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STOCHASTIC VARIABLES IN TEN DIMENSIONS ?

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A B S T R A C T

A generator of infinitesimal Nicolai transformations is employed to reformulate their existence and polynomiality conditions. Their perturbative analysis in a four-dimensional chiral model reveals its complete variety of Nicolai mappings which, however, does not contain polynomial ones. By a reduction and truncation argument the ten-dimensional super-Yang-Mills theory is found to share this property. Hence, the list of stochastic gauge theories cannot be extended further to $N = 1, d = 10$, $N = 2, d = 6$ or $N = 4, d = 4$ super-Yang-Mills.

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I. INTRODUCTION

It is of considerable interest to regard supersymmetric theories as local formulations of purely bosonic models that exhibit a parameter-independent, eventually zero, vacuum energy density¹⁾. This remarkable property is at the origin of their improved (and sometimes even finite²⁾) ultraviolet behaviour and finds its quantitative expression in the so-called Nicolai map.

The latter, first suggested by Nicolai³⁾ and constructed by Flume, Dietz and the present author⁴⁾, constitutes a non-local transformation of the bosonic fields which, after integrating out the fermionic fields, maps the functional measure to that of the corresponding free theory. Conversely, Parisi and Sourlas⁵⁾ demonstrated that some supersymmetric theories, namely quantum mechanics and 2-dimensional $N=2$ extended chiral models, can be generated by imposing stochastic equations in real time for a white noise variable. In these cases, called "stochastic" hereafter, the associated Nicolai map is a closed expression linear in the coupling constants (and involves only a single non-locality). Recently, de Alfaro, Fubini, Furlan and Veneziano have successfully extended this program to the $N=1$ super-Yang-Mills theory (SYM) in light cone gauges in 4 dimensions⁶⁾, even coupled to free chiral multiplets⁷⁾, and in 6 dimensions⁷⁾, with the cost of one additional non-locality and cubic terms in the two latter cases. This was accomplished by a direct "quadratrization" of the bosonic action made possible by the underlying quaternionic and octonionic structures, respectively. However, the final step to the remaining 10-dimensional or, under reduction to 4 dimensions, maximally extended $N=4$ SYM⁸⁾ has still to be made.

In earlier works Flume, Dietz and the present author⁴⁾ developed a systematic and feasible method to explicitly construct the Nicolai transformation T perturbatively for any flat space off-shell supersymmetric theory. Starting from the vanishing of the vacuum energy¹⁾ and exploiting the supersymmetry Ward identity we derived a generator R for infinitesimal parameter shifts that served to build up the Nicolai map in a fermionic tree expansion. Although the breakdown of the latter in the stochastic cases of polynomial maps was recovered (and seen to be restricted to the light cone gauge in SYM)^{4,9)} our approach cannot yield the complete set of possible mappings for a considered model. However, this information is (at least partially) needed to decide the possibility of a polynomial transformation in, for instance, 4-dimensional Wess-Zumino models.

In this paper we set out to achieve this goal by deducing and analyzing general conditions for the existence of an (infinitesimal, hence perturbative) Nicolai transformation T . The derivation of those requirements in terms of the

generator R or, more convenient, an associated integral kernel K is presented in the second section without employing supersymmetry. Section 3 then is devoted to a thorough study of the integral equations that arise for the generic case of the massless Wess-Zumino model in 4 dimensions. In its course we are able to substantially generalize the Nicolai transformation found earlier to a whole class and elucidate the entire variety of mappings. In section 4 we formulate the general conditions for the polynomiality of a Nicolai map. These are then seen to be not satisfiable by complex Wess-Zumino models in 4 or real ones in 2 dimensions. This result carries over to 10-dimensional SYM because this theory goes into a massless interacting chiral model after dimensional reduction and consistent truncation.

So we can finally rule out the possibility of a polynomial mapping and, hence, give a negative answer to the stochasticity question for all the cases that had remained open.

