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HAMILTONIAN TREATMENT OF FREE STRING FIELD THEORY

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ABSTRACT

We present a Hamiltonian analysis of field theory for free open or closed bosonic strings. The BRS formulation is treated in detail and the light front formulation commented upon.

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An important advance last year was the construction of conventional field theory actions describing the particle spectrum of free strings, as reported, e.g., in Refs. 1)-3) and references therein. Papers about interacting strings have begun to appear, too^{4),5)}. Due to the rapid publication rate, some smoke was generated as well. Therefore, we think that a Hamiltonian analysis might be of some interest.

We start from the action given for open bosonic strings in Refs. 3) and 4), viz.

$$\mathcal{L} = -\langle \chi | Q | \chi \rangle \quad (1)$$

Here Q is the BRS charge of the first quantized theory⁶⁾ and $|\chi\rangle$ is a vector in the Fock space of the same. In the conventional representation, Q is built up of harmonic oscillators a_n^μ , c_n and $\bar{c}_n$, of which the latter two are fermionic (ghosts and antighosts of the first quantized theory), and the field can be expanded as

$$|\chi\rangle = [A_{-1} + A_0 \cdot a_1^\dagger + A_1 \cdot a_2^\dagger + A_{\mu\nu} a_{\mu\nu}^\dagger a_{1\nu}^\dagger + A_i c_i^\dagger \bar{c}_i^\dagger + \dots] |0\rangle \quad (2)$$

The field is subject to the constraint

$$Q_c = \frac{1}{2}(c_0 \bar{c}_0 - \bar{c}_0 c_0) + \sum_{n=1}^{\infty} (c_n^\dagger \bar{c}_n - \bar{c}_n^\dagger c_n) | \chi \rangle = -\frac{1}{2} | \chi \rangle \quad (3)$$

Q_c is the ghost number operator of the first quantized string. The literature also contains actions where the field is constrained further^{1),2)}. We will avoid explicit mode expansions in this letter, both for convenience and as a matter of principle, since specifying a representation for Q forms part of the task of solving the equations, and one might imagine surprises here when one comes to the interacting theory.

For our purposes, we expand Q as

$$Q = \frac{1}{2} c_0 \partial^0 \partial^0 + c_0 h - \bar{c}_0 \eta + i a \partial^0 + q \quad (4)$$

Here, c_0 , $\bar{c}_0$ are ghost zero modes obeying

$$\{c_0, \bar{c}_0\} = 1 \quad (5)$$

Thus there are two distinct ghost vacua obeying

$$\bar{c}_0|+\rangle = |-\rangle \quad c_0|+\rangle = 0 \quad c_0|-\rangle = |+\rangle \quad \bar{c}_0|-\rangle = 0 \quad (6)$$

$$\langle +|+\rangle = \langle -|-\rangle = 0 \quad \langle +|-\rangle = \langle -|+\rangle = 1$$

We know that Q^2 is zero in 26 dimensions⁶⁾, which means that

$$[h, a] = [h, \eta] = [h, q] = [\eta, a] = [\eta, q] = 0 = \{a, q\} \quad (7)$$

$$a^2 = -\frac{1}{2}\eta \quad q^2 = \eta h$$

If we write

$$|\chi\rangle = \varphi|+\rangle + \psi|-\rangle \quad (8)$$

we can write the action (1) in somewhat more detail:

$$\mathcal{L} = \frac{1}{2} \dot{\psi} c_0 \dot{\psi} - \psi c_0 h \psi - \varphi \bar{c}_0 \eta \varphi - 2\varphi i a \dot{\psi} - 2\varphi q \psi \quad (9)$$

