } i = 5, 6, 11, 12, 17, 18,$$

$$\eta_i = -1 \quad \text{otherwise}$$
(20)

where the vector $i_s(j)$ is given in Table 9.

Notice that equations similar to eq.(17) exist also for the single independent non-vanishing helicity amplitudes of the $4g, 2g\tilde{g}$ and $5g, 3g\tilde{g}$ subprocesses, respectively. The simplicity of the relations of eqs.(17–18) is rather striking.

Recall that the $4g\tilde{g}$ amplitudes differ from the $4g2q$ amplitudes only in the actual values of the colour entries $Q(i)_q$ and $Q(i)_g$ (see eq.(7) and Table 3). The number of the supersymmetric relations is very large since the relations as given by eqs.(17–18) hold for each independent colour structures and many other relations exist for different helicity combinations. In the right hand side of eq.(18) presumably many terms cancel when combining the two terms.

Up to parity relations there are 15 helicity amplitudes for helicities $\{++++--\}$ and 10 amplitudes for the helicity combinations $\{+++---\}$. They can be obtained from the amplitudes of eq.(17–18) permuting the four momenta as given in Table 10.

4. DISCUSSION

In order to check the results I also have calculated the amplitudes and cross-sections for the $6g$, $4g2q$ and $4g\tilde{g}$ subprocesses with a completely numerical program. This way I have checked gauge invariance and the validity of the supersymmetry relations. The numerical program is about 10 times slower than the Fortran program based on the formulae presented in this paper.

The simultaneous evaluation of the colour- and spin-summed cross-sections of the $4g2q$ and $6g$ subprocesses requires ~ 0.1 sec IBM 3081 CPU time. Although rather long, it is still acceptable to do realistic phenomenological calculations where all the $2 \rightarrow 4$ subprocesses are taken into account. The computer time found in ref.[5] appears to be somewhat longer. Of course the computer time is significantly reduced using vectorised machines.

Phenomenological implications of the results of this paper will be discussed in a forthcoming publication [14]. Here I would like only to emphasize that we are finally in a position to carry out the phenomenological cross section calculations based on all subprocesses without resorting to any approximations.

Note added

After completing this work I have been informed by S. Parke and T. Taylor that they have also computed the amplitudes of the $4g2q$ subprocesses (Fermilab – Pub – 85/162 – T). We have compared our results numerically for both the subprocesses involving six gluons and four gluons plus two quarks and we have found complete agreement. I thank S. Parke for correspondence.

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FIGURE CAPTIONS

Fig.1 Feynman diagrams of topology (A) and (B)

TABLE CAPTIONS

Table 1	Definition of the labels of the 105 effective amplitudes in terms of permutations of the external legs of diagrams (A) and (B) in Fig.1.
Table 2	Labels defining the non vanishing matrix elements of the matrix $R(i,j)$. (See eqs. (5a – 5c)).
Table 3	The independent matrix elements of the colour matrices $C_{q,g}(i,j)$ (see eqs. (7a – 7b)). N denotes the $SU(N)$ colour and $V = N^2 - 1$.
Table 4	The map $m(r)$ which defines the matrix elements of $C_v(i,j)$ in terms of the independent elements $Q_v(k)$ (see Table 3 and eqs. (7a – 7b)).
Table 5	The label map of the numerator functions into themselves under the 5 – 6 interchange (see eq. (11)).
Table 6	Short hand notation of the inner products $\langle ij \rangle$.
Table 7	The map of the labels of the non vanishing numerator functions $n_{+-}(i)$ to the labels of the standard numerator functions $N_{+-}(k)$ (see eqs. (14 – 15)). $pmap(i), i = 1, \dots, 48$ gives the map for $n_{+}(i)$, while $nmap(i), i = 1, \dots, 72$ gives the map for $n_{-}(i)$.
Table 8	The non vanishing numerator functions $n_{-}(i), i = 1, \dots, 72$. j_1 has values of 1,2,3,4, while ℓ_1 and j_2 have values 1,2 corresponding to the permutations given by eq.(15b).
Table 9	The label map defining the new matrix elements of the colour decomposition matrix $R(i,j)$ under the interchange of the colour indices of particles 4 and 5 (see eqs.(19) – 2(20)).
Table 10	Momentum label permutations defining the 15 (of type $+++--+$) and the 10 (of type $++----+$) helicity amplitudes of the 6g parton process, correspondingly.