2. THE GENERATOR OF THE INFINITESIMAL TRANSFORMATION

In this section we shall establish the concept an infinitesimal Nicolai transformation, i.e. a generator R of the finite mapping, and derive its main properties without making recourse to an underlying supersymmetry. Our notation is as follows: We collectively denote bosonic quantum fields by $\phi(x)$ and coupling constants by g therewith suppressing internal indices. The brackets $\langle \dots \rangle$ and $\langle \dots \rangle_0$ indicate (euclidean) vacuum expectation values in the interacting ($g \neq 0$) and free ($g=0$) theory, respectively. The latter are characterized by actions

$$\begin{aligned} S[\varphi] &= S_b[\varphi] + S_f[\varphi] \\ S^{(0)}[\varphi] &= S_b^{(0)}[\varphi] + S_f^{(0)}[\varphi] \end{aligned} \tag{1.1}$$

that consist of a local part, $S_b (S_b^{(0)})$, and a non-local one, $S_f (S_f^{(0)})$, which have to be related in a very specific way to allow for a Nicolai mapping. S_b and S_f may be thought of originating from a supersymmetric action S_s after integration of fermionic and auxiliary fields, but we shall not make use of this possibility. In this framework we shall call a field transformation $\phi(x) \rightarrow (T\phi)(x)$ a Nicolai map if it satisfies the two following properties:

$$\langle X[\varphi] \rangle = \langle X[T^{-1}\varphi] \rangle_0 \quad \text{for any functional } X[\phi] \quad (1.2a)$$

$$S_b[\varphi] = S_b^{(0)}[T\varphi] \quad (1.2b)$$

In the original definition³⁾ (1.2a) was replaced by

$$S_f[\varphi] = S_f^{(0)}[T\varphi] - \ln \det \frac{\delta(T\varphi)}{\delta\varphi} \quad (1.2c)$$

which, assuming (1.2b), is seen to be equivalent:

$$e^{-S[\varphi]} = e^{-S^{(0)}[T\varphi]} \det \frac{\delta(T\varphi)}{\delta\varphi} \quad (1.3)$$

In formulating (1.2a) we have tacitly implied the existence of a well-defined inverse transformation T^{-1} . This is justified as long as we restrict ourselves to the "neighbourhood" of the perturbative vacuum configuration $\phi(x)=0$ so that

$$\begin{aligned} (T\varphi)(x) &= \varphi(x) + \mathcal{O}(g) \\ (T^{-1}\varphi)(x) &= \varphi(x) + \mathcal{O}(g) \end{aligned} \quad (1.4)$$

and, hence, any perturbative T gives rise to a unique T^{-1} (and vice versa).

As the Nicolai transformation may be considered to effect a finite shift in the coupling parameters of the theory a possible generator should be related to infinitesimal changes in g . We therefore consider the response of an arbitrary vacuum expectation value to a variation in the coupling constant:

$$\begin{aligned} \frac{\partial}{\partial g} \langle X[\varphi] \rangle &\stackrel{(1.2a)}{=} \frac{\partial}{\partial g} \langle X[T^{-1}\varphi] \rangle_0 \\ &= \left\langle \left(\frac{\partial X}{\partial g} \right) [\varphi] \right\rangle + \left\langle \int dx \frac{\partial (T^{-1}\varphi)(x)}{\partial g} \frac{\delta X}{\delta \varphi(x)} [T^{-1}\varphi] \right\rangle_0 \\ &= \left\langle \frac{\partial X}{\partial g} \right\rangle + \left\langle \int dx \left\{ \left(\frac{\partial T^{-1}}{\partial g} \circ T \right) (T^{-1}\varphi) \right\} (x) \frac{\delta X}{\delta \varphi(x)} [T^{-1}\varphi] \right\rangle_0 \\ &\stackrel{(1.2a)}{=} \left\langle \frac{\partial X}{\partial g} \right\rangle + \left\langle \int dx \left\{ \left(\frac{\partial T^{-1}}{\partial g} \circ T \right) \varphi \right\} (x) \frac{\delta X[\varphi]}{\delta \varphi(x)} \right\rangle \end{aligned}$$

$$=: \langle (RX)[\varphi] \rangle \quad (1.5)$$

where the derivative operator

$$\begin{aligned} R \equiv R[g; \varphi] &= \frac{\partial}{\partial g} + \int dx \left\{ \left(\frac{\partial T^{-1}}{\partial g} \circ T \right) \varphi \right\} (x) \frac{\delta}{\delta \varphi(x)} \\ &=: \frac{\partial}{\partial g} + \int dx K[\varphi; x] \frac{\delta}{\delta \varphi(x)} \end{aligned} \quad (1.6)$$

apparently generates the inverse Nicolai map when exponentiated:

$$\begin{aligned} \langle X[\varphi] \rangle &= \exp \left(g \frac{\partial}{\partial g'} \right) \langle X[\varphi] \rangle (g') \Big|_{g'=0} \\ &\stackrel{(1.5)}{=} \left\langle \exp(g R(g')) X[\varphi; g'] \Big|_{g'=0} \right\rangle \\ &= \left\langle X[\exp(g R(g')) \varphi]_{g'=0; g} \right\rangle, \end{aligned} \quad (1.7)$$

i.e.

