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COVARIANT STRING THEORY FEYNMAN AMPLITUDES

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ABSTRACT

We consider the covariant string amplitudes on a strip with the associated Feynman rules. The contribution of the Faddeev-Popov ghost are evaluated. The overlap vertex (coming from a δ -function interaction) is constructed. The corresponding scattering diagrams are shown to give the standard Veneziano model answer. The ghosts provide the necessary contribution to the measure.

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At present the covariant and gauge invariant field theory of strings is being developed¹⁾⁻¹⁰⁾. It is supposed to provide a field theoretic basis for the dual amplitudes and to allow for off-shell investigations. Up to now, it was only in the light cone gauge that the space-time scattering process of strings was computable and shown to give the dual amplitudes²⁾⁻⁶⁾. It is the purpose of the present paper to consider a covariant version of this process. It is based on the covariant string action with the Faddeev-Popov ghosts and the associated BRST symmetry^{7),1)}. The scattering diagram can be thought of in the strip picture with new arbitrary lengths $\{\alpha_i; i = 1, 2, 3, 4\}$ replacing the light cone momenta p^+ . This is given heuristically by the string functional integral

$$S = \int dT \int dX^\mu d\bar{C} dC \exp \left[- \int d\tau d\sigma \mathcal{L} \right] \quad (1)$$

The Lagrangian

$$\mathcal{L} = \partial_\sigma X \cdot \partial_\sigma X + \partial_\tau X \cdot \partial_\tau X + \bar{C}^a D_a{}^b C_b \quad (2)$$

with the Faddeev-Popov ghosts was investigated in detail in Ref. 7). In (1), one integrates over the interaction time $T = T_2 - T_1$ which is positive for the diagram that represents the s-channel process; the t-channel diagram is given by the opposite order with $T_2 < T_1$. The counting on the strip and the Feynman diagram interpretation is exactly identical to the well-known light cone picture. The time here represents the proper time of the relativistic propagator.

This strip picture is presently developed as the basis for a gauge invariant interacting field theory of strings⁸⁾. The above functional integral and what follows can then be thought of as a gauge fixed expression.

The basic problem is to reduce the integral to the expected dual model form where the Koba-Nielsen variables and their complex region (upper half plane) are to be obtained through the conformal transformation $\rho(z) = \sum \alpha_i \ln(z - Z_i)$ where $\rho = \tau + i\sigma$ parametrizes the strip. Taking for initial and final states momentum eigenstates given by the factors $\exp(i \sum_{i=1}^4 p_i^\mu X_\mu(\rho_i))$, the Gaussian integrals would clearly give

$$\int \prod dZ_i |Z_i - Z_j|^{2 p_i^\mu p_j^\mu} \mathcal{M}(Z, \alpha) \quad (3)$$

with the following main effects contributing to the measure

$$\left| \frac{\partial \mathcal{L}}{\partial Z} \right| (\det \Delta_L)^{-\frac{D}{2}} (\det \mathcal{D}_a^b) \quad (4)$$

the Jacobian of the mapping, the $D = 26$ dimensional bosonic Laplacian determinant and the Faddeev-Popov determinant. The total effect has to be shown to equal the standard Koba-Nielsen integration measure. This is actually one of the main problems that concerns this paper.

One could consider to compute the determinants explicitly as was done recently by Mandelstam⁴⁾ for the bosonic case, the ghost piece, however, is slightly more involved. It is instructive to transfer to the operator language deriving the operator Feynman rules. They correspond to the field theory.

The idea is to write on the strip the vertex and the propagator terms obtained by slicing the functional integral. What follows is an application and simple extension to the ghosts of the procedures established in the original papers^{4),5)}. The Feynman integral separated into five propagating regions reads

$$\int \prod_{i=1}^6 [dX^{(i)} dC^{(i)} d\tilde{C}^{(i)}] \prod_{i=3}^4 D(\infty; X^{(i)}, C^{(i)}) \cdot \\ \cdot V(X^{(3)}, C^{(3)}; X^{(4)}, C^{(4)}; X^{(6)}, C^{(6)}) D(6; 5) \cdot \\ \cdot V(X^{(5)}, C^{(5)}; X^{(1)}, C^{(1)}; X^{(2)}, C^{(2)}) D(X^{(1)}, 0) D(X^{(2)}, \infty) \quad (5)$$