dl	perm	dl	perm	dl	perm
1	123456	36	134652	71	126453
2	124356	37	142653	72	136254
3	132456	38	124653	73	136452
4	134256	39	241653	74	146253
5	142356	40	126534	75	146352
6	143256	41	136524	76	236154
7	231456	42	146523	77	236451
8	234156	43	346512	78	246153
9	241356	44	246513	79	246351
10	243156	45	236514	80	346152
11	341256	46	162534	81	346251
12	342156	47	261534	82	612345
13	123456	48	163524	83	612435
14	132456	49	361524	84	613245
15	142356	50	164523	85	613425
16	234561	51	461523	86	614235
17	342561	52	263514	87	614325
18	243561	53	362514	88	621345
19	341562	54	264513	89	621435
20	143562	55	462513	90	623145
21	134562	56	364512	91	623415
22	142563	57	463512	92	624135
23	124563	58	613425	93	624315
24	241563	59	623415	94	631245
25	123564	60	612435	95	631425
26	231564	61	632415	96	632145
27	312564	62	612345	97	632415
28	123654	63	642315	98	634125
29	231654	64	621435	99	634215
30	132654	65	631425	100	641235
31	234651	66	621345	101	641325
32	342651	67	641325	102	642135
33	243651	68	631245	103	642315
34	341652	69	641235	104	643125
35	143652	70	126354	105	643215

Table 1

$i_a(k), k = 1, \dots, 60$

1, 2, 3, 4, 5, 6, 3, 3, 5, 3, 6, 1, 1, 3, 5,
15, 15, 9, 9, 11, 9, 9, 5, 11, 5, 7, 6, 8, 12, 10,
21, 17, 21, 11, 15, 11, 19, 10, 13, 12, 12, 12, 18, 14, 15,
23, 21, 23, 23, 17, 17, 24, 24, 18, 18, 17, 18, 23, 21, 20,

$j_a(k), k = 1, \dots, 60$

7, 8, 7, 7, 8, 8, 1, 1, 2, 2, 4, 2, 2, 4, 6,
13, 13, 13, 12, 14, 10, 15, 6, 14, 4, 8, 4, 7, 11, 9,
19, 18, 20, 14, 16, 14, 20, 16, 16, 16, 10, 10, 17, 13, 16,
24, 19, 22, 20, 19, 20, 22, 22, 21, 22, 23, 24, 24, 22, 19,

$i_b(k), k = 1, \dots, 60$

1, 2, 2, 8, 8, 7, 14, 13, 14, 19, 20, 21, 6, 4, 4,
10, 10, 12, 16, 18, 16, 24, 22, 23, 1, 3, 3, 10, 12, 12,
6, 6, 4, 5, 2, 5, 15, 9, 9, 24, 18, 18, 17, 17, 23,
11, 21, 11, 1, 3, 5, 17, 11, 9, 8, 14, 20, 24, 22, 16,

$j_b(k), k = 1, \dots, 60$

3, 1, 3, 7, 9, 9, 15, 15, 13, 21, 19, 20, 5, 6, 5,
12, 11, 11, 17, 17, 18, 23, 24, 22, 7, 1, 7, 16, 10, 16,
4, 14, 14, 8, 8, 2, 13, 15, 13, 22, 24, 22, 23, 20, 20,
19, 19, 21, 2, 4, 6, 18, 12, 10, 7, 13, 19, 23, 21, 15,

$i_c(k), \dots, 36$

1, 7, 3, 13, 5, 19, 9, 15, 11, 21, 17, 23,
4, 10, 2, 16, 1, 22, 8, 14, 7, 20, 13, 19,
1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 17, 18,

$j_c(k), k = 1, \dots, 36$

2, 8, 4, 14, 6, 20, 10, 16, 12, 22, 18, 24,
6, 12, 5, 18, 3, 24, 11, 17, 9, 23, 15, 21,
7, 8, 13, 14, 19, 20, 15, 16, 21, 22, 23, 24,

Table 2

i	$(16 \cdot N^3/V) \cdot Q_q(i)$	$(1/N^2 \cdot V) \cdot Q_g(i)$
1	V^3	N^2
2	$-V^2$	$N^2/2$
3	V	$N^2/4$
4	-1	$N^2/8$
5	$V(V+2)$	0
6	$-V-2$	0
7	V^2+V+1	$(N^2+12)/8$
8	V^2-2	$3/2$
9	$-3 \cdot V-2$	$3/2$