$$(T^{-1} \varphi)(x) = \exp(g R(g')) \varphi(x) \Big|_{g'=0} \quad (1.8)$$

the last equality in (1.7) following from the derivative property of R . Eqs.(1.6) and (1.8) clearly demonstrate the equivalence of information hidden in T , T^{-1} , R or K . With the knowledge of K , for instance, we could certainly set out to recover T via Eq.(1.8) and subsequent iterative inversion. However, there happens to be a more economic way to achieve this goal. From the observation

$$\begin{aligned} \left\langle \frac{\partial X}{\partial g} [\varphi] \right\rangle_0 &= \frac{\partial}{\partial g} \langle X[\varphi] \rangle_0 \stackrel{(1.2a)}{=} \frac{\partial}{\partial g} \langle X[T\varphi] \rangle \\ &= \langle R(X \circ T)[\varphi] \rangle \\ &\stackrel{(1.2a)}{=} \left\langle T^{-1} \{ R(X \circ T)[\varphi] \} \right\rangle_0. \end{aligned} \quad (1.9)$$

we conclude

$$R(X \circ T)[\varphi] = T\left\{\frac{\partial X}{\partial q}[\varphi]\right\} \quad (1.10)$$

which yields, specifying $X[\phi] = \phi$,

$$RT[\varphi] = 0 \quad (1.11)$$

This identity will be used in due course to construct T from K directly.

Because it is more convenient to deal with the generator R rather than with the finite shift T we shall now translate the defining properties (1.2) into conditions on R or, equivalently, on the kernel K . Comparing

$$\frac{\partial}{\partial q} \langle X \rangle = \left\langle \frac{\partial X}{\partial q} \right\rangle - \left\langle X \cdot \frac{\partial S}{\partial q} \right\rangle \quad (1.12)$$

to

$$\begin{aligned} \frac{\partial}{\partial q} \langle X \rangle &= \langle RX \rangle \equiv \left\langle \frac{\partial X}{\partial q} \right\rangle + \left\langle \int K(x) \frac{\delta X}{\delta \varphi(x)} \right\rangle \\ &= \left\langle \frac{\partial X}{\partial q} \right\rangle + \left\langle X \int K(x) \frac{\delta S}{\delta \varphi(x)} \right\rangle - \left\langle X \int \frac{\delta K(x)}{\delta \varphi(x)} \right\rangle \end{aligned} \quad (1.13)$$

we find

$$RS[\varphi] = \int \frac{\delta K[\varphi; x]}{\delta \varphi(x)} \quad (1.14a)$$

On the other hand it follows from Eqs. (1.2b) and (1.11) that

$$RS_b[\varphi] = RS_b^{(0)}[T\varphi] = \int \left(\frac{\delta S_b^{(0)}}{\delta \varphi(x)} \right) [T\varphi] \cdot (RT)\varphi(x) = 0 \quad (1.14b)$$

which gives

$$RS_f[\varphi] = \int \frac{\delta K[\varphi; x]}{\delta \varphi(x)} \quad (1.14c)$$

Eq. (1.14c) could as well be derived directly from (1.2c):

$$\begin{aligned}
 R S_f[\varphi] &\stackrel{(1.11)}{=} R \left(S_f[\varphi] - S_f^{(0)}[\tau_\varphi] \right) = -R \operatorname{Tr} \ln \frac{\delta(T_\varphi)}{\delta\varphi} \\
 &= -\operatorname{Tr} \left[\left(\frac{\delta T_\varphi}{\delta\varphi} \right)^{-1} R \left(\frac{\delta T_\varphi}{\delta\varphi} \right) \right] = -\iint \left(\frac{\delta T_\varphi}{\delta\varphi} \right)^{-1}_{(y,z)} R \left(\frac{\delta(T_\varphi)(z)}{\delta\varphi(y)} \right) \\
 &\stackrel{(1.11)}{=} \iint \left(\frac{\delta T_\varphi}{\delta\varphi} \right)^{-1}_{(y,z)} \left(\frac{\delta R}{\delta\varphi(y)} \right) (T_\varphi)(z) \stackrel{(1.6)}{=} \iiint \left(\frac{\delta T_\varphi}{\delta\varphi} \right)^{-1}_{(y,z)} \frac{\delta K[\varphi; x]}{\delta\varphi(y)} \cdot \frac{\delta(T_\varphi)(z)}{\delta\varphi(x)} \\
 &= \iint \delta(y-x) \frac{\delta K[\varphi; x]}{\delta\varphi(y)} = \int \frac{\delta K[\varphi; x]}{\delta\varphi(x)} . \tag{1.15}
 \end{aligned}$$