Clearly $D(6;5)$ represents the propagator

$$D(6;5) = \int_0^\infty dT \langle 6 | e^{-\frac{T}{\alpha_s} (L_0 + L_0^{FP} - 1)} | 5 \rangle \quad (6)$$

with the three-point vertex given in terms of δ -functions

$$\begin{aligned}
 V(1;2;3) = & \prod_{\sigma} \prod_{\mu=1}^D \delta \left(X_{\mu}^{(3)}(\sigma) - X_{\mu}^{(1)} - X_{\mu}^{(2)} \right) \cdot \\
 & \prod_{\sigma} \delta \left(C^{(3)}(\sigma) - C^{(1)}(\sigma) - C^{(2)}(\sigma) \right) \\
 & \cdot \delta \left(\tilde{C}^{(3)}(\sigma) + \tilde{C}^{(1)}(\sigma) + \tilde{C}^{(2)}(\sigma) \right)
 \end{aligned} \tag{7}$$

where we have taken for notational reasons $C(\sigma) = C^0(\sigma)$, $\tilde{C}(\sigma) = \tilde{C}_1(\sigma)$. The precise meaning of the overlap δ -functions will be defined shortly. The presence of the δ -function ghost contributions was mentioned in earlier work¹⁾ and recently. The basic quantity to construct is the δ -function overlap vertex in the creation-annihilation operator space. Here one has a separation with terms coming from the bosonic co-ordinates (X_{μ}) and the fermionic Faddeev-Popov ghost contribution ($C, \tilde{C}$):

$$|V\rangle = \mu \exp \left[\Delta^X + \Delta^C \right] |V_0\rangle \tag{8}$$

The bosonic part has the well-known quadratic form which we state for notational reasons. From Refs. 4) and 5), it reads

$$\begin{aligned}
 \Delta^X = & \frac{1}{2} \sum_{r,s=1}^3 \sum_{m,n} a_{-m}^{(r)\mu} \bar{N}_{mn}^{rs} \sqrt{mn} a_{-n}^{(s)\mu} \\
 & + a_{-m}^{(r)\mu} \sqrt{m} \bar{N}_m^{(r)} P^{\mu} + \kappa P^2
 \end{aligned} \tag{9}$$

with the fact that here one has $D = 26$ oscillators and not only the transverse oscillators

$$[a_m^{(r)\mu}, a_{-n}^{(s)\nu}] = \delta_{m,n} \delta_{r,s} \eta^{\mu\nu} \tag{10}$$

We determine the ghost part essentially following the previous works. One parametrizes the three strings as

$$\begin{aligned}
 \sigma_1 &= \sigma, & \theta_1 &= \theta (\pi \alpha_1 - \sigma) \\
 \sigma_2 &= \pi \alpha_1 - \sigma, & \theta_2 &= \theta (\sigma - \pi \alpha_1) \\
 \sigma_3 &= \pi (\alpha_1 + \alpha_2) - \sigma,
 \end{aligned} \tag{11}$$

with the following Fourier expansion for ghost co-ordinates

$$\begin{aligned}
 C^{(r)}(\sigma) &= \frac{1}{\sqrt{\pi}} \sum_n \tilde{\beta}_n^r \cos\left(\frac{n\sigma}{\alpha_r}\right) \\
 \text{with} \\
 \tilde{C}^{(r)}(\sigma) &= \frac{-i}{\sqrt{\pi}} \frac{1}{|\alpha_r|} \sum_n \beta_n^r \sin\left(\frac{n\sigma}{\alpha_r}\right)
 \end{aligned} \tag{12}$$

One could take the δ -functions directly, and, after Fourier transforms, change into the coherent state basis, thus obtaining the vertex. There is also a second method which is simpler in the present case. One imposes and solves the projected overlap equations for the vertex [the method of Goto and Naka²⁾] which we follow. The conjugate co-ordinates for C and $\tilde{C}$ are expanded as

$$\begin{aligned}
 \Pi^{(r)}(\sigma) &\equiv \frac{\delta}{\delta C(\sigma)} = \frac{1}{\sqrt{\pi}} \frac{1}{|\alpha_r|} \sum_n \beta_n^{(r)} \cos \frac{n\sigma}{\alpha_r} \\
 \tilde{\Pi}^{(r)}(\sigma) &\equiv \frac{\delta}{\delta \tilde{C}(\sigma)} = \frac{-i}{\sqrt{\pi}} \sum_n \tilde{\beta}_n^{(r)} \sin \frac{n\sigma}{\alpha_r}
 \end{aligned} \tag{13}$$