Table 3

$m(r)$

2,2,3,3,5,2,3,3,4,4,6,3,4,5,6,7,
8,4,6,6,8,8,9,3,5,2,3,3,2,4,6,3,
4,4,6,6,8,8,9,3,4,5,6,7,2,5,3,3,
4,5,6,7,8,2,3,3,4,4,6,6,4,8,9,3,
2,4,6,6,8,8,9,3,2,4,6,3,4,4,3,7,
2,4,3,7,8,5,6,6,4,8,9,6,8,2,3,6,
4,8,9,6,8,4,3,7,8,5,6,3,2,2,3,3,
5,5,6,3,4,8,7,3,5,2,3,6,8,4,6,9,
2,5,3,3,4,2,3,3,2,4,6,3,2,8,9,7

Table 4

$\Theta(\ell) \cdot \text{IM}(\ell)$

– 31, – 32, – 33, – 34, – 35, – 36, – 37, – 38, – 39, – 28,
– 29, 30, – 43, – 44, – 45, 81, 80, 79, 78, 77,
76, 75, 74, 73, 72, 71, 70, 59, 61, 63,
65, 67, 69, – 105, – 99, – 103, – 93, – 97, – 91, – 104,
– 98, – 101, – 95, – 102, – 100

Table 5

		<ij>					
i	j	1	2	3	4	5	6
	1						
2		x					
3		w	v				
4		u	s	r			
5		q	p	o	g		
6		f	d	c	b	a	

Table 6

pmap(i)

1, 3, 7, 2, 4, 8, 13, 14, 15, 25, 27, 26,
38, 36, 31, 43, 44, 42, 57, 55, 51, 69, 67, 63,
5, 6, 10, 9, 11, 12, 37, 35, 33, 39, 34, 32,
74, 75, 79, 78, 80, 81, 100, 101, 103, 102, 104, 105

nmap(i)

94, 96, 100, 102, 95, 97, 101, 103, 98, 99, 104, 105,
72, 76, 74, 78, 73, 77, 75, 79, 61, 65, 63, 67,
49, 53, 51, 55, 41, 45, 42, 44, 29, 30, 39, 37,
31, 36, 33, 35, 22, 24, 27, 26, 3, 7, 5, 9,
8, 4, 10, 6, 80, 81, 34, 32, 11, 12, 23, 25,
68, 69, 56, 57, 28, 38, 14, 15, 1, 2, 43, 13