Being equivalent to Eqs. (1.2a-c), Eqs. (1.14a-c) will from now on serve as definitions of the (perturbative) Nicolai map. Extensive use will be made of the corresponding relations for the kernel $K[\phi; x]$:

$$\frac{\partial S_b}{\partial g}[\varphi] + \int dx K[\varphi; x] \frac{\delta S_b[\varphi]}{\delta\varphi(x)} = 0 \tag{1.16b}$$

$$\frac{\partial S_f}{\partial g}[\varphi] + \int dx K[\varphi; x] \frac{\delta S_f[\varphi]}{\delta\varphi(x)} - \int dx \frac{\delta K[\varphi; x]}{\delta\varphi(x)} = 0. \tag{1.16f}$$

Apparently, a model can only admit a Nicolai transformation if these equations are solvable. It is in general not an easy task to find the solution space of this inhomogeneous linear system, especially because of the appearance of the functional derivative in Eq. (1.16f). In particular, a complete answer requires the knowledge of all solutions of the homogeneous system

$$\int dx \tilde{K}[\varphi; x] \frac{\delta S_b[\varphi]}{\delta\varphi(x)} = 0 \tag{1.17b}$$

$$\int dx \tilde{K}[\varphi; x] \frac{\delta S_f[\varphi]}{\delta\varphi(x)} - \int dx \frac{\delta \tilde{K}[\varphi; x]}{\delta\varphi(x)} = 0. \tag{1.17f}$$

As we restricted ourselves to the perturbative regime it is legitimate to proceed by expanding the functional kernel $K[\phi; x]$ in the coupling constant g :

$$K[\varphi; x] = \sum_{n=0}^{\infty} q^n K_n[\varphi; x] \quad (1.18)$$

This will provide us with a useful tool to attack the system (1.16) in the following section. But, first of all, let us reconstruct the Nicolai map in terms of K_n . Decomposing also

$$(T_\varphi)(x) = \sum_{n=0}^{\infty} q^n T_n[\varphi; x] \quad (1.19)$$

Eq. (1.11) generates a recursion relation

$$T_{n+1}[\varphi; x] = \frac{-1}{n+1} \sum_{\ell=0}^n \int dy K_{n-\ell}[\varphi; y] \frac{\delta T_\ell[\varphi; x]}{\delta \varphi(y)}, \quad T_0[\varphi; x] = \varphi(x) \quad (1.20)$$

which is suitable to compute T_n iteratively. The result takes the following form:

$$T_n(x) = \sum_{\{n\}} c_{\{n\}} \int dx_r K_{n_r-1}(x_r) \frac{\delta}{\delta \varphi(x_r)} \dots \int dx_1 K_{n_1-1}(x_1) \frac{\delta}{\delta \varphi(x_1)} K_{n_0-1}(x)$$

$$\{n\} \equiv (n_0, n_1, \dots, n_r) \quad , \quad \sum_{i=0}^r n_i = n \quad , \quad n_i \in \mathbb{N} \quad (1.21)$$

$$c_{\{n\}} = (-1)^{r+1} [n_0 \cdot (n_0 + n_1) \cdot (n_0 + n_1 + n_2) \dots (n_0 + \dots + n_r)]^{-1}$$

which, after invoking the Leibnitz rule, represents a sum over tree structures as is depicted in Fig. 1. We conclude this section by working out explicitly the the first four orders:

$$T_0(x) = \varphi(x)$$

$$T_1(x) = - \int K_0(y) \frac{\delta T_0(x)}{\delta \varphi(y)} = - K_0(x)$$

$$\begin{aligned} T_2(x) &= -\frac{1}{2} \int \left[K_1(y) \frac{\delta T_0(x)}{\delta \varphi(y)} + K_0(y) \frac{\delta T_1(x)}{\delta \varphi(y)} \right] \\ &= -\frac{1}{2} K_1(x) + \frac{1}{2} \int K_0(y) \frac{\delta K_0(x)}{\delta \varphi(y)} \end{aligned}$$

$$\begin{aligned} T_3(x) &= -\frac{1}{3} \int \left[K_2(y) \frac{\delta T_0(x)}{\delta \varphi(y)} + K_1(y) \frac{\delta T_1(x)}{\delta \varphi(y)} + K_0(y) \frac{\delta T_2(x)}{\delta \varphi(y)} \right] \\ &= -\frac{1}{3} K_2(x) + \frac{1}{3} \int K_1(y) \frac{\delta K_0(x)}{\delta \varphi(y)} + \frac{1}{6} \int K_0(y) \frac{\delta K_1(x)}{\delta \varphi(y)} - \frac{1}{6} \iint K_0(z) \frac{\delta}{\delta \varphi(z)} \left(K_0(y) \frac{\delta K_0(x)}{\delta \varphi(y)} \right) \end{aligned}$$

