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A SIMPLIFIED ALGORITHM FOR MOMENTUM ESTIMATION
IN MAGNETIC SPECTROMETERS

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ABSTRACT

The Wind quintic spline algorithm is conceptually and algebraically simplified by separating from each other the estimates of initial energy, direction, and position of the particle. The covariance matrix of the fitted parameters is explicitly evaluated. The stopping power is taken into account but multiple scattering and energy-loss fluctuations remain as yet neglected.

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1. INTRODUCTION

Among numerous methods of momentum estimation in non-focusing magnetic spectrometers [1,2], the 'quintic spline' algorithm of Wind [3] has enjoyed increasing popularity in recent years. It is a rapidly converging iterative procedure, originally conceived for air-gap spectrometers and particle energies such that energy loss and multiple scattering may be neglected. Subsequently, these restrictions have been removed [4] and the algorithm has been applied even in the extreme case of a magnetized iron spectrometer [5].

However, the Wind algorithm has remained a rather formidable tool for computerized data analysis. It is not clear how it responds to approximations which are nearly unavoidable when energy loss and multiple scattering are important. Also, it is not very suitable for a simple qualitative evaluation of spectrometer resolution. These shortcomings are due mainly to the fact that it estimates, at each step of the iterative procedure, the initial momentum of the particle and its initial position in a given plane by a five-dimensional least squares fit.

The purpose of the present paper is to simplify the Wind algorithm by estimating first the initial momentum magnitude, then its direction, and finally the initial position of the particle. The corresponding spectrometer resolution will also be calculated. From the beginning, the algorithm will be generalized to the (non-physical) case of continuous non-stochastic energy loss. In later publications (called P2), the results of the present paper will form the basis for the development of a more general algorithm taking into account multiple scattering and energy-loss fluctuations.

A simple algorithm, based on an early version [6,7] of the work described here and in P2, has been used for a number of years in the analysis of deep inelastic muon scattering data by the BCDMS Collaboration [8]. It has proved to be practical, but even more important perhaps is its transparency and usefulness for qualitative reasoning.

2. THE WIND ALGORITHM WITH CONTINUOUS ENERGY LOSS

Wind's algorithm uses a Cartesian coordinate system with the z-axis along a suitable direction (e.g. the mean direction of particles accepted by the spectrometer). Its fundamental equations (derived in Appendix A) are

$$\begin{aligned} x(z) &= x_0 + x'_0 (z-z_0) + K X(z,K) \\ y(z) &= y_0 + y'_0 (z-z_0) + K Y(z,K) . \end{aligned} \quad (2.1)$$

Here, $\vec{r} = (x, y, z)$ is the radius vector of the particle anywhere along its trajectory. The initial position of the particle is characterized by $\vec{r}_0 = (x_0, y_0, z_0)$ and its initial direction (i.e. the initial slope of its trajectory) by $x'_0 \equiv (dx/dz)_{z=z_0}$ and $y'_0 \equiv (dy/dz)_{z=z_0}$; $K = p_0^{-1}$ is the inverse of the initial momentum magnitude. The quantities $X(z, K)$ and $Y(z, K)$, which we shall call field integrals, are defined by

$$X(z, K) = \int_{z_0}^z dv \int_{z_0}^v du X''(u, K) \quad (2.2)$$

$$Y(z, K) = \int_{z_0}^z dv \int_{z_0}^v du Y''(u, K),$$

with

$$X''(z, K) = \frac{es'}{F(z, K)} \left[x'y'B_x - (1+x'^2)B_y + y'B_z \right] \quad (2.3)$$

$$Y''(z, K) = \frac{es'}{F(z, K)} \left[(1+y'^2)B_x - x'y'B_y - x'B_z \right] .$$

Here, e is the charge of the particle, $\vec{B} = (B_x, B_y, B_z)$ is the magnetic induction at the point $\vec{r}$, while $x' \equiv dx/dz$ and $y' \equiv dy/dz$ characterize the slope of the trajectory at that point. The other quantities in eqs. (2.3) are defined by

$$s' = (1 + x'^2 + y'^2)^{1/2} \quad (2.4)$$

$$F(z, K) = \left[1 - K^2 \int_{z_0}^z f s' dz (2E_0 - \int_{z_0}^z f s' dz) \right]^{1/2}. \quad (2.5)$$

Here, $E_0 = (K^{-2} + m^2)^{1/2}$ is the initial total energy of the particle with mass m , and f is the ideal continuous non-stochastic energy loss per unit length. In general, f is a function of K and of $s(z) = \int_{z_0}^z s' dz$. For simplicity, we shall call it 'stopping power'; the question as to which physical quantity should be considered as equivalent to f will be discussed in P2. It can easily be seen that $F(z, K)p_0$ is the local momentum magnitude of the particle at the point $\vec{r}$. For extremely relativistic (er) particles the momentum magnitude is equal to the total energy and $F(z, K)$ simplifies to $F_{er}(z, K)$ given by

$$F_{er}(z, K) = 1 - K \int_{z_0}^z f s' dz. \quad (2.5')$$

The Wind algorithm considers the z coordinates as fixed constants and, therefore, is most directly applicable to spectrometers with parallel detector planes perpendicular to the z -axis. The x and y coordinates of the points on the trajectory are random quantities, but Wind avoids the formidable problem of the random nature of $\vec{B}$, x' and y' which are functions of $\vec{r}$, in principle, by the following stratagem. At each step of the iterative procedure, the quantities $X''(z, K)$ and $Y''(z, K)$ are evaluated on the basis of a magnetic field map, of the knowledge of the stopping power f as a function of particle energy, and of the previously calculated trajectory and its characterizing parameters x_0 , y_0 , x'_0 , y'_0 , K . However, the computational effort is greatly reduced by evaluating $X''(z, K)$ only for a restricted number of z -values z_r : those corresponding to the z -coordinates of the detector planes and, if the latter are too sparse, a finite number of intermediate z -values (so-called inserts [9]). The functions $X''(z, K)$ and $Y''(z, K)$ are each simulated by a third-order cubic spline fit [10] to the finite set of calculated

values $X''(z_r, K)$ and $Y''(z_r, K)$. Equation (2.1) then yields the new description of the trajectory as two fifth-order (quintic) spline functions $x(z)$ and $y(z)$.

