SUPERSPACE FORMULATION OF THE TEN-DIMENSIONAL COUPLED
EINSTEIN-YANG-MILLS SYSTEM

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ABSTRACT

We demonstrate explicitly the consistency of the superspace formulation of $N=1$ supergravity coupled to the supersymmetric Yang-Mills theory in ten dimensions. This is done by solving the superspace Bianchi identities subjected to a particular set of constraints on the super field strengths. As a by-product, we obtain the complete set of field equations for the coupled theory. We briefly discuss how this approach may be used to construct a manifestly supersymmetric field theory of the type generated by the heterotic string.

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While most supergravity-matter systems have already been formulated in superspace, an explicit construction of such a formulation in the case of the ten-dimensional coupled $N=1$ supergravity-Yang-Mills theory [1,2,3,4] is still lacking. It is certainly of some importance to demonstrate in detail how this theory can be embedded in superspace, in particular in view of the recent developments in superstring theories [5,6,7,8]. For instance, as is well known, the field theory obtained in the low energy limit of the type I open superstring [5] and the heterotic string [7], i.e. the Chapline-Manton theory [2], is infested with anomalies. However, it is shown in a remarkable paper by Green and Schwarz [5] that these anomalies can be removed. This is achieved at the expense of introducing supersymmetry breaking higher derivative terms (which however can be seen to be generated by the superstring [5,6]). Up to date all attempts [9,10] to supersymmetrize these terms have been made in terms of components (as opposed to superspace). This is an extremely cumbersome approach and it might be more fruitful to use a manifestly supersymmetric technique like superspace. The starting point would then be a superspace formulation of the Chapline-Manton theory [2]. As already mentioned, this theory has not yet been constructed in full detail in superspace. It is the purpose of this paper to remedy this by "solving" the superspace Bianchi identities [11,12,13] of this theory.

In two recent papers [12,13] it was explained how this is to be accomplished. The superspace Bianchi identities, as

well as the constraints, which when combined will lead to the field equations of the coupled $d=10$ supergravity Yang-Mills system, were presented in [12]. Due to the presence of the Yang-Mills Chern-Simons terms in the Chapline-Manton theory [1,2], the constraints on the superspace torsion tensor have to satisfy certain requirements. Most importantly, when introduced into the Bianchi identities of pure superspace supergravity they must not lead to any conditions on the curl of the third rank antisymmetric tensor field strength. This requirement is shown to be fulfilled in [13]. In fact, it is proved in this reference that these torsion constraints, which were first introduced in [11] to derive on-shell $N=1$ supergravity, together with the supertorsion Bianchi identities do not imply the on-shell theory. This being the case, we can now proceed and construct in full detail the Chapline-Manton theory in superspace. Since the presentation here will be rather brief we hope to give the details in a future publication.

We start by introducing $d=10$ chiral superspace [11] parametrized by

$$z^M = (x^\mu, \theta^m) \quad , \quad (1)$$

where the world index M takes ten bosonic (μ) and sixteen fermionic values (m). In this superspace we define in the standard fashion the vielbein $E_M^A(z)$, the Lie-algebra valued

spin connection $\omega_{MA}^B(z)$, the second rank antisymmetric tensor potential $B_{MN}(z)$ and the matrix-valued Yang-Mills potential $A_M(z)$. Here the tangent superspace indices $A, B, \dots$ split into bosonic ones $\alpha, \beta, \dots$ and fermionic ones $a, b, \dots$. Furthermore, we will use the following conventions. The tangent space metric $\eta_{\alpha\beta} = \text{diag}(-, +, +, +, +, +, +, +, +, +)$ and the γ -matrices satisfy $\{\gamma_\alpha, \gamma_\beta\} = -2\eta_{\alpha\beta}$. All quantities with fermionic indices satisfy the Weyl-condition as well as the Majorana condition. Below we will give the Bianchi identities in the language of differential forms in which the potentials are $E^A = dz^M E_M^A$, $\omega_A^B = dz^M \omega_{MA}^B$, $B = \frac{1}{2} dz^N \wedge dz^M B_{MN}$ and $A = dz^M A_M$. A covariant exterior derivative can then be constructed, i.e.,

$$D = d + \omega + A \quad , \quad (2)$$

and to each of the connections introduced above there is the corresponding field strength; namely the torsion

$$T^A = DE^A \quad , \quad (3)$$

the curvature

$$R_A^B = d\omega_A^B - \omega_A^C \wedge \omega_C^B \quad , \quad (4)$$

the three form [2,12]

$$H = dB + \text{tr}(A \wedge F + \frac{1}{3} A \wedge A \wedge A) \quad , \quad (5)$$