$$\begin{aligned}
 T_4(x) &= -\frac{1}{4} \int \left[K_3(y) \frac{\delta T_0(x)}{\delta \varphi(y)} + K_2(y) \frac{\delta T_1(x)}{\delta \varphi(y)} + K_1(y) \frac{\delta T_2(x)}{\delta \varphi(y)} + K_0(y) \frac{\delta T_3(x)}{\delta \varphi(y)} \right] \\
 &= -\frac{1}{4} K_3(x) + \frac{1}{4} \int K_2(y) \frac{\delta K_0(x)}{\delta \varphi(y)} + \frac{1}{8} \int K_1(y) \frac{\delta K_1(x)}{\delta \varphi(y)} \\
 &\quad + \frac{1}{12} \int K_0(y) \frac{\delta K_2(x)}{\delta \varphi(y)} - \frac{1}{8} \iint K_1(z) \frac{\delta}{\delta \varphi(z)} \left(K_0(y) \frac{\delta K_0(x)}{\delta \varphi(y)} \right) \\
 &\quad - \frac{1}{12} \iint K_0(z) \frac{\delta}{\delta \varphi(z)} \left(K_1(y) \frac{\delta K_0(x)}{\delta \varphi(y)} \right) - \frac{1}{24} \iint K_0(z) \frac{\delta}{\delta \varphi(z)} \left(K_0(y) \frac{\delta K_1(x)}{\delta \varphi(y)} \right) \\
 &\quad + \frac{1}{24} \iiint K_0(w) \frac{\delta}{\delta \varphi(w)} \left(K_0(z) \frac{\delta}{\delta \varphi(z)} \left(K_0(y) \frac{\delta K_0(x)}{\delta \varphi(y)} \right) \right) \quad (1.22)
 \end{aligned}$$

3. ANALYSIS OF THE MASSLESS WESS-ZUMINO MODEL

To demonstrate the reliability of our approach we like to devote this section to a detailed study of the simplest 4-dimensional model that is known to exhibit a Nicolai transformation: a single massless $N=1$ Wess-Zumino multiplet, consisting only of a complex bosonic field $\phi(x)$ after elimination of the fermionic partner and the auxiliary field. We adopt the notation of Ref. 10) and specify

$$\begin{aligned}
 S_b[\varphi] &= - \int d^4x \left[(\partial_\mu \varphi)(\partial^\mu \varphi^*) + g^2 (\varphi \varphi^*)^2 \right] (x) \\
 S_f[\varphi] &= -\frac{1}{2} \text{Tr} \ln \mathcal{F}[\varphi] \quad , \quad \begin{bmatrix} \mathcal{F}_{\alpha\beta} & \mathcal{F}_{\alpha\dot{\beta}} \\ \mathcal{F}_{\dot{\alpha}\beta} & \mathcal{F}_{\dot{\alpha}\dot{\beta}} \end{bmatrix} = \begin{bmatrix} 2g\varphi & -i\delta \\ i\delta & 2g\varphi^* \end{bmatrix} \quad (2.1)
 \end{aligned}$$

and deduce

$$\begin{aligned}
 \frac{\partial S_b}{\partial g}[\varphi] &= -g \int d^4x (\varphi \varphi^*)^2 (x) + \text{h.c.} \\
 \frac{\partial S_f}{\partial g}[\varphi] &= - \int d^4x \varphi(x) (\mathcal{F}^{-1})_{\alpha}^{\alpha} (x,x) + \text{h.c.} \quad , \quad (2.2)
 \end{aligned}$$

$$\begin{aligned}\frac{\delta S_b[\varphi]}{\delta \varphi(x)} &= \square \varphi^*(x) - 2g^2 (\varphi \varphi^{*2})(x) \\ \frac{\delta S_f[\varphi]}{\delta \varphi(x)} &= -g (\mathcal{F}^{-1})_\alpha^\alpha(x, x)\end{aligned}\quad (2.3)$$