The new parameters x_0, y_0, x'_0, y'_0, K are obtained by a least squares fit of the values $x(z_i), y(z_i)$ as calculated from eq. (2.1) to the corresponding values x_i, y_i as measured by the detector planes. As long as the finite spatial resolution of detectors is the only source of randomness and as long as the x- and y-coordinates are measured independently of each other, diagonal coordinate error (covariance) matrices x_V and y_V are appropriate for this fit. Their matrix elements ${}^x_{V_{i,j}}$ and ${}^y_{V_{i,j}}$ are defined by, for example,

$${}^x_{V_{i,j}} = E \left[(x_i - x_{i,0})(x_j - x_{j,0}) \right] = \sigma_x^2 \delta_{i,j} \quad (2.6)$$

Here, the symbol E indicates the expectation value with respect to the N_x -dimensional distribution of measured values x_i , N_x being the number of measured x-coordinates of a trajectory; $\delta_{i,j}$ is the Kronecker symbol. The 'parent mean' [11] $x_{i,0}$ is the average x_i -value of a population of infinitely many trajectories of particles with identical sets of characteristic parameters x_0, y_0, x'_0, y'_0 , and K. As long as only random measuring errors are considered, $x_{i,0}$ is equal to the true value of the x-coordinate of the trajectory at $z = z_i$. However, when multiple scattering and energy loss fluctuations are treated in analogy to measuring errors, the two concepts -- parent mean and true value -- have to be distinguished.

Strictly speaking, the expectation values (2.6) do not exist if the distributions of measured coordinates around their parent means possess long tails, e.g. of Cauchy form. In practice, any such tails are very weak and the central parts of the distributions are Gaussian to a good approximation [12]. Then it is sensible to take the variance of the Gaussian as σ_x^2 and, if necessary, identify exceptionally deviating coordinates by statistical tests and reject them. In general, the variance σ_x^2 also depends on the angle of particle traversal through the detector, but often this angle and the effective thickness of detectors are small and the spatial

resolution is limited by other factors. The variance σ_x^2 can differ from detector to detector, but often identical detectors are used everywhere in the spectrometer so that $\sigma_x^2 = \sigma_y^2 = \text{const.}$ We shall follow Wind in making this simplifying assumption which, however, could be rather easily avoided.

Wind's algorithm can also be easily generalized to the situation (realized in the BCDMS spectrometer), where the detectors measuring the x-coordinates are displaced along the z-axis with respect to those measuring the y-coordinates. By coordinate transformations and introduction of non-diagonal error matrices, Wind's algorithm can be generalized to the case of non-parallel detector planes, non-independent coordinate measurements, and strongly curved or even returning trajectories [9]. On the other hand, the interpolation of the functions (2.3) by cubic splines leading to the representation of field integrals by quintic splines, though in general an advisable procedure, is sometimes unnecessarily complicated. If particles are only weakly bent in nearly homogeneous magnetic fields, lower-order splines or even low-order polynomials approximating the entire trajectory may be quite satisfactory, especially when multiple scattering makes the true trajectory wiggly anyway. Thus, calling Wind's algorithm the quintic spline method puts too much emphasis on one of its ancillary features. Rather, its outstanding characteristic is the implicit claim that the field integrals $X(z,K)$ and $Y(z,K)$ are subject to relatively much smaller random fluctuations than the measured coordinates x_i and y_i . This claim is justified in most practical situations even, in a certain sense, when multiple scattering and energy-loss fluctuations are important (see P2).

Wind's iterative procedure starts from a geometrical fit to the measured coordinates on the trajectory and a suitable guess of the parameter K . In numerical examples, Wind has shown that his iterative procedure converges very fast. With finite stopping power f , the normal equations arising from the least squares fit are non-linear and also have to be solved by iterative methods. The convergence of such methods is discussed in any textbook on numerical mathematics.

3. FIRST AND SECOND DIFFERENCES OF COORDINATES

In this section, we shall explicitly treat only the x-coordinate and define the corresponding symbols concerning the y-coordinate in table 1. As applied to the impact point in the i-th detector, the first of eqs. (2.1) reads

$$x(z_i) = x_0 + x'_0 (z_i - z_0) + K X_i \quad (3.1)$$

with

$$X_i \equiv X(z_i, K). \quad (3.2)$$

We define quantities $t(z_i)$ which we call 'pseudoslopes' by

$$t(z_i) \equiv \frac{x(z_{i+1}) - x(z_i)}{z_{i+1} - z_i} \quad (3.3)$$

and obtain from expression (3.1)

$$t(z_i) = x'_0 + K T_i, \quad (3.4)$$

with the field integrals T_i defined by

$$T_i \equiv \frac{X_{i+1} - X_i}{z_{i+1} - z_i}. \quad (3.5)$$

Similarly, we introduce 'pseudocurvatures' $a(z_i)$ by the definition

$$a(z_i) \equiv t(z_{i+1}) - t(z_i). \quad (3.6)$$

obtaining from eq. (3.4)

$$a(z_i) = K A_i \quad (3.7)$$

with the field integrals A_i defined by

$$A_i \equiv T_{i+1} - T_i. \quad (3.8)$$

Hopefully, our practice of indiscriminately calling X_i , T_i and A_i field integrals is not too confusing though T_i and A_i have the dimension of momentum while X_i has the dimension momentum $\times$ length.

From the definition (2.2) of X one easily obtains

$$A_i = \int_{z_i}^{z_{i+2}} dz H(z) \int_{z_0}^z du X''(u, K), \quad (3.9)$$

with

$$H(z) = \frac{1}{z_{i+2} - z_{i+1}} \quad \text{for} \quad z_{i+1} < z < z_{i+2} \quad (3.10)$$

$$H(z) = -\frac{1}{z_{i+1} - z_i} \quad \text{for} \quad z_i < z < z_{i+1}.$$

Since

$$\int_{z_i}^{z_{i+2}} H(z) dz = 0,$$

we can subtract a constant from the integrand of the outer integral in eq. (3.9), i.e. choose instead of z_0 another constant lower limit for the inner integral. With the choice of z_{i+1} for this limit we obtain

$$\begin{aligned} A_i &= \int_{z_i}^{z_{i+2}} dz H(z) \int_{z_{i+1}}^z du X''(u, K) = \\ &= \int_{z_i}^{z_{i+1}} \frac{dz}{z_{i+1} - z_i} \int_z^{z_{i+1}} du X''(u, K) + \int_{z_{i+1}}^{z_{i+2}} \frac{dz}{z_{i+2} - z_{i+1}} \int_{z_{i+1}}^z du X''(u, K). \end{aligned} \quad (3.11)$$

Expression (3.11) may seem more complicated than (3.9) but it has the advantage of yielding A_i as a sum of two terms of (normally) equal

sign, while expression (3.9) may easily lead to a subtraction of comparably large numbers -- an obvious drawback for numerical computations.

