THE $\overline{MS}$ AND ON-SHELL RENORMALIZATION SCHEMES IN THE ANALYSIS OF
NEUTRINO SCATTERING EXPERIMENTS

Robin G. Stuart

CERN - Geneva

ABSTRACT

The relationship between the $\overline{MS}$ and on-shell renormalization schemes is discussed and the correction, for finite top quark mass, to the formula connecting $\sin^2\theta_W = 1 - M_W^2/M_Z^2$ and $\sin^2\hat{\theta}_W(M_W)$ is given. A table is presented to allow easy conversion. The relative sensitivity, to the top quark and Higgs masses, of the two definitions, when extracted from semi-leptonic neutrino scattering experiments is considered.

Introduction

At the present time the most precise measurements of $\sin^2\theta_W$ come from semi-leptonic neutrino scattering experiments. With the advent of the new generation of e^+e^- machines (SLC and LEP) the best value of $\sin^2\theta_W$, within the context of the standard model, will be calculated from the measured M_Z , G_μ and α . Then the results of neutrino scattering experiments will provide an important test of the standard model if electroweak radiative corrections are properly taken into account. It is the purpose of this paper to describe some of the properties of these corrections and, in particular, the relationship between $\sin^2\theta_W=1-M_W^2/M_Z^2$ and $\sin^2\hat{\theta}_W(\mu)$ as defined in the $\overline{\text{MS}}$ scheme. For historical reasons certain groups have implemented programs which extract $\sin^2\theta_W$ from data and others $\sin^2\hat{\theta}_W(\mu=M_W)$. Each is equally valid although the former is relatively insensitive to the top quark and Higgs masses. The conversion between the two definitions has been the source of some confusion. A table is provided to allow ready conversion in the future.

The layout of the paper is as follows. In section 1, the relationship between $\sin^2\theta_W$ and $\sin^2\hat{\theta}_W(\mu)$ is derived and a formula for the correction from finite top quark mass effects is given.

In section 2 the the leading top quark and Higgs mass dependence is examined for $\sin^2\theta_W$ and $\sin^2\hat{\theta}_W(M_W)$ as extracted from fixed experimental data.

1. Relationship between $\overline{\text{MS}}$ and on-shell renormalization schemes

In their calculation of radiative corrections to lepton scattering, Wheater and Llewellyn-Smith [1] used the $\overline{\text{MS}}$ scheme. In this scheme the counter-terms at one loop contain only infinite parts which leads to the property that the Ward identities are automatically satisfied. In practice this is not such an advantage since physical and renormalized quantities are, in general, different and one must convert between the two in order to obtain a measurable prediction. On the other hand, in the on-shell scheme, physical and renormalized quantities are arranged to be equal and one must check that the renormalization conditions are consistent with the Ward identities or impose them explicitly. Having done this at the outset, all subsequent calculations, directly yield physically measurable predictions.

Throughout what follows, Sirlin [2] will be followed by defining

$$\sin^2 \theta_w = 1 - \frac{M_W^2}{M_Z^2} \quad (1.1)$$

where M_W and M_Z are the physical masses of the vector bosons. In the $\overline{\text{MS}}$ renormalization scheme one defines,

$$\sin^2 \hat{\theta}_w(\mu) = 1 - \frac{M_W^2(\mu)}{M_Z^2(\mu)} \quad (1.2)$$

where $M_W(\mu)$ and $M_Z(\mu)$ are the renormalized masses and μ is an arbitrary scale that arises naturally in dimensional regularization [3] It can be shown that this definition is equivalent to the ratio of the couplings,

$$\sin^2 \hat{\theta}_w(\mu) = \frac{e^2(\mu)}{g^2(\mu)} \quad (1.3)$$

In order to find a relation between the two, the bare (W or Z) boson mass squared is written

$$\begin{aligned} (M^0)^2 &= M^2(\mu) + \delta \hat{M}^2 && \overline{\text{MS}} \text{ Scheme} \\ &= M^2 + \delta M^2(\mu) && \text{On-shell scheme} \end{aligned} \quad (1.4)$$

Here $M^2(\mu)$ is the renormalized mass in the $\overline{\text{MS}}$ scheme

$\delta \hat{M}^2$ is the $\overline{\text{MS}}$ counter-term that to one-loop contains only an infinite part