Inserting (2.2), (2.3) and (1.18) into the definitions (1.16) we end up with infinite sets of coupled equations for the even and odd kernels separately. As only the latter involve inhomogeneities, a particular solution may be found by setting $K_{2n}=0$ and analyzing ($n>1$, odd)

$$\int K_1[\varphi; x] \square \varphi^*(x) = \int (\varphi \varphi^*)^2(x) \quad (2.4b)$$

$$\int \frac{\delta K_1[\varphi; x]}{\delta \varphi(x)} = - \int \varphi(x) L_2[\varphi; x] \quad , \quad (2.4f)$$

$$\int K_n[\varphi; x] \square \varphi^*(x) = 2 \int K_{n-2}[\varphi; x] (\varphi \varphi^{*2})(x) \quad (2.5b)$$

$$\int \frac{\delta K_n[\varphi; x]}{\delta \varphi(x)} = - \sum_{\ell=1}^{[n/2]} \int K_{n-2\ell}[\varphi; x] L_{2\ell}[\varphi; x] - \int \varphi(x) L_{n+1}[\varphi; x] \quad (2.5f)$$

where we introduced the "fermion loops"

$$(\mathcal{F}^{-1})_\alpha^\alpha(x, x) =: \sum_{\ell=1}^{\infty} g^{2\ell-1} L_{2\ell}[\varphi; x] \quad (2.6)$$

For a graphical representation see Fig. 2. However, to exhaust the variety of Nicolai maps it is necessary to solve as well the homogeneous system:

$$\left. \begin{aligned} \int \tilde{K}_n[\varphi; x] \square \varphi^*(x) &= 2 \int \tilde{K}_{n-2}[\varphi; x] (\varphi \varphi^{*2})(x) \\ \int \frac{\delta \tilde{K}_n[\varphi; x]}{\delta \varphi(x)} &= - \sum_{\ell=1}^{[n/2]} \int \tilde{K}_{n-2\ell}[\varphi; x] L_{2\ell}[\varphi; x] \end{aligned} \right\} \begin{aligned} n \in \mathbb{N}_o, \\ \tilde{K}_{-1} \equiv 0 \equiv \tilde{K}_{-2} \end{aligned} \quad (2.7b)$$

$$(2.7f)$$

It should be remarked that on both sides of Eqs. (2.4), (2.5) and (2.7) a hermitian conjugate contribution from

$$R = \frac{\partial}{\partial g} + \int K[\varphi; x] \frac{\delta}{\delta \varphi(x)} + \int K^*[\varphi; x] \frac{\delta}{\delta \varphi^*(x)} \quad (2.8)$$

has been omitted so that, strictly speaking, only their real parts need to be valid. Without loss of generality, however, we fix the arbitrariness of their imaginary parts by taking these equations as they stand, i.e. treating K_n and K_n^* separately.

Let us first seek a solution of Eqs. (2.4), (2.5). A short inspection reveals that K_{2n-1} may be assumed to be homogeneous in ϕ and ϕ^* of order $(\phi^{n+1} \phi^{*n})$. The structure of Eq. (2.5) then strongly suggests a linear tree ansatz for K_{2n-1} : In (2.5b) the d'Alembertian acting on the left effects a tree shortening whereas the short-circuiting in (2.5f) produces a series of trees singly attached to loops as is visualized schematically in Fig. 3. Furthermore, the inhomogeneities of Eqs. (2.4f), (2.5f) dictate, modulo K_{2n-1} , the very details of K_{2n-1} : Due to

$$\begin{aligned} L_{2n} [\varphi; x_i] &= \ell^{\mu_1 \nu_1 \mu_2 \nu_2 \dots \mu_n \nu_n} \Lambda_{\mu_1 \nu_1 \mu_2 \nu_2 \dots \mu_n \nu_n} [\varphi; x_i] \\ \Lambda_{\mu_1 \dots \nu_n} [\varphi; x_i] &:= \int dx_1 \int dx_2 \dots \int dx_n \partial_{\mu_1} \bar{\square}^{-1}(x_1 - y_1) \varphi^*(y_1) \partial_{\nu_1} \bar{\square}^{-1}(y_1 - x_2) \varphi(x_2) \dots \varphi^*(y_n) \partial_{\nu_n} \bar{\square}^{-1}(y_n - x_i) \\ \ell^{\mu_1 \nu_1 \mu_2 \nu_2 \dots \mu_n \nu_n} &:= 2^{2n-1} \text{tr} (\sigma^{\mu_1} \bar{\sigma}^{\nu_1} \sigma^{\mu_2} \bar{\sigma}^{\nu_2} \dots \sigma^{\mu_n} \bar{\sigma}^{\nu_n}) \end{aligned} \quad (2.9)$$