Table 7

Numerator functions of type $n_{-}(i)$

$$\begin{aligned}
 n_{-}(j_1) &= -d36 \cdot c \cdot w \cdot g \cdot a \cdot p \cdot c \\
 n_{-}(j_1 + 4) &= -d36 \cdot c \cdot p \cdot c \cdot b1 \cdot a1 \\
 n_{-}(j_1 + 8) &= -d36 \cdot p \cdot c \cdot r \cdot b2 \cdot a2 \\
 n_{-}(j_1 + 12) &= -n_{-}(j_{-}) \\
 n_{-}(j_1 + 16) &= c \cdot p \cdot c \cdot b1 \cdot (d36 \cdot a1 - 0.5 \cdot f \cdot c \cdot b \cdot d136) \\
 n_{-}(j_1 + 20) &= d36 \cdot q \cdot c \cdot (b \cdot (b2 \cdot r \cdot s \cdot c - 0.5 \cdot d234 \cdot d \cdot c \cdot c) - d26 \cdot r \cdot v \cdot c \cdot c) \\
 n_{-}(j_1 + 24) &= -d36 \cdot c \cdot (a1 \cdot (s \cdot c \cdot a \cdot c \cdot b - d26 \cdot p \cdot c) - 0.5 \cdot b \cdot q \cdot c \cdot d \cdot d136) \\
 n_{-}(j_1 + 28) &= c \cdot (-d36 \cdot w \cdot c \cdot (p \cdot c \cdot (r \cdot d46 - s \cdot c \cdot d \cdot c) - 0.5 \cdot a4 \cdot b \cdot d \cdot c) - 0.5 \cdot b4 \cdot \\
 &\quad f \cdot c \cdot (b \cdot (d46 \cdot p \cdot c + 0.5 \cdot d \cdot c \cdot a3))) \\
 n_{-}(j_1 + 32) &= c \cdot g \cdot a \cdot c \cdot (d36 \cdot (c \cdot q \cdot c \cdot v \cdot c - 0.5 \cdot b \cdot x \cdot c \cdot g \cdot c) - 0.25 \cdot d \cdot c \cdot b \cdot g \cdot c \cdot a5) \\
 \\
 zv1 &= -b \cdot d \cdot c \cdot a6 \cdot (v \cdot b \cdot d \cdot c + 0.5 \cdot c \cdot (r \cdot c \cdot c - s \cdot d \cdot c)) \\
 zv2 &= -a7 \cdot c \cdot (c \cdot v \cdot c \cdot b5 + 0.5 \cdot d \cdot c \cdot (c \cdot a14 - d \cdot a8)) \\
 zv3 &= -b \cdot b5 \cdot (d \cdot c \cdot (0.5 \cdot c \cdot (d34 - d24) + v \cdot b \cdot s \cdot c) + c \cdot v \cdot c \cdot b \cdot c \cdot r) \\
 n_{-}(j_1 + 36) &= q \cdot c \cdot (zv1 + zv2 + zv3) \\
 \\
 zb2(j_1 + 36) &= 0.5 \cdot d23 \cdot b5 \cdot b \cdot c \cdot d \cdot c \cdot q \cdot c \\
 zw1 &= 0.5 \cdot b \cdot f \cdot c \cdot ((b \cdot s \cdot c + f \cdot x \cdot c) \cdot a17 + a9 \cdot (r \cdot g \cdot c + w \cdot q \cdot c) - (d24 - d12) \cdot c \cdot p \cdot c) \\
 zw2 &= -u \cdot f \cdot c \cdot c \cdot (q \cdot c \cdot a9 + p \cdot c \cdot d \cdot x \cdot c) \\
 zw3 &= b \cdot u \cdot c \cdot (-b \cdot d \cdot c \cdot a17 + r \cdot a \cdot c \cdot a9 + c \cdot p \cdot c \cdot s \cdot d \cdot c) \\
 zb1(j_1 + 40) &= 0.5 \cdot d36 \cdot b \cdot c \cdot d14 \cdot q \cdot c \cdot d \cdot c \\
 n_{-}(j_1 + 40) &= -d36 \cdot (zw1 + zw2 + zw3) + zb1(j_1 + 40) \\
 \\
 vz1 &= -0.5 \cdot c \cdot f \cdot c \cdot (0.5 \cdot b \cdot a \cdot c \cdot b3 + a \cdot c \cdot g \cdot (-f \cdot q \cdot c + c \cdot o \cdot c) + b \cdot g \cdot c \cdot (-f \cdot c \cdot u + c \cdot c \cdot r)) \\
 vz2 &= -0.5 \cdot b \cdot a \cdot c \cdot (f \cdot c \cdot w \cdot (-a \cdot q \cdot c + b \cdot u \cdot c) + c \cdot w \cdot c \cdot (r \cdot b \cdot c - o \cdot a \cdot c)) \\