M^2 is the physical boson mass

In the on-shell scheme one finds that to $O(\alpha)$ in the Euclidean metric,

$$\delta M^2(\mu) = \text{Real } \Pi(q^2 = -M^2, \mu) \quad (1.5)$$

in which Π is the one particle irreducible (1PI) self energy evaluated at the physical mass. It follows immediately that

$$M^2(\mu) = M^2 + [\delta M^2(\mu) - \delta \hat{M}^2] \quad (1.6)$$

where $[\delta M^2(\mu) - \delta \hat{M}^2]$

is a finite quantity. Applying (1.6) to (1.2) one then obtains to $O(\alpha)$,

$$\sin^2 \theta_w = \sin^2 \hat{\theta}_w(\mu) \left\{ 1 + \frac{\cos^2 \theta_w}{\sin^2 \theta_w} \text{Real} \left[\frac{\hat{\Pi}_{ww}(-M_w^2, \mu)}{M_w^2} - \frac{\hat{\Pi}_{zz}(-M_z^2, \mu)}{M_z^2} \right] \right\} \quad (1.7)$$

Here $\hat{\Pi}$ denotes the finite part of Π . Sarantakos et al. [4] have also given an expression relating $\sin^2 \theta_w$ and $\sin^2 \hat{\theta}_w(M_w)$. Explicit substitution of the expressions (A11) and (B9) given by Marciano and Sirlin [5] for the 1PI self energies into (1.7) yields the relation (9) of reference [4] directly. It is important to notice that their relation is valid only for a top quark of $m_t=0$. For $\sin^2 \theta_w=0.215$, $M_Z=M_H$ the authors correctly state that

$$\frac{\sin^2 \theta_w}{\sin^2 \hat{\theta}_w(M_w)} = 1.0059$$

The ratio however is quite sensitive to the exact value of m_t . For example for the same values of the parameters except that $m_t=40\text{GeV}$ the ratio is 1.0088; for $m_t=60\text{GeV}$, 1.0108 and for $m_t=240\text{GeV}$ it is 0.9326. In order to correct the relation for finite top quark mass effects, note that the fermionic contributions to the 1PI vector boson self energies are,

$$\begin{aligned}\Pi_{ww}^{(f)}(q^2, \mu) &= \frac{g^2}{16\pi^2} \sum_D \left\{ 2q^2 B_3(q; m_1^D, m_2^D) \right. \\ &\quad \left. + (m_1^D)^2 B_1(q; m_2^D, m_1^D) + (m_2^D)^2 B_1(q; m_1^D, m_2^D) \right\} \\ \Pi_{zz}^{(f)}(q^2, \mu) &= \frac{g^2}{16\pi^2} \frac{1}{2C_\theta^2} \sum_f \left\{ [1 + (2t_{3,f} - 4Q_f s_\theta^2)^2] q^2 B_3(q; m_f, m_f) \right. \\ &\quad \left. - m_f^2 B_0(q; m_f, m_f) \right\}\end{aligned}\quad (1.8)$$

where the B form-factors are defined by Passarino and Veltman [6] and $B_3=B_{21}+B_1$. $\sum_f$ sums over fermions and $\sum_D$ sums over isospin doublets with masses m_1^D and m_2^D of the $t_3=+1/2, -1/2$ fermions respectively. From this it may be deduced that each massless ($m_f \ll M_W$) generation gives a contribution ($s_\theta \equiv \sin\theta_W, c_\theta \equiv \cos\theta_W$)

$$\frac{\hat{\Pi}_{ww}^{(f)}(-M_W^2, M_W)}{M_W^2} - \frac{\hat{\Pi}_{zz}^{(f)}(-M_Z^2, M_W)}{M_Z^2} = \frac{g^2}{16\pi^2} \frac{4}{3} \left\{ \frac{s_\theta^2}{C_\theta^2} \left(1 - \frac{8}{3} s_\theta^2 \right) \left[\ln c_\theta^2 + \frac{5}{3} \right] - \ln c_\theta^2 \right\}\quad (1.9)$$

and a third generation in which $m_b \ll m_t, M_W$ receives an additional contribution of

$$\begin{aligned}\Delta &= \frac{g^2}{16\pi^2} \frac{4}{3} \frac{s_\theta^2}{C_\theta^2} \left(\frac{4}{3} \frac{s_\theta^2}{C_\theta^2} - 1 \right) \left\{ \ln v - (1+2v) \text{Re} \sqrt{1-4v} \ln \left(\frac{\sqrt{1-4v}-1}{\sqrt{1-4v}+1} \right) - 4v \right\} \\ &\quad + \frac{g^2}{16\pi^2} \frac{1}{2C_\theta^2} \left\{ \ln v + (v-1) \text{Re} \sqrt{1-4v} \ln \left(\frac{\sqrt{1-4v}-1}{\sqrt{1-4v}+1} \right) \right\} \\ &\quad - \frac{g^2}{16\pi^2} \left\{ \ln u + \frac{1}{2} (1-u)^2 (u+2) \ln \left| 1 - \frac{1}{u} \right| + \frac{1}{2} u^2 \right\}\end{aligned}\quad (1.10)$$

where $u=m_t^2/M_W^2$ and $v=u \cos^2 \theta_W$

Thus the relation (9) of reference [4] must be modified to

$$\sin^2 \theta_w = \sin^2 \theta_w \Big|_{m_t=0} + \Delta \cos^2 \theta_w \quad (1.11)$$

The detailed dependence on top quark and Higgs masses is displayed in Table 1 which should be sufficiently dense to allow interpolation.