$T_{ab}^{\gamma} = -i(\gamma^{\gamma})_{ab}$	$H_{abc} = 0$	$F_{ab} = 0$
$T_{ab}^c = i\delta_{(a}^c T_{b)}$	$H_{ab\gamma} = i(\gamma_{\gamma})_{ab}$	$F_{a\beta} = i(\gamma_{\beta})_a^b \chi_b$
$T_{a\beta}^{\gamma} = i\delta_{\beta}^{\gamma} T_a$	$H_{a\beta\gamma} = -2i(\gamma_{\beta\gamma})_a^b \Lambda_b$	
$T_{\alpha\beta}^{\gamma} = 0$		

Table: On-shell constraints for the coupled Einstein-Yang-Mills system in superspace [12].

and the Yang-Mills field strength

$$F = dA - A \wedge A \quad . \quad (6)$$

From (2) to (6) we obtain the Bianchi identities

$$DT^A = -E^B \wedge R_B^A \quad , \quad (7)$$

$$DR_A^B = 0 \quad , \quad (8)$$

$$DH = \text{tr} F \wedge F \quad , \quad (9)$$

$$DF = 0 \quad , \quad (10)$$

where (8) follows from (7) [14] and is thus not discussed any further.

In order to derive the field equations we now introduce the constraints presented in [12] (see Table) into the Bianchi identities. As explained in [12], if we insert the torsion constraints into (7) we will find an off-shell formulation of d=10 supergravity. By also solving (9) with $F = 0$ the equations of motion will emerge. This was demonstrated in [11]. (The constraints used in [11] differ from the ones used here by a Fierz transformation on $T_{a\beta}^{\gamma}$ followed by a field redefinition and are thus entirely equivalent. A different set of constraints were used in [15].)

We now introduce the full set of constraints into (9) and (10) and combine the results so obtained with the results previously derived from (7) in Ref. [13]. The tangent space versions of (9) and (10) read

$$\sum_{ABCD} (D_A^H BCD + T_{AB}^E H_{ECD} - 2\text{tr} F_{AB} F_{CD}) = 0 \quad , \quad (11)$$

$$\sum_{ABC} (D_A^F BC + T_{AB}^D F_{DC}) = 0 \quad , \quad (12)$$

where we sum over all independent permutations. We denote these equations by (ABCD) and (ABC) respectively, and consider them in the order: i) (abcd); ii) (abc) and (αbcd) ; iii) (αbc) and $(\alpha\beta cd)$; iv) $(\alpha\beta\gamma d)$ and $(\alpha\beta c)$; and finally v) $(\alpha\beta\gamma\delta)$ and $(\alpha\beta\gamma)$.

i) and ii). $(abcd)$ and (abc) are immediately satisfied, while from (αbcd) it follows that $\Lambda_a = T_a$. Thus, by using our knowledge from the pure supergravity sector [13], we get

$$\Lambda_a = T_a = \hat{T}_a \quad . \quad (13)$$

iii). In [13] the torsion $T_{\alpha b}^c$ was found to be

$$T_{\alpha b}^c = -\frac{1}{2}(\gamma^\beta \gamma_\alpha)_b^c D_\beta \phi + (\gamma_\alpha^{\beta\gamma\delta} - 6\delta_\alpha^\beta \gamma^{\gamma\delta})_b^c Y_{\beta\gamma\delta} \quad , \quad (14)$$

where the superfield ϕ is defined by

$$iT_a = D_a \phi \quad , \quad (15)$$

and where $Y_{\alpha\beta\gamma}$ is an arbitrary third rank antisymmetric tensor superfield. From (15) it follows that

$$D_a D_b \phi = \frac{1}{2}(\gamma^\alpha)_{ab} D_\alpha \phi + i(\gamma^{\alpha\beta\gamma})_{ab} K_{\alpha\beta\gamma} \quad , \quad (16)$$

where $K_{\alpha\beta\gamma}$ is another arbitrary third rank antisymmetric tensor superfield [13]. $(\alpha\beta cd)$ now expresses these three forms in terms of previously introduced fields [12]

$$Y_{\alpha\beta\gamma} = -\frac{1}{72}H_{\alpha\beta\gamma} - \frac{2i}{3}\chi_{\alpha\beta\gamma} \quad , \quad (17)$$

$$K_{\alpha\beta\gamma} = -\frac{1}{72}H_{\alpha\beta\gamma} + \frac{i}{3}\chi_{\alpha\beta\gamma} - \frac{5i}{2}\Lambda_{\alpha\beta\gamma} \quad . \quad (18)$$