we inevitably arrive at

$$K_{2n-1}^{(x)} = k^{\mu_1 \nu_1 \dots \mu_n \nu_n} \int dx_1 \int dx_2 \dots \int dx_n \varphi^2(x_1) \partial_{\mu_1} \bar{\square}^{-1}(x_1 - y_1) \varphi^*(y_1) \partial_{\nu_1} \bar{\square}^{-1}(y_1 - x_2) \varphi(x_2) \dots \partial_{\nu_n} \bar{\square}^{-1}(y_n - x) \quad (2.10)$$

which is depicted in Fig. 4. A delocalization of $\phi(x_i)$ or $\phi^*(y_i)$ from the trunk or a point-splitting of $\phi^2(x_1)$ is incompatible with (2.9), and a second possible ordering of fields along the tree, $(\phi\phi^*)(x_1) \phi(y_1) \phi^*(x_2) \dots \phi(y_n)$, is excluded by the homogeneous piece in Eq. (2.5f). We shall already remark here that the situation is more intricate in the case of gauge fields where the latter conclusion does not necessarily apply due to the connection of internal and space-time indices. Certainly Eqs. (2.4), (2.5) lead to restrictions on the coefficients $k^{\mu_1 \nu_1 \dots \mu_n \nu_n}$. It is instructive to observe how K_{2n-1} of Eq. (2.10), inserted into the left hand side of Eq. (2.5b), reduces to K_{2n-3} after some partial integrations, if

$$k^{\mu_1 \nu_1 \dots \mu_{n-1} \nu_{n-1} \{\mu_n \nu_n\}} = 4 k^{\mu_1 \nu_1 \dots \mu_{n-1} \nu_{n-1}} \eta^{\mu_n \nu_n} \quad (2.11)$$

(we symmetrize with weight 1/2; $\eta^{\mu\nu}$ denotes the Minkowskian metric). In addition,

Eq. (2.5f) yields

$$\begin{aligned}
 k^{\mu_1 \nu_1 \dots \mu_n \nu_n} &= k^{\mu_1 \nu_1 \dots \mu_{n-1} \nu_{n-1}} \ell^{\mu_n \nu_n} + \bar{k}_{(1)}^{\mu_1 \nu_1 \dots \mu_n \nu_n} \\
 &= k^{\mu_1 \nu_1 \dots \mu_{n-2} \nu_{n-2}} \ell^{\mu_{n-1} \nu_{n-1} \mu_n \nu_n} + \bar{k}_{(2)}^{\mu_1 \nu_1 \dots \mu_n \nu_n} \\
 &\quad \vdots \\
 &= k^{\mu_1 \nu_1} \ell^{\mu_2 \nu_2 \dots \mu_n \nu_n} + \bar{k}_{(n-1)}^{\mu_1 \nu_1 \dots \mu_n \nu_n} \\
 &= \frac{1}{2} \ell^{\mu_1 \nu_1 \dots \mu_n \nu_n} + \bar{k}_{(n)}^{\mu_1 \nu_1 \dots \mu_n \nu_n}
 \end{aligned} \tag{2.12}$$

where

$$\bar{k}_{(n-l)}^{\mu_1 \nu_1 \dots \mu_n \nu_n} \Lambda_{\mu_{l+1} \nu_{l+1} \dots \mu_n \nu_n}^{(x)} = 0, \quad l=0,1,\dots,n-1. \tag{2.13}$$

Compatibility of Eqs. (2.11) and (2.12) requires

$$\bar{k}_{(n-l)}^{\mu_1 \nu_1 \dots \mu_{n-1} \nu_{n-1} \{\mu_n \nu_n\}} = 4 \bar{k}_{(n-l-1)}^{\mu_1 \nu_1 \dots \mu_{n-1} \nu_{n-1}} \eta^{\mu_n \nu_n}, \quad \bar{k}_{(0)} \equiv 0. \tag{2.14}$$

Eqs. (2.11) and (2.12) now allow for an iteration whose starting point stems from Eq. (2.4b) or (2.4f):

$$k^{\mu\nu} = 2 \eta^{\mu\nu} + \bar{k}^{[\mu\nu]}. \tag{2.15}$$

There exist many solutions to (2.11-15) due to the arbitrariness of $\bar{k}$ as can already be seen at $n=2$:

$$\begin{aligned}
 k^{\mu\nu\rho\lambda} &= 8 \eta^{\mu\nu} \eta^{\rho\lambda} + 4 \bar{k}^{[\mu\nu]} \eta^{\rho\lambda} + \bar{k}_{(1)}^{\mu\nu\rho\lambda} \\
 &= 8 [\eta^{\mu\nu} \eta^{\rho\lambda} + \eta^{\mu\lambda} \eta^{\nu\rho} - \eta^{\mu\rho} \eta^{\nu\lambda} + i \varepsilon^{\mu\nu\rho\lambda}] + \bar{k}_{(2)}^{\mu\nu\rho\lambda}, \\
 \bar{k}_{(2)}^{\mu\nu\{\rho\lambda\}} &= 4 \bar{k}^{[\mu\nu]} \eta^{\rho\lambda}.
 \end{aligned} \tag{2.16}$$