2. Extraction of $\sin^2\theta_W$ and $\sin^2\hat{\theta}_W(\mu)$ from neutrino scattering experiments and sensitivity to top quark and Higgs masses.

In this section the notation of Wheater and Llewellyn-Smith [1] will be used as far as possible except that a Euclidean metric is adopted. The ratio of charged to neutral current cross-sections for neutrinos only will be considered

$$R_\nu = \frac{\sigma(\nu_\mu N \rightarrow \nu_\mu X)}{\sigma(\nu_\mu N \rightarrow \mu X)} \quad (2.1)$$

as it is about 30 times as sensitive to shifts in $\sin^2\theta_W$ as the corresponding quantity for anti-neutrinos. In lowest order for the valence quarks only of an isoscalar target,

$$R_\nu(\sin^2\theta_W) = \frac{1}{2} \left(1 - 2\sin^2\theta_W + \frac{40}{27}\sin^4\theta_W \right) \quad (2.2)$$

One may define $\sin^2\theta_W^{\text{exp}}$ to be that value which is extracted from the measured value of R_ν taking account of QED corrections, the anti-quark sea, scaling violations etc. but not purely weak radiative corrections.

The top quark and Higgs mass dependence comes from the Z self-energy corrections and Z- γ mixing in the neutral current sector and from W self-energy corrections in the charged current sector. One may write

$$R_\nu(\sin^2\theta_W^{\text{exp}}) = R_\nu(\sin^2\hat{\theta}_W(\mu)) \frac{1+r_{nc}}{1+r_{cc}} \quad (2.3)$$

For isoscalar targets with $|q^2| \ll M_W^2$ and keeping only terms of interest,

$$\frac{1}{2} r_{nc} = \frac{\hat{\Pi}_{zz}(q^2, \mu)}{M_Z^2} + \frac{\left(1 - \frac{40}{27}s_\theta^2\right)}{2R_\nu(s_\theta^2)} \frac{\hat{\Pi}_{zy}^{(f)}(q^2, \mu)}{q^2} \quad (2.4)$$

$$\frac{1}{2} r_{cc} = \frac{\hat{\Pi}_{WW}(q^2, \mu)}{M_W^2}$$

again in the Euclidean metric. In the above, only the fermionic part of the Z- γ mixing needs to be considered because ignoring terms $\propto q^2/M_W^2$ the bosonic part is,

$$\Pi_{Z\gamma}^{(b)}(q^2, \mu) = -\frac{g^2}{16\pi^2} 2M_Z^2 s_\theta c_\theta (\Delta - \ln M_W^2/\mu^2) \quad (2.5)$$

where $\Delta = \pi^{(n/2-2)}\Gamma(2-n/2)$ and n is the number of space-time dimensions. This infinite quantity is removed by the $\overline{\text{MS}}$ counter-term and the logarithm vanishes for $\mu=M_W$. To display the relative sensitivity of $\sin^2\theta_W$ and $\sin^2\hat{\theta}_W(M_W)$ to shifts in the top quark and Higgs masses, take a recently published value [7] $\sin^2\hat{\theta}_W(M_W)=0.226\pm0.012(\text{exp.})\pm0.006(\text{theor.})$ which was extracted assuming $m_t=30\text{GeV}$ and $M_H=100\text{GeV}$. The average q^2 will be taken as 20GeV^2 .

In many of the papers published to date, the authors fail to state the values of m_t and M_H used in the data analysis. Experimenters are urged to clearly record what model and parameter dependence is contained in published results so that they can be used in for later independent analysis.