Once the quantities A_i have been computed for all i , the quantities T_i may be obtained from T_1 and the recursion relation

$$T_{i+1} = T_i + A_i, \quad (3.12)$$

while the quantities X_i may be obtained from X_1 and the recursion relation

$$X_{i+1} = X_i + (z_{i+1} - z_i) T_i. \quad (3.13)$$

Using the symbol correspondence of table 1, the y -coordinate analogues of eqs. (3.1) to (3.13) are obtained in a trivial way. If detectors measuring respectively the x - and y -coordinates are placed at different z -positions (as is the case, for example, in the BCDMS spectrometer), the indexing of quantities in table 1 has to be suitably generalized.

Equations (3.1), (3.4) and (3.7) may be reinterpreted in a way which is sometimes conceptually advantageous. They can be rewritten in the form

$$x_c(z_i) \equiv x(z_i) - K X_i = x_0 + x'_0(z_i - z_0) \quad (3.14)$$

$$t_c(z_i) \equiv t(z_i) - K T_i = x'_0 \quad (3.15)$$

$$a_c(z_i) \equiv a(z_i) - K A_i = 0. \quad (3.16)$$

We have defined 'corrected' coordinates $x_c(z_i)$, pseudoslopes $t_c(z_i)$ and pseudocurvatures $a_c(z_i)$, respectively. Equations (3.14) to (3.16) satisfied by these corrected quantities are those of a straight line. In other words, the knowledge of K and of the field integrals permits the 'straightening' of the observed trajectory.

Even with a diagonal coordinate error matrix of the form (2.6), the pseudoslope and pseudocurvature error matrices t_y and a_y are non-diagonal. Their matrix elements are listed in Appendix B.

4. EVALUATION OF THE CHARACTERISTIC PARAMETERS AND THEIR COVARIANCE MATRIX

On the basis of eqs. (3.1), (3.4), and (3.7), we define three x^2 quantities

$$\begin{aligned} x_x^2 = & \text{}^T [\underline{x} - x_0 \underline{1} - x_0' (\underline{z} - z_0 \underline{1}) - K\underline{x}] \text{}^x \underline{W} [\underline{x} - x_0 \underline{1} - x_0' (\underline{z} - z_0 \underline{1}) - K\underline{x}] + \\ & + \text{}^T [\underline{y} - y_0 \underline{1} - y_0' (\underline{z} - z_0 \underline{1}) - K\underline{y}] \text{}^y \underline{W} [\underline{y} - y_0 \underline{1} - y_0' (\underline{z} - z_0 \underline{1}) - K\underline{y}] \end{aligned} \quad (4.1)$$

$$\begin{aligned} x_t^2 = & \text{}^T (\underline{t} - x_0' \underline{1} - K\underline{T}) \text{}^t \underline{W} (\underline{t} - x_0' \underline{1} - K\underline{T}) + \\ & + \text{}^T (\underline{u} - y_0' \underline{1} - K\underline{U}) \text{}^u \underline{W} (\underline{u} - y_0' \underline{1} - K\underline{U}) \end{aligned} \quad (4.2)$$

$$x_a^2 = \text{}^T (\underline{a} - K\underline{A}) \text{}^a \underline{W} (\underline{a} - K\underline{A}) + \text{}^T (\underline{b} - K\underline{B}) \text{}^b \underline{W} (\underline{b} - K\underline{B}) . \quad (4.3)$$

Here, we have adopted a notation from matrix calculus designating the set of measured coordinates x_i for each track by the vector $\underline{x}$:

$$\underline{x} \equiv x_1, x_2, x_3, \dots, x_{N_x}.$$

The vector $\underline{x}$ has to be considered as a single-column matrix; the corresponding transposed vector is a single-row matrix $\text{}^T \underline{x}$. An analogous notation applies to all other vectors in expressions (4.1) to (4.3). One has to remember that the number of impact points observed in the (yz)-projection N_y is different from N_x , in general. The vectors $\underline{t}$ and $\underline{T}$ are $(N_x - 1)$ -dimensional, the vectors $\underline{a}$ and $\underline{A}$ are $(N_x - 2)$ -dimensional, etc. The dimensionality of the vectors $\underline{1} \equiv 1, 1, 1, \dots, 1$ depends in an obvious way on the context in which they are used. Any weight matrix $\underline{W}$ is the inverse of the corresponding error matrix $\underline{V}$. e.g.

$$\text{}^x \underline{W} = (\text{}^x \underline{V})^{-1}$$

and is quadratic with the same dimension as the corresponding vector. If any of the dimensions is smaller than 1, the corresponding term in the respective χ^2 quantity is absent. All error and weight matrices are symmetric.

The parameters to be determined by minimizing the corresponding χ^2 are:

$$\text{for } \chi_x^2 : x_0, y_0, x'_0, y'_0, K$$

$$\text{for } \chi_t^2 : x'_0, y'_0, K$$

$$\text{for } \chi_a^2 : K.$$

All five parameters can be estimated only if $N_x \geq 2$, $N_y \geq 2$, and $N_x + N_y \geq 5$; the number of degrees of freedom ν is the same for all three χ^2 quantities:

$$\nu = N_x + N_y - 5. \quad (4.4)$$

Intuitively, it is clear that the minimization of the three different χ^2 quantities (4.1) to (4.3) should lead to the same values of the parameters K , x'_0 , y'_0 . The situation is entirely analogous to the familiar χ^2 -fit to points expected to lie on a straight line. If one is not interested in the intercept of the straight line with a coordinate axis one can find its slope directly in a one-parameter fit to measured pseudo-slopes with a non-diagonal error matrix [13]. However, for the benefit of the mathematically minded sceptic we present a formal equivalence proof in Appendix C.