We impose the following overlap equations for the vertex

$$\begin{aligned}
 (C^{(3)}(\sigma) - C^{(1)}(\sigma)\theta_1 - C^{(2)}(\sigma)\theta_2) |V\rangle &= 0 \\
 (\Pi^{(3)}(\sigma) + \Pi^{(1)}(\sigma)\theta_1 + \Pi^{(2)}(\sigma)\theta_2) |V\rangle &= 0
 \end{aligned} \tag{14}$$

and

$$\begin{aligned}
 & (\tilde{C}^{(3)}_{(0)} + \tilde{C}^{(1)}_{(2)} \theta_1 + \tilde{C}^{(2)}_{(2)} \theta_2) |V\rangle = 0 \\
 & (\tilde{\pi}^{(3)}_{(0)} - \tilde{\pi}^{(1)}_{(2)} \theta_1 - \tilde{\pi}^{(2)}_{(2)} \theta_2) |V\rangle = 0
 \end{aligned} \tag{15}$$

The change in signs in Eq. (15) is consistent with the definitions of Eq. (12) and Eq. (13). In terms of normal modes, Eq. (14) simply becomes:

$$\sum_{r=1}^3 \alpha_r A_{mn}^{(r)} \frac{1}{\sqrt{n}} (\tilde{\beta}_{-n}^{(r)} + \tilde{\beta}_n^{(r)}) + \sum_{i=1}^2 \alpha_i B_m^{(i)} \tilde{\beta}_0^{(i)} = 0 \tag{16a}$$

$$\sum_{r=1}^3 A_{mn}^{(r)} \frac{1}{\sqrt{n}} (\beta_n^{(r)} + \beta_{-n}^{(r)}) + \sum_{i=1}^2 B_m^{(i)} \beta_0^{(i)} = 0 \tag{16b}$$

Here we adopt the overlap notation of Ref. 6). For the Fourier representation of the other two constraints $\tilde{C}$, $\tilde{\pi}$, one has to take into account that they have sin functions instead of cosines which can be compensated for example by imposing constraints for $\partial_\sigma \tilde{C}$ and $\partial_\sigma \tilde{\pi}$. This is consistent since the functions vanish at the joining points and ends. One obtains then for Eq. (15) the following normal mode representations

$$\sum_{r=1}^3 \frac{1}{\alpha_r} A_{mn}^{(r)} \sqrt{n} (\beta_n^{(r)} - \beta_{-n}^{(r)}) = 0 \tag{17a}$$

$$\sum_{r=1}^3 A_{mn}^{(r)} \sqrt{n} (\tilde{\beta}_m^{(r)} - \tilde{\beta}_{-m}^{(r)}) = 0 \tag{17b}$$

One looks for a solution of a double set of constraints (16a)-(16b) and (17a)-(17b) in terms of a quadratic form. By direct verification using the basic anti-commutators $\{\beta_m^r, \tilde{\beta}_n^s\} = \delta_{m+n,0} \delta_{rs}$, we take the following expression

$$\begin{aligned} \Delta^C = & -2 \sum (A^{(r)} C^{\frac{1}{2}} \tilde{\beta}^{(r)})_m \Gamma_{mn}^{-1} (A^{(s)} C^{-\frac{1}{2}} \beta^{(s)}) \\ & + \sum \tilde{\beta}_{-n}^{(r)} \beta_{-n}^{(r)} \\ & - \sum (A^{(r)} C^{\frac{1}{2}} \tilde{\beta}^{(r)})_m \Gamma_{mm}^{-1} B_m S_0 \end{aligned} \quad (18)$$

Here

$$\Gamma = \sum_{r=1}^3 A^{(r)} A^{(r)T}, \quad C_{mn} = n \delta_{m,n} \quad (19)$$

while the zero mode term involves

$$S_0 = \alpha_1 \beta_0^{(2)} - \alpha_2 \beta_0^{(1)} = \alpha_2 \beta_0^{(3)} - \alpha_3 \beta_0^{(2)} = \alpha_3 \beta_0^{(1)} - \alpha_1 \beta_0^{(3)} \quad (20)$$

the cyclic symmetry being guaranteed on a state $|V_0\rangle$ which obeys the zero mode conservation and is built by applying $\Sigma \beta_0^{(r)}$ on diagonal state of $\tilde{\beta}$'s.