Here $\chi_{\alpha\beta\gamma}$ and $\Lambda_{\alpha\beta\gamma}$ are defined by $\text{tr}\chi_a\chi_b = (\gamma^{\alpha\beta\gamma})_{ab}\chi_{\alpha\beta\gamma}$ and $\Lambda_a\Lambda_b = (\gamma^{\alpha\beta\gamma})_{ab}\Lambda_{\alpha\beta\gamma}$.

From the equation $(\alpha\beta\gamma)$ we obtain an expression for the spinor derivative of the Yang-Mills superfield χ :

$$\hat{D}_a\chi_b = -\frac{i}{4}(\gamma^{\alpha\beta})_{ab}F_{\alpha\beta}, \quad (19)$$

where $\hat{D}_a = D_a + \frac{3i}{2}\Lambda_a$. Note the simple form of (19) which is basically the supersymmetry variation of the Yang-Mills spinor field in the component formulation [1,2].

iv). At this level we find for the first time equations of motion. Namely, from Ref. [13] and $(\alpha\beta\gamma d)$ it follows directly that

$$\begin{aligned} (\gamma^\alpha)^{ab}D_\alpha\Lambda_b &= 2(\gamma^\alpha)^{ab}(D_\alpha\phi)\Lambda_b + (\gamma^{\alpha\beta\gamma})^{ab}Y_{\alpha\beta\gamma}\Lambda_b + \\ &\quad \frac{i}{12}(\gamma^{\alpha\beta})^{ab}\text{tr}(\chi_bF_{\alpha\beta}), \end{aligned} \quad (20)$$

where $Y_{\alpha\beta\gamma}$ is given in (17). Also, by using (19) and $(\alpha\beta c)$ we find

$$(\gamma^\alpha)^{ab}D_\alpha\chi_b = 3(\gamma^\alpha)^{ab}(D_\alpha\phi)\chi_b + 3(\gamma^{\alpha\beta\gamma})^{ab}Y_{\alpha\beta\gamma}\chi_b. \quad (21)$$

From (21) and (17) we thus infer the existence of a $(\text{tr}\chi\gamma^{\mu\nu\rho}\chi)^2$ -term in the x-space Lagrangian (see Refs. [3,4] for other derivations of this result).

Finally, the Rarita-Schwinger field equation is easily derived if we first decompose $T_{\alpha\beta}{}^c$ into irreducible parts [11]

$$T_{\alpha\beta}{}^c = G_{\alpha\beta}{}^c + 2(G_{[\alpha}\gamma_{\beta]})^c + (G\gamma_{\alpha\beta})^c, \quad (22)$$

where $G_{\alpha\beta}\gamma^\beta = G_\alpha\gamma^\alpha = 0$, and recall that

$$T_{\alpha\beta}{}^c(\gamma^{\alpha\beta\gamma})_c{}^d = -72G\gamma^d, \quad (23)$$

since G^a vanishes due to the torsion Bianchi identity as is shown in [13]. The analysis carried out in this reference also expressed $G_\alpha{}^b$ in terms of a spinor derivative on $Y_{\alpha\beta\gamma}$. Combining this with $(\alpha\beta\gamma d)$ yields

$$\begin{aligned} G_\alpha{}^b = & -\frac{3i}{14}(\hat{D}_a Y_{\alpha\beta\gamma})(\gamma^{\beta\gamma})^{ab} = \frac{1}{4}D_\alpha\Lambda^b + \frac{5}{8}(D_\alpha\phi)\Lambda^b - \\ & \frac{1}{8}(\gamma_\alpha{}^\beta)^{bc}(D_\beta\phi)\Lambda_c - \frac{1}{4}(\gamma_\alpha\gamma^{\beta\gamma\delta} + 9\delta_\alpha^\beta\gamma^{\gamma\delta})^{bc}\Lambda_c Y_{\beta\gamma\delta} + \\ & \frac{1}{96}(\gamma_\alpha\gamma^{\beta\gamma} + 12\delta_\alpha^\beta\gamma^\gamma)^{bc}\text{tr}(\chi_c F_{\beta\gamma}), \end{aligned} \quad (24)$$

where $\hat{D}_a = D_a + i\Lambda_a$, which together with (23) gives the explicit form of the Rarita-Schwinger field equation.