The minimization of χ_a^2 as given by eq. (4.3) leads to the normal equation

$${}^T(\underline{a}-K\underline{A}) \underline{a}_w \frac{d}{dK} (K\underline{A}) + {}^T(\underline{b}-K\underline{B}) \underline{b}_w \frac{d}{dK} (K\underline{B}) = 0 \quad \text{at } K = K_{\text{est}}. \quad (4.5)$$

If the energy loss may be neglected ($f = 0$), the vectors of field integrals $\underline{A}$ and $\underline{B}$ do not depend on K and the solution of the normal equation (4.5) is

$$K_{\text{est}} = \frac{T_{\underline{a}} \underline{a}_{\underline{W}} \underline{A} + T_{\underline{b}} \underline{b}_{\underline{W}} \underline{B}}{T_{\underline{A}} \underline{a}_{\underline{W}} \underline{A} + T_{\underline{B}} \underline{b}_{\underline{W}} \underline{B}} \quad (\text{for } f = 0) \quad (4.6)$$

In general, the normal equation (4.5) is non-linear and has to be solved by an appropriate method, e.g. the Newton-Raphson iteration method [14]. (Clearly, only one solution can be physically meaningful.) However, in any case, the vectors $\underline{a}$ and $\underline{b}$ appear linearly in the normal equation (4.5). This means that the different pseudocurvatures a_i and b_k contribute to the final estimate K_{est} with well-defined weights (which depend on K_{est}). We shall see in P2 that this is not the case for the most efficient energy estimate in the presence of multiple scattering.

In the analysis of the BCDMS experiments, Brun [5] has introduced a modification of eq. (4.5) by putting

$$\begin{aligned} \frac{d}{dK} (K\underline{A}) &\simeq \underline{A} \\ \frac{d}{dK} (K\underline{B}) &\simeq \underline{B} \end{aligned} \quad (4.7)$$

(or, rather, the equivalent of these approximations for the field integrals $\underline{X}$ and $\underline{Y}$). The resulting approximate normal equation

$$T(\underline{a} - K\underline{A}) \underline{a}_{\underline{W}} \underline{A} + T(\underline{b} - K\underline{B}) \underline{b}_{\underline{W}} \underline{B} = 0 \quad \text{at } K = K_{\text{est}} \quad (4.8)$$

is only apparently linear in K , since $\underline{A}$ and $\underline{B}$ are K -dependent. Thus, it has to be solved by iteration as well and one seems not to have gained anything by substituting it for eq. (4.5) but to have lost efficiency (i.e. spoiled the energy resolution). While the original motivation for introducing approximation (4.7) was the simplification of the five-dimensional minimization of χ_x^2 , eq. (4.8) turns out to be sensible for a quite different reason. Relatively to eq. (4.5) it assigns less weight to the end of the trajectory than to its beginning. This is apparent from

$$\frac{d}{dK} (K\underline{A}) = \underline{A} + K \frac{d\underline{A}}{dK}$$

and from an approximate evaluation of dA/dK on the basis of expressions (2.3), (2.5'), and (3.11), which shows that in all practically relevant cases dA_i/dK has the same sign as A_i but increases in magnitude relative to A_i with increasing index i .

Such a shift in relative weights of pseudocurvatures is justifiable on the following grounds. Either the energy loss is small in which case $K(dA/dK) \ll A$ and eq. (4.5) is practically equivalent to eq. (4.8) since, as can be shown in numerical examples, the efficiency of the energy estimate is then not very strongly dependent on the pseudocurvature weighting. Or else the energy loss is important (e.g. if the particle stops in the spectrometer). In this case, if energy loss were a non-stochastic process and the stopping power f a known function of energy, the end of the trajectory would indeed be much more valuable for the estimation of K than its beginning since its curvature there is larger. Equation (4.5) puts correspondingly large weights on the last pseudocurvatures of the trajectory. However, in reality, energy-loss fluctuations decrease this usefulness of the end of the trajectory to a considerable extent but, for several reasons, it is difficult to take them properly into account when estimating K . Therefore, an expedient such as replacing eq. (4.5) by eq. (4.8) is quite welcome (see P2).

Once the estimate K_{est} has been obtained, it is easy to estimate the remaining parameters x_0, y_0, x'_0, y'_0 . The minimization of χ_t^2 of eq. (4.2) leads to

$$\begin{aligned} x'_{0,est} &= \langle t \rangle - K_{est} \langle T \rangle \\ y'_{0,est} &= \langle u \rangle - K_{est} \langle U \rangle \end{aligned} \quad (4.9)$$

and the minimization of χ_x^2 of eq. (4.1) to

$$\begin{aligned} x_{0,est} &= \langle x \rangle - x'_{0,est} (\langle z \rangle_x - z_0) - K_{est} \langle X \rangle \\ y_{0,est} &= \langle y \rangle - y'_{0,est} (\langle z \rangle_y - z_0) - K_{est} \langle Y \rangle \end{aligned} \quad (4.10)$$

Here, the following notation has been introduced

$$\langle t \rangle = S_t^{-1} \begin{pmatrix} T_1 & t_W & t \end{pmatrix}$$

$$\langle T \rangle = S_t^{-1} \begin{pmatrix} T_1 & t_W & T \end{pmatrix}$$

$$S_t = \begin{pmatrix} T_1 & t_W & 1 \end{pmatrix}$$

$$\langle x \rangle = S_x^{-1} \begin{pmatrix} T_1 & x_W & x \end{pmatrix}$$

$$\langle X \rangle = S_x^{-1} \begin{pmatrix} T_1 & x_W & X \end{pmatrix}$$

$$\langle z \rangle_x = S_x^{-1} \begin{pmatrix} T_1 & x_W & z \end{pmatrix}$$

$$S_x = \begin{pmatrix} T_1 & x_W & 1 \end{pmatrix} ,$$

with completely analogous definitions for the quantities pertaining to the other projection of the trajectory, e.g.

$$\langle z \rangle_y = S_y^{-1} \begin{pmatrix} T_1 & y_W & z \end{pmatrix}.$$

Clearly, the quantity $\langle t \rangle$ may be interpreted as the properly averaged pseudoslope of the (xz)-projection of the trajectory; in this case, the relevant weight matrix is t_W . Analogous statements apply to the other quantities introduced above. Only in the case of $\langle z \rangle$ we have to specify the choice of the weight matrix (x_W or y_W) by the index x or y. In view of the definitions (3.14) and (3.15), expressions (4.9) and (4.10) may be interpreted in the following suggestive way: the best estimate of the initial particle direction is the average pseudoslope of the 'straightened' trajectory and the best estimate of the initial position is the intercept with the plane $z = z_0$ of the straight line fit to the 'straightened' trajectory.