In verifying Eqs. (17) and (18), two non-trivial identities obeyed by $A_{mn}^{(r)}$'s and B 's are to be used. They read:

$$\sum_{r=1}^3 \alpha_r (A^{(r)} C^{-1} A^{(r)T})_{mn} = \frac{1}{2} \alpha_1 \alpha_2 \alpha_3 B_m B_n \quad (21)$$

(which is needed in the bosonic case already) and

$$\sum_{r=1}^3 \frac{1}{\alpha_r} (A^{(r)} C A^{(r)T})_{mn} = 0 \quad (22)$$

These identities are given in Appendix of Ref. 6). One way to understand them is the following. Equation (21) is simply the consistency condition for constraints C and π [(16a) and (16b)] and similarly Eq. (22) is the consistency condition for the constraints $\tilde{C}$ and $\tilde{\pi}$ [(17a) and (17b)] obtained by their anticommutation. This makes the origin of the quadratic form (18) clear. It can be finally written as:

$$\Delta^C = \sum_{rs} \tilde{\beta}_{-m}^{(r)} N_{mn}^{rs} \beta_{-n}^{(s)} - \sum_r \tilde{\beta}_{-m}^{(r)} N_m^r S_0 \quad (23)$$

with the standard Mandelstam coefficients. In addition to the above, there is one more term we shall consider. The vertex $\exp(\Delta_F)$ can be multiplied by

$C^{(\sigma)}(\pi\sigma_{int})$ at the interaction point. This can be thought as coming from the issue of continuity, namely the Fourier transforms [Eqs. (16) and (17)] of the delta function conditions [Eq. (7)] are actually weaker since agreement of Fourier modes does not assure continuity⁵⁾. One could enforce continuity at the interaction point explicitly through the additional constraints. In a recent paper¹⁰⁾, a factor

$$F = C^{(3)}(\pi\sigma_{int}) C^{(2)}_{(0)} C^{(1)}(\pi) \sim C^{(3)}(\pi\sigma_{int}) \rho$$

was suggested. So to compare with Ref. 10), one should consider

$$F e^{\Delta^c} |V_0\rangle \quad (24)$$

We also mention a similarity with the supersymmetric problem solved by Green and Schwarz⁶⁾. In fact, by a change of variables one can establish a relationship [the basis of Ref. 6) however is different]. Consider namely Ref. 6) and then the algebra $\{Q_m^a, Q_n^b\} = \alpha \delta_{m+n,0} \delta_{ab}$ for $a, b = 1, 2$ can be achieved by

$$Q_n^1 = \frac{1}{\sqrt{2}} (\beta_n + \alpha \tilde{\beta}_n) \quad , \quad Q_n^2 = \frac{i}{\sqrt{2}} (\beta_n - \alpha \tilde{\beta}_n) \quad (25)$$

A slight difference is that here $Q_n^1 = Q_{-n}^1$ but $Q_n^{2+} = -Q_{-n}^2$, but that should not influence solving the projection equations. One can easily see that the constraint

$$\sum_{r=1}^2 Q_a^{(r)}(\pm\sigma) |V\rangle = 0 \quad (26)$$

has the same content as the constants of Eqs. (16) and (17). Namely the operator $Q^a(\sigma)$ given as

$$Q^a(\sigma) = \frac{1}{\pi} \frac{1}{|\alpha|} \sum_n Q_n^a e^{i \frac{n\sigma}{\alpha}} \quad (27)$$

is represented by

$$Q^1(\sigma) = \hat{C}(\sigma) + \frac{\delta}{\delta \hat{C}(\sigma)}$$

$$Q^2(\sigma) = \hat{C}(\sigma) - \frac{\delta}{\delta \hat{C}(\sigma)}$$
(28)

with

$$\hat{C}(\sigma) = \frac{1}{\sqrt{\pi}} \frac{1}{|\alpha|} \sum_n \beta_n e^{-i \frac{n\sigma}{\alpha}} = \pi(\sigma) + \tilde{C}(\sigma)$$

$$\frac{\delta}{\delta \hat{C}(\sigma)} = \frac{1}{\sqrt{\pi}} \sum_n \tilde{\beta}_n e^{-i \frac{n\sigma}{\alpha}} = c(\sigma) + \tilde{\pi}(\sigma)$$
(29)

The comparison of Eqs. (28) and (29) with the constraints of Eqs. (16) and (17) establishes the explicit correspondence.