 vz3 &= -b \cdot b \cdot f \cdot c \cdot f \cdot w \cdot g \cdot c - c \cdot c \cdot a \cdot c \cdot a \cdot c \cdot g \cdot w \cdot c \\
 v1e2 &= -c \cdot (c \cdot w \cdot c \cdot d \cdot c - 0.5 \cdot f \cdot c \cdot (c \cdot v \cdot c + f \cdot x \cdot c)) \\
 v2e2 &= b \cdot (-b \cdot g \cdot c \cdot d \cdot c - 0.5 \cdot a \cdot c \cdot (b \cdot s \cdot c - a \cdot p \cdot c)) \\
 v1k2 &= -c \cdot (-v \cdot w \cdot c \cdot d \cdot c + 0.5 \cdot f \cdot c \cdot (d23 - d12)) - w \cdot d \cdot x \cdot c \cdot f \cdot c \\
 v2k2 &= b \cdot (s \cdot g \cdot c \cdot d \cdot c + 0.5 \cdot a \cdot c \cdot (d24 - d25 - d26)) + g \cdot d \cdot p \cdot c \cdot a \cdot c \\
 n_{-}(j_1 + 44) &= a10 \cdot (vz1 + vz2 + vz3) + v1e2 \cdot (v2k2 - 0.5 \cdot d56 \cdot b \cdot a \cdot c) - v1k2 \cdot v2e2 \\
 za1(j_1 + 44) &= 0.5 \cdot c \cdot b \cdot d \cdot c \cdot (d13 \cdot (-b \cdot g \cdot c \cdot f \cdot c + 0.5 \cdot a \cdot c \cdot (a \cdot q \cdot c - b \cdot u \cdot c)) \\
 &\quad + d123 \cdot (-c \cdot w \cdot c \cdot a \cdot c - 0.5 \cdot f \cdot c \cdot (c \cdot o \cdot c - f \cdot q \cdot c))) \\
 n_{-}(j_1 + 44) &= n_{-}(j_1 + 44) + za1(j_1 + 44) \\
 \\
 za1(j_1 + 48) &= 0.5 \cdot c \cdot b \cdot d \cdot c \cdot q \cdot c \cdot (d23 \cdot b5 + 0.5 \cdot d234 \cdot (d26 - d36)) \\
 n_{-}(j_1 + 48) &= n_{-}(j_1 + 36) + za1(j_1 + 48) \\
 n_{-}(j_1 + 36) &= n_{-}(j_1 + 36) + zb2(j_1 + 36) \\
 n_{-}(\ell_1 + 52) &= -r \cdot p \cdot c \cdot b2 \cdot b2 \cdot a2 \\
 n_{-}(\ell_1 + 54) &= n_{-}(\ell_1 + 52) \\
 n_{-}(\ell_1 + 56) &= n_{-}(\ell_1 + 52) \\
 \\
 zb1(j_2 + 58) &= -0.5 \cdot d36 \cdot d12 \cdot c \cdot b \cdot (p \cdot c \cdot f \cdot c - q \cdot c \cdot d \cdot c) \\
 n_{-}(j_2 + 58) &= d36 \cdot x \cdot c \cdot (b \cdot b13 \cdot (-r \cdot g \cdot c + 0.5 \cdot c \cdot a \cdot c) \cdot c \cdot a11 \cdot a12 - b \cdot a \cdot c \cdot r \cdot a13) + zb1(j_2 + 58) \\
 n_{-}(j_2 + 60) &= d36 \cdot c \cdot g \cdot a \cdot c \cdot x \cdot c \cdot a12 \\
 n_{-}(j_2 + 62) &= d36 \cdot r \cdot x \cdot c \cdot a12 \cdot b2 \\
 n_{-}(j_2 + 64) &= -c \cdot a \cdot c \cdot x \cdot c \cdot g \cdot (d36 \cdot a12 + 0.5 \cdot b13 \cdot g \cdot c \cdot b) \\
 \\
 uz1 &= 0.5 \cdot c \cdot f \cdot c \cdot (0.5 \cdot b \cdot d \cdot c \cdot b19 + d \cdot c \cdot s \cdot (c \cdot v \cdot c + f \cdot x \cdot c) + b \cdot s \cdot c \cdot (u \cdot f \cdot c - r \cdot c \cdot c)) \\
 uz2 &= 0.5 \cdot b \cdot d \cdot c \cdot (f \cdot c \cdot w \cdot (-d \cdot x \cdot c + b \cdot u \cdot c) + c \cdot w \cdot c \cdot (r \cdot b \cdot c + v \cdot d \cdot c)) \\
 uz3 &= -b \cdot b \cdot f \cdot c \cdot f \cdot c \cdot w \cdot s \cdot c - c \cdot c \cdot d \cdot c \cdot d \cdot c \cdot s \cdot w \cdot c
 \end{aligned}$$