Of course, one has to remember that the χ^2 minimizations take place at each step of the Wind iteration which is needed, in principle, to find the best description of the trajectory and thus the best values of the field integrals.

Standard statistical methods [14] lead to an evaluation of the covariance matrix $\underline{V}$ of the estimated parameters. Defining the weight matrix $\underline{W}$ by $\underline{W} \equiv \underline{V}^{-1}$, the matrix element $W_{p,r}$ related to the parameters p and r is obtained by

$$W_{p,r} = \frac{1}{2} E \left(\frac{\partial^2 \chi^2}{\partial p \partial r} \right). \quad (4.11)$$

Here, the derivative has to be taken at the true values of all parameters. The successful separation of energy, direction, and position estimation greatly facilitates also the inversion of the weight matrix $\underline{W}$. First, from eq. (4.3) $V_{K,K}$ is obtained as

$$V_{K,K} = \left[\left[\frac{d}{dK} \right]^T (KA) \right] a_W \frac{d}{dK} (KA) + \left[\frac{d}{dK} \right]^T (KB) \right] b_W \frac{d}{dK} (KB) \right]^{-1}. \quad (4.12)$$

Then, the elements of $\underline{W}$ related to x'_0, y'_0, K are obtained from eq. (4.2) and the corresponding elements of $\underline{V}$ are easily evaluated using the fact that $V_{K,K}$ is already known from eq. (4.12). Finally, the elements of $\underline{W}$ related to x_0, y_0, x'_0, y'_0, K are obtained from eq. (4.1). These, together with the previously evaluated part of $\underline{V}$, permit the calculation of the remaining elements of $\underline{V}$ leading to the final result given in table 2, where only one out of any two equal elements of the symmetric matrix $\underline{V}$ is listed.

We see that the matrix $\underline{V}$ is remarkably simple. It is the sum of a covariance matrix of the parameters x_0, y_0, x'_0, y'_0 as pertaining to trajectories in the absence of magnetic field and a covariance matrix proportional to $V_{K,K}$. The latter describes the energy variance and its effect on the resolution of any physical quantity which is a function of the characteristic parameters.

If, instead of the normal equations (4.5), eq. (4.8) is used for the energy estimate, the covariance matrix of the fitted parameters $\underline{V}$ is modified. However, this problem can only be treated in P2 since, whenever the difference between eqs. (4.5) and (4.8) is important, so are the energy-loss fluctuations.

5. CONCLUSIONS

The popular Wind 'quintic spline' algorithm has been simplified by considering, apart from impact-point coordinates, also their first and second differences. The resulting expressions for the estimates of the characteristic parameters -- the initial three-momentum and the initial position of the particle -- and for their covariance matrix are indeed simple, as evidenced by eqs. (4.5), (4.9), (4.10) and (4.12), and table 2, even if a finite stopping power is taken into account. Wind's iterative procedure is still necessary, in general, to find the optimal trajectory and the optimal values of the field integrals. However, we have pointed out that the quintic spline method is not the most essential ingredient of the algorithm. Rather, we discern the essence of the algorithm in its implicit assumption that the field integrals are error-free. This aspect will become even more important in subsequent articles, to be published, which will treat the conceptually and algebraically more complicated problems of multiple scattering and energy-loss fluctuations.

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Table 1

Correspondence between symbols used to designate coordinates, pseudoslopes, pseudocurvatures, the corresponding field integrals, and the number of measured coordinates, in the two projections of the trajectory, respectively.

(xz)-plane :	x	t	a	X	T	A	N_x
(yz)-plane :	y	u	b	Y	U	B	N_y

Table 2

Covariance matrix of the fitted parameters

	x_0	y_0	x'_0	y'_0	K
x_0	$S_x^{-1} + (\langle z \rangle_x - z_0)^2 S_t^{-1} + C_x^2 V_{x,K,K}$	$C_x C_y V_{K,K}$	$-S_t^{-1} (\langle z \rangle_x - z_0) - C_x \langle \frac{d(KT)}{dK} \rangle V_{K,K}$	$-C_x \langle \frac{d(KU)}{dK} \rangle V_{K,K}$	$C_x^V V_{K,K}$
y_0		$S_y^{-1} + (\langle z \rangle_y - z_0)^2 S_u^{-1} + C_y^2 V_{y,K,K}$	$-C_y \langle \frac{d(KT)}{dK} \rangle V_{K,K}$	$-S_u^{-1} (\langle z \rangle_y - z_0) - C_y \langle \frac{d(KU)}{dK} \rangle V_{K,K}$	$C_y^V V_{K,K}$
x'_0			$S_t^{-1} + \langle \frac{d(KT)}{dK} \rangle^2 V_{K,K}$	$\langle \frac{d(KT)}{dK} \rangle \langle \frac{d(KU)}{dK} \rangle V_{K,K}$	$-\langle \frac{d(KT)}{dK} \rangle V_{K,K}$
y'_0				$S_u^{-1} + \langle \frac{d(KU)}{dK} \rangle^2 V_{K,K}$	$-\langle \frac{d(KU)}{dK} \rangle V_{K,K}$
K					$V_{K,K}$

with $C_x = (\langle z \rangle_x - z_0) \langle \frac{d(KT)}{dK} \rangle - \langle \frac{d(KX)}{dK} \rangle$

$C_y = (\langle z \rangle_y - z_0) \langle \frac{d(KU)}{dK} \rangle - \langle \frac{d(KY)}{dK} \rangle$

$\langle \frac{d(KX)}{dK} \rangle = S_x^{-1} (1_x^T x_w \frac{d(KX)}{dK})$, etc.