Here we have suppressed the bras and kets. The Lagrangian equation of motion for ϕ is

$$\eta \varphi + i c_0 a \dot{\psi} + q \psi = 0 \quad (10)$$

If this is solved for ϕ and the result inserted in the action, we obtain the action of Ref. 1). However, this requires that the left inverse of the operator η exists, which is not the case unless the field is constrained further [i.e., in addition to the constraint (3)]. Such an extra constraint was proposed in Ref. 1), which thus gives a truncated version of the action (1). Another, more drastic, truncation was proposed in Ref. 2), and claimed to be wrong in Ref. 7). We have not investigated the consistency of any of the truncated actions; as we will see, ϕ is a gauge field, not an auxiliary field, so that there is no need to solve for it. It plays a rôle analogous to that of A^0 in the conventional treatment of electrodynamics. Let us now go from the action (9) to the Hamiltonian. The canonical momenta are

$$p_\psi = c_0 \dot{\psi} + 2i a \varphi$$

$$p_\varphi = 0 \quad (11)$$

and the Hamiltonian becomes

$$\mathcal{H} = \frac{1}{2} p_\psi \bar{c}_0 p_\psi + \psi c_0 h \psi + 2\phi [i a \bar{c}_0 p_\psi + q \psi] \quad (12)$$

Due to the existence of the primary constraint $p_\phi = 0$, we generate [applying Dirac's procedure⁸⁾] the following secondary constraints:

$$\begin{aligned} \phi_1 &= q \psi + i a \bar{c}_0 p_\psi = 0 \\ \phi_2 &= -2 i a h c_0 \psi - q p_\psi = 0 \end{aligned} \quad (13)$$

As it turns out, all the constraints are first class, i.e., they generate gauge symmetries. Clearly we can use $p_\phi = 0$ to set $\phi = 0$ as a gauge choice. Equation (13) then summarizes all the gauge symmetry. It is interesting to see explicitly what these formulae imply for the massless sector, i.e., the field A_0^μ in (2). The choice of surface terms in the action is unconventional, and hence the constraints differ slightly from the usual ones. Conventionally, we have

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} & H_c &= \frac{1}{2} p^i p_i + \frac{1}{4} F_{ij} F^{ij} \\ p_0 &= 0 & \partial^i p_i &= 0 \end{aligned} \quad (14)$$

However, what we get from the action (9) is

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu + \phi \partial \cdot A - \frac{1}{2} \phi^2 \\ H_c &= \frac{1}{2} p^\mu p_\mu + \frac{1}{2} \partial_i A_\mu \partial^i A^\mu \\ p_0 - \partial_i A_i &= 0 & \partial_i p_i - \partial^i \partial_i A_0 &= 0 \end{aligned} \quad (15)$$

Let us return to the analysis of Eq. (13). These constraints are not independent but obey certain conditions, which we can write in matrix form:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} q & i a \bar{c}_0 \\ -2 i a h c_0 & -q \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0 \quad (16)$$

These conditions obey further conditions in their turn; in fact we get an infinite set of conditions

$$\begin{pmatrix} \phi_1^n \\ \phi_2^n \end{pmatrix} = \begin{pmatrix} q & ia\bar{c}_0 \\ -2iahc_0 & -q \end{pmatrix} \begin{pmatrix} \phi_1^{n-1} \\ \phi_2^{n-1} \end{pmatrix} \quad (17)$$

where n is an arbitrary integer. Now, if we want to write down the BRS charge for the field theory, we must ensure that its nilpotency is equivalent to all the structure equations; $[\phi_i, \phi_j] = 0$ and (16), (17). This requires the introduction of an infinite set of ghosts and antighosts $\eta^n, \bar{\eta}^n$, together with their canonical momenta $P_{\eta}^n, \bar{P}_{\eta}^n$ [a helpful review of the Hamiltonian approach to BRS is given in Ref. 9)]. It is easy to check that the BRS charge of the field theory has to be

$$Q_{FT} = (\bar{P}_{\eta}^0 \eta^0) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} + \sum_{n=1}^{\infty} (\bar{P}_{\eta}^n \eta^n) \begin{pmatrix} q & ia\bar{c}_0 \\ -2iahc_0 & -q \end{pmatrix} \begin{pmatrix} \eta^{n-1} \\ P_{\eta}^{n-1} \end{pmatrix} \quad (18)$$

We can also read off the Grassmann properties of the ghosts, as well as their ghost number, in the sense of Eq. (3). What we find then is that the ghosts and antighosts can be collected together with the physical fields into one big string field subject to no constraint at all.