$$\begin{aligned}
 u1e5 &= -c \cdot (c \cdot wc \cdot ac + 0.5 \cdot fc \cdot (c \cdot oc - f \cdot qc)) \\
 u2e5 &= -b \cdot (b \cdot sc \cdot ac + 0.5 \cdot dc \cdot (b \cdot gc - d \cdot pc)) \\
 u2k5 &= -b \cdot (g \cdot sc \cdot ac + 0.5 \cdot dc \cdot b14) - s \cdot a \cdot pc \cdot dc \\
 u1k5 &= -c \cdot (o \cdot wc \cdot ac + 0.5 \cdot fc \cdot b15) - w \cdot a \cdot qc \cdot fc \\
 n_{(j_2 + 66)} &= -a3 \cdot (uz1 + uz2 + uz3) - u1e5 \cdot u2k5 + u2e5 \cdot u1k5 \\
 za2(j_2 + 66) &= -0.5 \cdot ac \cdot b \cdot c \cdot (d24 \cdot (c \cdot dc \cdot wc - 0.5 \cdot fc \cdot (c \cdot vc + f \cdot xc)) \\
 &\quad - d13 \cdot (b \cdot fc \cdot sc - 0.5 \cdot dc \cdot (b \cdot uc - d \cdot xc))) \\
 n_{(j_2 + 66)} &= n_{(j_2 + 66)} + za2(j_2 + 66) \\
 av1 &= -b \cdot b13 \cdot gc \cdot (r \cdot d36 + a16 \cdot b) \\
 av2 &= -g \cdot ac \cdot c \cdot c \cdot (oc \cdot b13 + ac \cdot a15) \\
 av3 &= 0.5 \cdot b \cdot ac \cdot (c \cdot (b16 \cdot b13 + a15 \cdot (o \cdot ac - r \cdot bc)) + a16 \cdot (a \cdot a12 - b \cdot a13)) \\
 za1(j_2 + 68) &= 0.5 \cdot xc \cdot b \cdot c \cdot ac \cdot (d123 \cdot b13 + 0.5 \cdot d12 \cdot (d56 - d46)) \\
 n_{(j_2 + 68)} &= xc \cdot (av1 + av2 + av3) + za1(j_2 + 68) \\
 n_{(71)} &= r \cdot xc \cdot b2 \cdot b2 \cdot a12 \\
 n_{(72)} &= -n_{(71)} \\
 n_{(43)} &= -n_{(43)} \\
 \\
 a1 &= b \cdot fc - r \cdot wc & a2 &= -d \cdot xc - a \cdot qc & a3 &= f \cdot qc + c \cdot oc \\
 a4 &= c \cdot ac + w \cdot qc & a5 &= -d \cdot xc - c \cdot wc & a6 &= d \cdot sc + c \cdot rc \\
 a7 &= s \cdot dc + r \cdot cc & a8 &= -b \cdot sc - c \cdot vc & a9 &= -b \cdot sc + f \cdot xc \\
 a10 &= -c \cdot vc + f \cdot xc & a11 &= u \cdot fc + s \cdot dc & a12 &= f \cdot qc + d \cdot pc \\
 a13 &= f \cdot uc + d \cdot sc & a14 &= d \cdot vc - b \cdot rc & a15 &= f \cdot wc + d \cdot vc \\
 a16 &= w \cdot fc + v \cdot dc & a17 &= 0.5 \cdot c \cdot ac + v \cdot pc & b3 &= 2 \cdot (d34 - d14) + d23 - d12 \\
 b1 &= d26 + d56 & b2 &= d36 + d46 & b4 &= d13 + d16 - d36 \\
 b5 &= d16 + d56 & b13 &= d16 + d26 & b14 &= d45 + d46 - d25 - d26 \\
 b15 &= d35 + d36 - d15 - d16 & b16 &= d35 + d36 + d56 - d34 & b19 &= d34 + d12 - d14 - d23 \\
 d136 &= d13 + d16 + d36 & d234 &= d24 + d23 + d34 & d123 &= d12 + d13 + d23
 \end{aligned}$$

Table 8

$i_s(j)$

82,46,84,48,17,18,88,47,90,52,19,20,
94,49,96,53,24,22,12,10,11, 6, 9, 5

Table 9

momentum labels for the helicity amplitudes of type A(+ + + - - +)

1,2,3,4,5,6, 1,2,4,3,5,6, 1,4,3,2,5,6,
4,2,3,1,5,6, 1,2,3,4,6,5, 1,2,4,3,6,5,
1,4,3,2,6,5, 4,2,3,1,6,5, 4,5,6,2,3,1,
4,5,6,1,2,3, 4,5,6,1,3,2, 1,6,5,2,4,3,
1,2,5,3,4,6, 1,2,3,5,6,4, 5,6,3,1,4,2

momentum labels for the helicity amplitudes of type A(+ + - - - +)

1,2,3,4,5,6, 1,2,3,4,6,5, 1,5,2,4,3,6,
1,4,2,5,3,6, 1,3,4,2,5,6, 1,2,4,5,6,3,
4,1,6,5,3,2, 1,3,2,5,6,4, 1,3,4,2,6,5,
1,4,2,3,6,5

Table 10

(a)

(b)

- Figure 1 -