DERIVATION OF WIND'S FORMULAE IN THE PRESENCE OF CONTINUOUS ENERGY LOSS

Let the particle motion be described by the radius vector $\vec{r}(t)$ which is a function of time t , and let us call the path length along the trajectory s . The fact that the particle velocity $\vec{v} \equiv d\vec{r}/dt$ is tangential to the trajectory is expressed by the equations

$$\begin{aligned} x' &\equiv \frac{dx}{dz} = \frac{v_x}{v_z} \\ y' &\equiv \frac{dy}{dz} = \frac{v_y}{v_z} \end{aligned} \quad (A.1)$$

while the fundamental metric equation

$$ds^2 = dx^2 + dy^2 + dz^2$$

can also be written in the form

$$s' \equiv \frac{ds}{dz} = (x'^2 + y'^2 + 1)^{1/2} \quad (A.2)$$

Equations (A.1) and (A.2) yield

$$s' = \frac{(v_x^2 + v_y^2 + v_z^2)^{1/2}}{v_z} = \frac{v}{v_z} \quad (A.3)$$

Differentiating eqs. (A.1) with respect to z and taking into account eq. (A.3), we obtain

$$\begin{aligned} x'' &\equiv \frac{d^2x}{dz^2} = \frac{1}{v_z} \left(v'_x - \frac{v_x v'_z}{v_z} \right) = \frac{s'}{v} (v'_x - x' v'_z) \\ y'' &= \frac{s'}{v} (v'_y - y' v'_z) \end{aligned} \quad (A.4)$$

The equation of motion of the particle with three-momentum $\vec{p}$ and total energy E , i.e. velocity $\vec{v} = \vec{p}/E$ (we use $c = 1$ throughout), is

$$\frac{d\vec{p}}{dt} = e(\vec{v} \times \mathbf{B}) - f \frac{\vec{v}}{v} \quad (A.5)$$

where e stands for the charge of the particle and $\vec{B}$ is the magnetic field. The second term on the right-hand side of eq. (A.5) is a 'friction' term due to the continuous energy loss. Scalar multiplication with $\vec{v}$ leads to

$$\vec{v} \cdot \frac{d\vec{p}}{dt} = \frac{1}{2E} \frac{dp^2}{dt} = \frac{dE}{dt} = -fv \quad (\text{A.6a})$$

or

$$\frac{dE}{ds} = -f \quad (\text{A.7})$$

We see that f is the absolute value of the 'stopping power', i.e. of the continuous energy loss per unit path length. It follows that

$$\frac{d\vec{p}}{dt} = v \left(E \frac{d\vec{v}}{ds} + \vec{v} \frac{dE}{ds} \right) = \frac{v}{s'} E \vec{v}' - v f \vec{v}.$$

Equations (A.5) and (A.7) lead to

$$\vec{v}' = \frac{s'}{Ev} \left[e(\vec{v} \times \vec{B}) - \frac{f}{v} (1 - v^2) \vec{v} \right]. \quad (\text{A.8})$$

Substituting eq. (A.8) into eq. (A.4) yields

$$\begin{aligned} x'' &= \frac{s'^2}{Ev^2} \left[e(\vec{v} \times \vec{B})_x - \frac{f}{v} (1 - v^2) v_x - x' e (\vec{v} \times \vec{B})_z \right. \\ &\quad \left. + x' \frac{f}{v} (1 - v^2) v_z \right] = \frac{es'^2}{Ev^2} \left[(\vec{v} \times \vec{B})_x - x' (\vec{v} \times \vec{B})_z \right] \\ y'' &= \frac{es'^2}{Ev^2} \left[(\vec{v} \times \vec{B})_y - y' (\vec{v} \times \vec{B})_z \right], \end{aligned}$$

i.e.

$$\begin{aligned} x'' &= \frac{es'}{p} \left[x' y' B_x - (1 + x'^2) B_y + y' B_z \right] \\ y'' &= \frac{es'}{p} \left[(1 + y'^2) B_x - x' y' B_y - x' B_z \right]. \end{aligned} \quad (\text{A.9})$$

The fact that the stopping power f does not appear explicitly in eq. (A.9) may seem remarkable. Actually, it follows simply from the fact that the local curvature of the trajectory is determined by the local magnetic field and the local magnitude of the particle momentum p , irrespective of the stopping power.

Let the particle have the energy E_0 , momentum $\vec{p}_0$, and velocity $\vec{v}_0$ at the starting point of the trajectory $\vec{r}_0 = (x_0, y_0, z_0)$. Then the trajectory has the initial slopes

$$x'_0 = \frac{v_0 x}{v_0 z}, \quad y'_0 = \frac{v_0 y}{v_0 z}$$

in the (x,z) and (y,z) projection, respectively. Designating the inverse of the initial momentum magnitude p_0 by K , i.e.

$$K \equiv p_0^{-1} \quad (\text{A.10})$$

the local momentum magnitude p may be written in the form

$$p = p_0 F(z,K) \quad (\text{A.11})$$

with

$$F(z,K) = \left[1 - K^2 \int_{z_0}^z f s' dz \left(2E_0 - \int_{z_0}^z f s' dz \right) \right]^{1/2} \quad (\text{A.12})$$

In the extreme relativistic limit $v = c$, $E = p$, eq. (A.12) simplifies to

$$F_{\text{er}}(z,K) = 1 - K \int_{z_0}^z f s' dz. \quad (\text{A.12}')$$

The integral form of eq. (A.9) is

$$\begin{aligned} x(z) &= x_0 + x'_0 (z-z_0) + K X \\ y(z) &= y_0 + y'_0 (z-z_0) + K Y, \end{aligned} \quad (\text{A.13})$$

where the quantities X and Y depend on z and K , and are given by

$$X = \int_{z_0}^z dv \int_{z_0}^v du X''(u, K) \quad (A.14)$$

$$Y = \int_{z_0}^z dv \int_{z_0}^v du Y''(u, K)$$

with

$$X''(z, K) = \frac{es'}{F(z, K)} \left[x'y'B_x - (1+x'^2) B_y + y'B_z \right] \quad (A.15)$$

$$Y''(z, K) = \frac{es'}{F(z, K)} \left[(1+y'^2) B_x - x'y'B_y - x'B_z \right] .$$