The stated value of $\sin^2\hat{\theta}_W(M_W)$ was obtained following Kim et al. [8] and using recent structure function data. This aspect need not concern us further which would not be the case if we started from the published value of R_V . Using (2.3) and (2.4) it is possible to straightforwardly divide out the contributions from radiative corrections used in the original data analysis and replace them with those appropriate to the top quark and Higgs masses of interest. A new value of $\sin^2\hat{\theta}_W(M_W)$ is thus obtained and may be converted to $\sin^2\theta_W$ by means of (1.11). The sensitivity of the two quan-

tities to m_t is displayed in Table 2. The Higgs mass dependence is rather weaker. For small m_t the variations of both is comparable and lie well within experimental errors whereas over a broad range $\sin^2\theta_W$ changes little but $\sin^2\hat{\theta}_W(M_W)$ increases monotonically. The reason for this is not difficult to discover. Following the procedure just described one easily obtains,

$$\sin^2\theta_w \approx \sin^2\theta_w^{\text{exp}} \left\{ 1 + \frac{c_\theta^2}{s_\theta^2} \Delta_1 - \frac{2R_\nu}{s_\theta^2 \left(1 - \frac{40}{27}s_\theta^2\right)} \Delta_2 \right\} \quad (2.6)$$

in which

$$\begin{aligned} \Delta_1 &= \frac{\hat{\Pi}_{ww}(-M_w^2, \mu)}{M_w^2} - \frac{\hat{\Pi}_{zz}(-M_z^2, \mu)}{M_z^2} = -\frac{3g^2}{64\pi^2} \frac{m_t^2}{M_w^2} + \dots \\ \Delta_2 &= \frac{\hat{\Pi}_{ww}(q^2, \mu)}{M_w^2} - \frac{\hat{\Pi}_{zz}(q^2, \mu)}{M_z^2} = -\frac{3g^2}{64\pi^2} \frac{m_t^2}{M_w^2} + \dots \end{aligned} \quad (2.7)$$

The term involving Z- γ mixing has been dropped as its dependence on m_t is logarithmic and it is not a function of Higgs mass. If (2.6) had been derived wholly in the on-shell scheme the term Δ_1 would have arisen from mass counterterms.

For $\sin^2\theta_W=0.23$,

$$\frac{2R_\nu(s_\theta^2)}{s_\theta^2 \left(1 - \frac{40}{27}s_\theta^2\right)} = 4.08 \quad ; \quad \frac{c_\theta^2}{s_\theta^2} = 3.35$$

Thus there is a suppression but not an exact cancellation of the quadratic in m_t contributing to $\sin^2\theta_W$. On the otherhand such terms in $\sin^2\hat{\theta}_W(M_W)$ experience an enhancement by $\cot^2\theta_W$. The effect is dependent on the exact value of $\sin^2\theta_W$ and on the quantity considered. In this sense it is purely fortuitous and in some ways analagous to the accidental suppression of the vector coupling of the Z to massive leptons.

The strong dependence on $\sin^2 \hat{\theta}_W(M_W)$ on m_t was noted by Wheater and Llewellyn-Smith [1] and the tendency toward cancellation was commented on by Marciano and Sirlin [5]

Dependence of $\sin^2 \hat{\theta}_W(M_W)$ on Higgs mass is logarithmic and is given in reference [1]. The authors find a significant effect only for M_H above 1000 GeV. For $M_H=1000\text{GeV}$ the values for $\sin^2 \hat{\theta}_W(M_W)$ in Table 2 are increased by up to 0.0014. The variation is largest for small m_t . Tabulated values of $\sin^2 \theta_W$ are unchanged. That the asymptotic behaviour is independent of q^2 and therefore the same for Δ_1 and Δ_2 in (2.6) can be seen as follows. All self-energies can be expressed as the sum of integrals of the form [6]

$$\int_0^1 dx \, x^n \ln [-q^2 x^2 + (q^2 + m_2^2 - m_1^2)x + m_1^2 - i\epsilon] \quad (2.8)$$

where n is some integer and m_1 and m_2 are the masses of the virtual particles in internal loops. If one is interested in the Higgs mass dependence then $m_2=M_H$ and $m_1=M_{W(Z)}$ for the $W(Z)$ self-energy. For $M_H^2 \gg M_W^2, |q^2|$, the integral (2.8) can be written

$$\int_0^1 dx \, x^n \ln \left[M_H^2 \left(1 - \frac{q^2}{M_H^2} x \right) \left(x + \frac{M_W^2}{M_H^2} \right) - i\epsilon \right] \quad (2.9)$$

and the result follows immediately. Thus using the same arguments as for the top quark mass it can be seen that $\sin^2 \theta_W$ has its leading Higgs mass dependence suppressed as well.

By the calculation described above one finds that the total theoretical uncertainty in $\sin^2\theta_W=1-M_W^2/M_Z^2$ due to uncertainties in the parameters of the standard model for $30\text{GeV}<m_t<240\text{GeV}$ and $10\text{GeV}<M_H<1000\text{GeV}$ is ± 0.0015 . For fixed M_H it is ± 0.0013 and for fixed m_t in the above range it is ± 0.0004 .