Let us now consider the scattering amplitude and exhibit the most important effect of the ghost contribution. In the operator form, the four-particle scattering amplitude is given by

$$S_4 = \int_0^\infty dT \langle 3,4 | \hat{V} | 5 \rangle \cdot \langle 5 | e^{-\frac{T}{\alpha_s} \hat{L}} | 6 \rangle \langle 6 | \hat{V} | 1,2 \rangle$$
(30)

The effect of the propagator can be transferred into a change of N_{mn}^{rs} 's in the right vertex. For the main determinant effect, which we are interested in, let us concentrate on

$$\langle 3,4 | \exp(\Delta^X + \Delta^C) \exp(\Delta^X(\omega) + \Delta^C(\omega)) | 1,2 \rangle$$
(31)

Commuting through the intermediate oscillations ($5 = 6$) (which have no external states) using the formula

$$\langle 0 | \exp(\frac{1}{2} a \cdot M_2 \cdot a) \exp(\frac{1}{2} a^+ \cdot M_1 \cdot a^+) | 0 \rangle = \det(\mathbb{1} - M_2 M_1)^{-\frac{1}{2}}$$
(32a)

for bosonic oscillators and

$$\langle 0 | \exp(\tilde{b} \cdot U_2 \cdot b) \exp(\tilde{b}^+ \cdot U_1 \cdot \tilde{b}^+) | 0 \rangle = \det(\mathbb{1} - U_2 U_1)$$
(32b)

for fermionic ones gives the determinants

$$\det (1 - \sqrt{c} \bar{N} c \bar{N}(\omega) \sqrt{c})^{-\frac{D}{2}} \quad (33)$$

from the X-co-ordinate vertex [Eq. (9)] and

$$\det (1 - \bar{N}(\omega) c \bar{N} c) \quad (34)$$

from the ghost vertex. With the addition of the (only length-dependent) normalizing determinants of each vertex (given by $\det \Gamma^{D-2}$), these determinants correspond to the Mandelstam determinants in the functional integral representation of Eq. (4). Clearly, through the effect of ghost, there is a cancellation and the bosonic factor is reduced from $D = 26$ to $D-2 = 24$ which is what is needed for the cancellation of the Jacobian in going from the strip to the upper half-plane representation:

$$\mathcal{M}(Z, \alpha) \sim \left| \frac{\partial \rho}{\partial Z} \right| \det (1 - \bar{N}(\omega) c \bar{N} c)^{-\frac{26-2}{2}} \sim \text{const.}$$

Furthermore, on mass-shell the length dependence also disappears, leaving one with the Koba-Nielsen form [Eq. (3)] and therefore the Veneziano amplitude.

Let us now discuss in some detail the scattering geometry and field theory proposed recently in the work of Witten⁹⁾. Here similarly a consideration of the above form can be considered. The scattering geometry being such that there are no arbitrary string lengths, should lead to a vertex which is even of a more symmetric form. For the ghosts, the suggestion was to use a bosonized co-ordinate avoiding anticommuting variables. For a complete computation and derivation of the dual amplitudes in this theory, it appears to us (in analogy with the present discussion) that the fermionic treatment of ghosts should be necessary. Namely, with bosonic ghosts the determinantal factor would come out enhanced (to $D+2$) instead of being reduced for $D-2$. It is possible that, except for this effect, the rest is uninfluenced and then one could systematically stay with the bosonized form. Most important, however, is a construction of the corresponding conformal mapping and the explicit form of the corresponding vertex. Work on this is in progress.

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NOTE

Near the completion of this paper, we have received the report of Hata, Itoh, Kugo, Kunitomo and Ogawa [Ref. 10)] on their vertex and BRS symmetric field theory.

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