A BRS invariant Hamiltonian is easily found:

$$H_{BRS} = \frac{1}{2} P_{\psi} \bar{c}_0 P_{\psi} + \psi c_0 h \psi + \sum_{n \neq 0} (\bar{P}_{\eta}^n \bar{c}_0 P_{\eta}^n + 2 \bar{\eta}^n c_0 h \eta^n) \quad (19)$$

This is in fact the BRS-Hamiltonian originally proposed in Ref. 10). One can now count the number of degrees of freedom that the action (1) contains, taking account of the constraints and the conditions on the constraints. Equivalently, one can count the ghosts. This check was carried out in Ref. 3). The number of degrees of freedom is precisely that of the open string, so that there can be no reasonable doubt that the action (1) is correct.

Let us now consider closed strings. The first quantized BRS charge is simply a sum of two open BRS charges, built up from two independent sets of oscillators $a_n^L, c_n^L, \bar{c}_n^L$ and $a_n^R, c_n^R, \bar{c}_n^R$:

$$Q = Q^L + Q^R = \frac{1}{2} c_0 \partial^0 \partial^0 + c_0 h + \bar{c}_0 \Delta H - \bar{c}_0 \eta - \bar{c}_0 \Delta h + ia \partial^0 + q \quad (20)$$

where

$$h = h^L + h^R, \quad \Delta H = h^L - h^R \quad \text{etc.} \quad (21)$$

The closed string field can now be expanded similarly to Eq. (2). However, a slight modification of the procedure from the open case is advisable, so that we avoid the problem noted in Ref. 4). We begin by imposing

$$\Delta H |x\rangle = 0 \quad (22)$$

as a second class constraint. Acting on such a field we find that

$$Q' = \frac{1}{2} c_0 \partial^0 \partial^0 + c_0 h + \bar{c}_0 \eta + i a \partial^0 + q \quad (23)$$

is nilpotent, so that only one pair of ghost zero modes is necessary. Hence we need only one pair of ghost vacua. We also impose the constraint

$$Q'_c = \frac{1}{2} (c_0 \bar{c}_0 - \bar{c}_0 c_0) + \sum_{n=1}^{\infty} (c_n^L + \bar{c}_n^L - \bar{c}_n^+ c_n^L + c_n^+ \bar{c}_n^R - \bar{c}_n^+ c_n^R) |x\rangle = -\frac{1}{2} |x\rangle \quad (24)$$

on the field. Here Q'_c is not quite the ghost number operator of the first quantized theory. Then we choose the action

$$\mathcal{L} = -\langle x | Q' | x \rangle \quad (25)$$

From here the analysis proceeds exactly as in the open case. A truncated version of the action (25) was given in Ref. 2) (the second paper).

About the interacting theory we have nothing to say. Since its interaction terms will contain arbitrarily high orders of time derivatives, its canonical analysis will be non-trivial and the question of causal behaviour important. In this connection, it is interesting to observe that the effective action for the massless sector of a U(1) string is the Born-Infeld theory¹¹⁾, which shows the above property, but is nevertheless known to have a sensible canonical structure as well as local causality¹²⁾.

Finally, a comment on the light front gauge. The correct Hamiltonian has been known for a long time¹³⁾. However, this formulation cannot be reached from the one considered in this paper by any gauge choice, since it entails not only a gauge choice but a completely different set of initial data¹⁴⁾. In fact, the

action (1) - or (15) for the massless sector - is not suitable for a light front formulation of dynamics. As far as we can see, this is no cause for worry. Light front quantization of a gauge theory always leads to second class constraints which break manifest covariance.

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