It may be concluded that $\sin^2\theta_W=1-M_W^2/M_Z^2$ as extracted from neutrino scattering experiments is comparatively free of theoretical uncertainties due to unknown parameters in the standard model (the top quark and Higgs masses). The quantity $\sin^2\hat{\theta}_W(M_W)$ defined in the $\overline{\text{MS}}$ scheme does not share this feature. When, however, the value of $\sin^2\theta_W$ is used as input for the prediction of other physically measurable quantities, such as M_W or M_Z , terms quadratic in m_t will, in general, reappear.

References

- [1] Wheeler, J.F. & Llewellyn-Smith, C.H., Nucl. Phys. B208, 27, (1982) ; errata: Nucl. Phys. B228, 547, (1983)
- [2] Sirlin, A., Phys. Rev. D22, 971, (1980)
- [3] Itzykson, C. & Zuber, J-B., Quantum Field Theory, New York:McGraw-Hill (1980)
- [4] Sarantakos S., Sirlin, A. & Marciano W.J., Nucl. Phys. B217, 84, (1983)
- [5] Marciano, W.J. & Sirlin, A., Nucl. Phys. B189, 442, (1981)
- [6] Passarino, G. & Veltman, M., Nucl. Phys. B160, 151, (1979)
- [7] Abramowicz, H. et al., Z. Phys. C - Particles & Fields 28, 51, (1985)

- [8] Kim, J.E., Langacker, M., Levine, M. & Williams, H.H., Rev. Mod. Phys. 53, 211, (1981)

Table 1: Relation between $\sin^2\theta_W$ and $\sin^2\hat{\theta}_W(M_W)$ for various values of m_t and M_H (all masses in GeV)

		$M_H=10$	$M_H=100$	$M_H=1000$
	$\sin^2\theta_W$	$\sin^2\hat{\theta}_W(M_W)$		
$m_t=30$	0.200	0.1989	0.1983	0.1963
	0.206	0.2049	0.2043	0.2024
	0.212	0.2110	0.2104	0.2084
	0.218	0.2170	0.2164	0.2145
	0.224	0.2231	0.2225	0.2205
	0.230	0.2291	0.2285	0.2265
$m_t=60$	0.200	0.1980	0.1974	0.1953
	0.206	0.2041	0.2035	0.2015
	0.212	0.2102	0.2096	0.2076
	0.218	0.2164	0.2157	0.2137
	0.224	0.2225	0.2218	0.2198
	0.230	0.2286	0.2280	0.2260
$m_t=90$	0.200	0.2013	0.2007	0.1989
	0.206	0.2075	0.2068	0.2050
	0.212	0.2136	0.2130	0.2111
	0.218	0.2197	0.2190	0.2172
	0.224	0.2258	0.2251	0.2233
	0.230	0.2318	0.2312	0.2294

$m_t=120$

0.200	0.2042	0.2036	0.2019
0.206	0.2102	0.2096	0.2079
0.212	0.2163	0.2156	0.2140
0.218	0.2223	0.2217	0.2200
0.224	0.2283	0.2277	0.2260
0.230	0.2343	0.2337	0.2320

 $m_t=180$

0.200	0.2099	0.2093	0.2080
0.206	0.2158	0.2152	0.2139
0.212	0.2217	0.2211	0.2198
0.218	0.2277	0.2271	0.2257
0.224	0.2336	0.2330	0.2316
0.230	0.2395	0.2389	0.2375

 $m_t=240$

0.200	0.2165	0.2160	0.2151
0.206	0.2223	0.2218	0.2209
0.212	0.2282	0.2276	0.2267
0.218	0.2340	0.2334	0.2325
0.224	0.2398	0.2392	0.2383
0.230	0.2456	0.2451	0.2441

Table 2: Dependence on m_t of values of $\sin^2\theta_W$ and $\sin^2\hat{\theta}_W(M_W)$ extracted from semi-leptonic neutrino scattering

	(M _H =100GeV)	
	$\sin^2\theta_W$	$\sin^2\hat{\theta}_W(M_W)$
$m_t=30$	0.227	0.226
$m_t=60$	0.229	0.227
$m_t=90$	0.227	0.228
$m_t=120$	0.226	0.230
$m_t=180$	0.226	0.235
$m_t=240$	0.226	0.242

Acknowledgements

The author wishes to thank F. Dydak, C. Geweniger, A. Blondel and L. Fayard for helpful discussions. Financial support through the European Exchange Fellowship Programme is gratefully acknowledged.