v). $(\alpha\beta\gamma\delta)$ and $(\alpha\beta\gamma)$ are the superspace versions of the ordinary x-space Bianchi identities for $H_{\alpha\beta\gamma}$ and $F_{\alpha\beta}$.

We can now derive the equations of motion for the bosonic fields as well. The Klein-Gordon equation for ϕ is obtained either by inserting (24) into $\hat{D}_c(G_\alpha\gamma^\alpha)^c = 0$ [11] or by

taking a spinor derivative on (20). The result is

$$\square \phi = 6(D_\alpha \phi)(D^\alpha \phi) + 2Y^{\alpha\beta\gamma} H_{\alpha\beta\gamma} + \frac{1}{12} \text{tr}(F_{\alpha\beta} F^{\alpha\beta}) \quad , (25)$$

where $Y_{\alpha\beta\gamma}$ is given by (17).

Turning to Einstein's equation we recall the expression for the Ricci tensor obtained from the torsion Bianchi identity in [13]

$$\begin{aligned} R_{(\alpha\beta)} = & -\eta_{\alpha\beta} \square \phi - 8D_{(\alpha} D_{\beta)} \phi + 8\eta_{\alpha\beta} (D_\gamma \phi)^2 - 8D_\alpha \phi D_\beta \phi - \\ & 144(\eta_{\alpha\beta} Y^{\gamma\delta\epsilon} Y_{\gamma\delta\epsilon} - 7Y_\alpha^{\gamma\delta} Y_{\beta\gamma\delta}) - \frac{144}{7} Q_{\gamma\delta(\alpha,\beta)}^{\gamma\delta} \quad , \end{aligned} \quad (26)$$

where $Q_{\alpha_1\alpha_2\alpha_3,\beta_1\beta_2\beta_3} \equiv -\frac{i}{96}(\gamma_{\beta_1\beta_2\beta_3})^{ab} D_a D_b Y_{\alpha_1\alpha_2\alpha_3}$. By inserting $\square \phi$ from (25) and $Q_{\gamma\delta(\alpha,\beta)}^{\gamma\delta}$, which can be derived from (24), we find

$$\begin{aligned} R_{(\alpha\beta)} = & -6D_{(\alpha} D_{\beta)} \phi - 4i(\gamma_{(\alpha})^{ab} D_{\beta)} \Lambda_b \Lambda_a + i(\gamma_{(\alpha})^{ab} \text{tr} D_{\beta)} \chi_b \chi_a + \\ & 384i\eta_{\alpha\beta} Y^{\gamma\delta\epsilon} \Lambda_{\gamma\delta\epsilon} - 18Y_{(\alpha}^{\gamma\delta} (H_{\beta)\gamma\delta} + 192i\Lambda_{\beta)\gamma\delta}) - \\ & \text{tr} F_{\alpha\gamma} F_\beta^\gamma - \frac{1}{6}(\eta_{\alpha\beta} Y^{\gamma\delta} - 12\delta_{(\alpha}^{\gamma} \gamma_{\beta)}^{\delta})^{ab} \Lambda_a \text{tr}(\chi_b F_{\gamma\delta}) \quad . \end{aligned} \quad (27)$$

Finally, in the Yang-Mills sector it is easily shown, for instance by acting with a spinor derivative on (21), that

$$D^\alpha F_{\alpha\beta} = 6(D^\alpha \phi)F_{\alpha\beta} + 36Y_{\beta\gamma\delta}F^{\gamma\delta} + 8G_\beta^c \chi_c, \quad (28)$$

where $Y_{\alpha\beta\gamma}$ is given in (17) and G_α^b in (24).