CALCULATION OF ERROR MATRICES

Defining

$$\Delta_i \equiv z_{i+1} - z_i \quad (\Delta_i > 0) \quad (B.1)$$

we obtain from the definition (3.3) of the pseudoslope t_i the general formula

$$t_{V_{i,j}} = \frac{x_{V_{i+1,j+1}} - x_{V_{i+1,j}} - x_{V_{i,j+1}} + x_{V_{i,j}}}{\Delta_i \Delta_j} \quad (B.2)$$

From the definition (2.6) of the pseudocurvature a_i follows the general formula

$$a_{V_{i,j}} = t_{V_{i+1,j+1}} - t_{V_{i+1,j}} - t_{V_{i,j+1}} + t_{V_{i,j}} \quad (B.3)$$

If, in particular, the matrix x_V is assumed to be a multiple of the unit matrix, i.e.

$$x_{V_{i,j}} = \sigma^2 \delta_{i,j} \quad (B.4)$$

we obtain from eqs. (B.1-4)

$$t_{V_{i,i}} = 2 \frac{\sigma^2}{\Delta_i^2} \quad (B.5)$$

$$t_{V_{i,i+1}} = - \frac{\sigma^2}{\Delta_i \Delta_{i+1}} \quad (B.6)$$

$$t_{V_{i,j}} = 0 \quad \text{for } j > i + 2 \quad (B.7)$$

$$a_{V_{i,i}} = 2\sigma^2 \left(\frac{1}{\Delta_i^2} + \frac{1}{\Delta_i \Delta_{i+1}} + \frac{1}{\Delta_{i+1}^2} \right) \quad (B.8)$$

$$a_{V_{i,i+1}} = -\sigma^2 \left(\frac{1}{\Delta_i \Delta_{i+1}} + \frac{2}{\Delta_{i+1}^2} + \frac{1}{\Delta_{i+1} \Delta_{i+2}} \right) \quad (\text{B.9})$$

$$a_{V_{i,i+2}} = \frac{\sigma^2}{\Delta_{i+1} \Delta_{i+2}} \quad (\text{B.10})$$

$$a_{V_{i,j}} = 0 \quad \text{for} \quad j \geq i + 3. \quad (\text{B.11})$$

The remaining elements of error matrices x_v , t_v and a_v are defined by the symmetry requirements $V_{i,j} = V_{j,i}$.

EQUIVALENCE OF PARAMETERS AND THEIR COVARIANCES
AS EVALUATED FROM DIFFERENT χ^2 QUANTITIES

Without loss of generality we limit ourselves to trajectories in the (xz)-plane. We consider only the transition from χ_t^2 to χ_a^2 , leaving the algebraically more complicated but basically similar case of the transition from χ_x^2 to χ_t^2 to the interested reader. We define the vector $\underline{c}$ by

$$\begin{aligned} c_1 &= x'_0 + K T_1 \\ c_{i+1} &= K A_i \quad \text{for } 1 \leq i \leq N_x - 2 \end{aligned} \quad (C.1)$$

and the vector $\underline{v}$ by

$$\begin{aligned} v_1 &= t_1 - c_1 \\ v_{i+1} &= a_i - c_{i+1} \quad \text{for } 1 \leq i \leq N_x - 2. \end{aligned} \quad (C.2)$$

The requirement $\underline{v} = \underline{L} (\underline{t} - x'_0 \underline{1} - K \underline{T})$ defines an $[(N_x - 1) \times (N_x - 1)]$ -dimensional non-singular matrix $\underline{L}$. Explicitly one finds

$$\begin{aligned} L_{i,i} &= 1 \quad \text{for } 1 \leq i \leq N_x - 1 \\ L_{i,i-1} &= -1 \quad \text{for } 2 \leq i \leq N_x - 1 \\ L_{i,j} &= 0 \quad \text{for } j < i - 1 \text{ or } j > i. \end{aligned} \quad (C.3)$$

The inverse matrix $\underline{M} = \underline{L}^{-1}$ satisfies $\underline{M}\underline{L} = \underline{L}\underline{M} = \underline{1}$, $\underline{1}$ being the unit matrix. Explicitly one finds

$$\begin{aligned} M_{i,j} &= 1 \quad \text{for } j \leq i \\ M_{i,j} &= 0 \quad \text{for } j > i. \end{aligned} \quad (C.4)$$

Then we obtain

$$\begin{aligned}
x_t^2 &= \underline{T} (\underline{t} - \underline{x}'_0 \underline{1} - \underline{KT}) \underline{t}_{\underline{W}} (\underline{t} - \underline{x}'_0 \underline{1} - \underline{KT}) = \\
&= \underline{T} (\underline{t} - \underline{x}'_0 \underline{1} - \underline{KT}) \underline{T}_{\underline{L}} \underline{T}_{\underline{M}} \underline{t}_{\underline{W}} \underline{M} \underline{L} (\underline{t} - \underline{x}'_0 \underline{1} - \underline{KT}) \\
&= \underline{T}_{\underline{V}} \underline{T}_{\underline{M}} \underline{t}_{\underline{W}} \underline{M} \underline{V} .
\end{aligned}$$

Defining the symmetric matrices $\underline{Z}$ and $\underline{U}$ by

$$\underline{Z} \equiv \underline{T}_{\underline{M}} \underline{t}_{\underline{W}} \underline{M} \quad (C.5)$$

$$\underline{U} \equiv \underline{Z}^{-1} = \underline{L} \underline{t}_{\underline{V}} \underline{T}_{\underline{L}} , \quad (C.6)$$

we have

$$x_t^2 = \underline{T}_{\underline{V}} \underline{Z} \underline{V} = \underline{T}_{\underline{V}} \underline{U}^{-1} \underline{V} \quad (C.7)$$