We have not yet given explicitly the Bianchi identity and equation of motion for $B_{\alpha\beta}$, although they have appeared implicitly in the above discussion. Since many of the novel features of the theory under consideration are related to these equations we reproduce them here. They are

$$D^\alpha H_{\alpha\beta\gamma} - 6(D^\alpha \phi)H_{\alpha\beta\gamma} + 48iD^\alpha \chi_{\alpha\beta\gamma} - 288i(D^\alpha \phi)\chi_{\alpha\beta\gamma} = 0, \quad (29)$$

$$D[\alpha H_{\beta\gamma\delta}] - 3iT[\alpha\beta\gamma\delta]^\Lambda - \frac{3}{2}\text{tr}F[\alpha\beta\gamma\delta] = 0. \quad (30)$$

The former equation is obtained by substituting (17) into

$$D^\alpha Y_{\alpha\beta\gamma} - 6(D^\alpha \phi)Y_{\alpha\beta\gamma} = 0, \quad (31)$$

which follows from equation (7) as explained in [13], while the latter equation is just $(\alpha\beta\gamma\delta)$. (Recall that no equation for the curl of $H_{\alpha\beta\gamma}$ in terms of other superfields is derivable from (7) [13].)

Before summarizing the results of this paper we want to emphasize that although the above presentation only discusses how to derive the field equations, we have of course verified explicitly that all irreducible tensor components of the various Bianchi identities are satisfied and that no further

conditions on the field strengths are derivable. In the process of doing this some of the irreducible tensor components of $Q_{\alpha_1\alpha_2\alpha_3,\beta_1\beta_2\beta_3}$ are determined. The equations for these irreducible tensors given in Ref. [13] are then seen to be satisfied identically. This provides a separate check on the consistency of the solution.

We end with a short summary and a few comments. By inserting a certain set of constraints on the superfield strengths T_{AB}^C , H_{ABC} and F_{AB} into their Bianchi identities, we derive the superspace field equations of the Einstein-Yang-Mills system, and, more importantly, we prove thereby that a superspace formulation of this theory exists. This approach constitutes a new and independent derivation of the theory. For instance, we have verified the presence of the term $(Y_{\mu\nu\rho})^2 \sim (H_{\mu\nu\rho} - \frac{i}{2}\text{tr}\chi\gamma_{\mu\nu\rho}\chi)^2$ in the Lagrangian [3,4]. (Note that this is not in contradiction with the field equations derived above. The reason for this is that the theory is obtained in its manifestly scale invariant form as is discussed extensively in Ref. [12].) Thus a situation obtains which is quite analogous to the one of Ref. [11]. Namely, the whole theory is expressible in terms of the two superfields $\phi(z)$ and $\chi(z)$ [12].

As a final comment, we return to the possible applicability of this approach to the field theory corresponding to the open $N = 1$ superstring and/or the heterotic string. The supersymmetrization of the higher derivative terms mentioned at the beginning of this paper is very likely to lead to a Lagrangian with infinitely many terms of progressively higher powers in the fields $R_{\alpha\beta\gamma}{}^{\delta}$, $R_{\alpha\beta}$, R and $H_{\alpha\beta\gamma}$. Starting from Bianchi identities in superspace, on the other hand, it is presumably correct to assume that these identities have only a finite number of terms. In fact, the difference to what has been presented above seems to be restricted to the Bianchi identity for $H_{ABC}(z)$ which in this case should read

$$dH = \text{tr } F^2 + \alpha \text{ tr } \hat{R}^2, \quad (32)$$

where $\hat{R}$ is a modified Riemann tensor containing dH and H^2 terms. Solving this new set of equations (which ought to be possible) will then indeed give rise to a Lagrangian with infinitely many terms. The reason for this is that the superspace equations must now be solved iteratively in the parameter α introduced in (32), since H appears on both sides of this equation. A preliminary investigation of (32) indicates that the modification of $R_{\alpha\beta\gamma\delta}$ is in terms of $\frac{1}{2}H_{\alpha\beta\gamma}$ which agrees with results obtained by Kaluza-Klein [16].

Note added When this paper was being typed, we received a paper by Atick, Dhar and Ratra (SLAC preprint) which contains many of the results derived here. However, the constraints used by these authors are the ones proposed by Witten [15]. Their results seem identical to ours, albeit expressed in field variables with canonical kinetic terms (see Ref. [12] for a discussion of this formulation as compared to our manifestly scale-invariant one).

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