Explicitly we obtain

$$\begin{aligned}
U_{i,j} &= \sum_{k=1}^{N_x-1} \sum_{l=1}^{N_x-1} L_{i,k} t_{V_{k,l}} T_{L_{1,j}} = L_{i,i} t_{V_{i,j}} T_{L_{j,j}} + \\
&+ L_{i,i-1} t_{V_{i-1,j}} T_{L_{j,j}} + L_{i,i} t_{V_{i,j-1}} T_{L_{j-1,j}} + \\
&+ L_{i,i-1} t_{V_{i-1,j-1}} T_{L_{j-1,j}} = t_{V_{i,j}} - t_{V_{i-1,j}} - t_{V_{i,j-1}} + t_{V_{i-1,j-1}}
\end{aligned}$$

or

$$\begin{aligned}
U_{1,1} &= t_{V_{1,1}} \\
U_{1,i} &= t_{V_{1,i}} - t_{V_{1,i-1}} \quad \text{for } i \neq 1 \\
U_{i,j} &= t_{V_{i-1,j-1}} \quad \text{for } i \neq 1, j \neq 1 .
\end{aligned} \quad (C.8)$$

Equation (C.7) can be written in the form

$$\begin{aligned}
x_t^2 &= z_{1,1} v_1^2 + 2v_1 \sum_{j=2}^{N_x-1} z_{1,j} v_j + \sum_{i=2}^{N_x-1} \sum_{j=2}^{N_x-1} z_{i,j} v_i v_j = \\
&= z_{1,1} \left[v_1 + \sum_{j=2}^{N_x-1} \frac{z_{1,j}}{z_{1,1}} v_j \right]^2 + \sum_{i=2}^{N_x-1} \sum_{j=2}^{N_x-1} \frac{z_{i,j} z_{1,1} - z_{i,1} z_{1,j}}{z_{1,1}} v_i v_j.
\end{aligned} \tag{C.9}$$

We can prove that for $i \geq 2, j \geq 2$

$$\frac{z_{i,j} z_{1,1} - z_{i,1} z_{1,j}}{z_{1,1}} = a_{w_{i-1,j-1}}. \tag{C.10}$$

The proof consists in showing that the matrix a_w defined by (C.10) is indeed the inverse of the error matrix a_v . In view of eq. (C.6) we have

$$\sum_{k=1}^{N_x-1} z_{i,k} u_{k,j} = z_{i,1} u_{1,j} + \sum_{k=2}^{N_x-1} z_{i,k} u_{k,j} = \delta_{i,j},$$

i.e.

$$\sum_{k=2}^{N_x-1} z_{i,k} u_{k,j} = \delta_{i,j} - z_{i,1} u_{1,j}. \tag{C.11}$$

In view of eqs. (C.6), (C.8), (C.10), and (C.11) we obtain

$$\begin{aligned}
\sum_{k=1}^{N_x-2} a_{w_{i,k}} a_{v_{k,j}} &= \sum_{k=1}^{N_x-2} \frac{z_{i+1,k+1} z_{1,1} - z_{i+1,1} z_{1,k+1}}{z_{1,1}} u_{k+1,j+1} \\
&= \delta_{i+1,j+1} - z_{i+1,1} u_{1,j+1} + z_{i+1,1} u_{1,j+1} = \delta_{i,j}.
\end{aligned}$$

Q.E.D.

Introducing

$$w_1 = v_1 + \sum_{j=2}^{N_x-1} \frac{z_{1,j}}{z_{1,1}} v_j \quad (C.12)$$

and considering the definitions (C.1) and (C.2), we can write (C.9) in the form

$$x_t^2 = z_{1,1} w_1^2 + x_a^2. \quad (C.13)$$

We have

$$\begin{aligned} \frac{1}{2} \frac{\partial x_t^2}{\partial x_0'} &= -z_{1,1} w_1 \\ \frac{1}{2} \frac{\partial x_t^2}{\partial K} &= z_{1,1} w_1 \frac{\partial w_1}{\partial K} + \frac{1}{2} \frac{dx_a^2}{dK}. \end{aligned} \quad (C.14)$$

Minimizing x_t^2 with respect to variations of x_0' requires

$$w_1 = 0$$

resulting in

$$\frac{\partial x_t^2}{\partial K} = \frac{dx_a^2}{dK} \quad \text{at } x_0' = x_{0,est}' \quad (C.15)$$

Therefore, the values of K_{est} as obtained from the minimization of x_t^2 and x_a^2 are the same.

Furthermore, since $E(w_1) = 0$, we have

$$\begin{aligned} E \left[\frac{1}{2} \frac{\partial^2 x_t^2}{\partial x_0'^2} \right] &= z_{1,1} \\ E \left[\frac{1}{2} \frac{\partial^2 x_t^2}{\partial x_0' \partial K} \right] &= -z_{1,1} \frac{\partial w_1}{\partial K} \\ E \left[\frac{1}{2} \frac{\partial^2 x_t^2}{\partial K^2} \right] &= z_{1,1} \left[\frac{\partial w_1}{\partial K} \right]^2 + \frac{1}{2} E \left[\frac{d^2 x_a^2}{dK^2} \right]. \end{aligned} \quad (C.16)$$

These quantities are the elements of the weight matrix $\underline{W}$ of the fitted parameters x'_0 and K . The elements of their error matrix $\underline{V} = \underline{W}^{-1}$ are

$$\begin{aligned}
 V_{K,K} &= \left[\frac{1}{2} E \left(\frac{d^2 \chi_a^2}{dK^2} \right) \right]^{-1} \\
 V_{K,x'_0} &= \frac{\partial w_1}{\partial K} V_{K,K} = - \left[\frac{d(KT_1)}{dK} + \sum_{j=2}^{N_x-1} \frac{z_{1,j}}{z_{1,1}} \frac{d(KA_{j-1})}{dK} \right] V_{KK} = \\
 &= - \left\langle \frac{d(KT)}{dK} \right\rangle V_{K,K} \tag{C.17}
 \end{aligned}$$

$$V_{x'_0, x'_0} = \frac{1}{z_{1,1}} + \left\langle \frac{d(KT)}{dK} \right\rangle^2 V_{K,K} = S_t^{-1} + \left\langle \frac{d(KT)}{dK} \right\rangle^2 V_{K,K} .$$

The first of eqs. (C.17) proves the equivalence of the estimates $V_{K,K}$ from χ_t^2 and χ_a^2 , while the other two reproduce the result listed in table 2. Equations (C.4) and (C.5) have been used for their derivation.