To obtain a Lorentz-covariant set of coefficients we may simply set $\bar{k}_{(n)}=0$:

$$k_{cov}^{\mu_1 \nu_1 \dots \mu_n \nu_n} = \frac{1}{2} \ell^{\mu_1 \dots \nu_n} = 2^{2(n-1)} \text{tr}(\sigma^{\mu_1} \bar{\sigma}^{\nu_1} \dots \bar{\sigma}^{\nu_n}). \quad (2.17)$$

As a slight generalization one can also choose

$$\bar{k}_{(n-l)}^{\mu_1 \dots \nu_n} = \bar{\bar{k}}^{\mu_1 \dots \nu_l} \bar{k}_{(n-l)}^{\mu_{l+1} \dots \nu_n}, \quad \bar{\bar{k}} \text{ arbitrary} \quad (2.18)$$

and further specialize $\bar{\bar{k}} \equiv k$:

$$\begin{aligned} k^{\mu_1 \dots \nu_n} &= k^{\mu_1 \dots \nu_{n-1}} (\ell + \bar{k}_{(1)})^{\mu_n \nu_n} \\ &= k^{\mu_1 \dots \nu_{n-2}} (\ell + \bar{k}_{(2)})^{\mu_{n-1} \nu_{n-1} \mu_n \nu_n} \\ &\quad \vdots \\ &= k^{\mu_1 \nu_1} (\ell + \bar{k}_{(n-1)})^{\mu_2 \nu_2 \dots \mu_n \nu_n} \\ &= \frac{1}{2} (\ell + \bar{k}_{(n)})^{\mu_1 \nu_1 \dots \mu_n \nu_n} \end{aligned} \quad (2.19)$$

with

$$\begin{aligned} \bar{k}_{(n-l)}^{\mu_{l+1} \nu_{l+1} \dots \mu_n \nu_n} \Lambda_{\mu_{l+1} \nu_{l+1} \dots \mu_n \nu_n}^{(x)} &= 0, \\ \bar{k}_{(n-l)}^{\mu_{l+1} \dots \nu_{n-1} \{ \mu_n \nu_n \}} &= 4 \bar{k}_{(n-l-1)}^{\mu_{l+1} \dots \nu_{n-1}} \gamma^{\mu_n \nu_n}, \quad \bar{k}_{(1)}^{\{ \mu \nu \}} = 0. \end{aligned} \quad (2.20)$$

Eqs. (2.19) and (2.20) are satisfied by taking for example

$$\bar{k}^{\mu_1 \dots \nu_n} = 2^{2n-1} \left[(\sigma^{\mu_1} \bar{\sigma}^{\nu_1} \dots \bar{\sigma}^{\nu_n})_1' - (\sigma^{\mu_1} \bar{\sigma}^{\nu_1} \dots \bar{\sigma}^{\nu_n})_2^2 \right] \cdot \omega \quad (2.21)$$

which amounts to

$$k_{\omega}^{\mu_1 \dots \nu_n} = 2^{2(n-1)} \left[(\sigma^{\mu_1} \dots \bar{\sigma}^{\nu_n})_1' (1+\omega) + (\sigma^{\mu_1} \dots \bar{\sigma}^{\nu_n})_2^2 (1-\omega) \right] \quad (2.22)$$

The special results (2.17) and (2.22), together with Eqs. (2.9), (1.18) and (1.6) may also be obtained by a more straightforward but supersymmetric procedure:

Starting from the observation that the supersymmetric action S_s of any model that admits a superspace formulation can be expressed as a supervariation δ_α ($\alpha=1,2$) of some fermionic functional Δ^α ,

$$\begin{aligned} S_s &= \delta_1 \Delta^1 + h.c. = \delta_2 \Delta^2 + h.c. \\ &= \frac{1}{2} \delta_\alpha \Delta^\alpha + \frac{\omega}{2} (\delta_1 \Delta^1 - \delta_2 \Delta^2) + h.c. , \end{aligned} \quad (2.23)$$

and exploiting the supersymmetry Ward identity an R-operation can be constructed directly proving the existence of a Nicolai map (not assuming it, cf. (1.5)):

$$R_\omega = \frac{\partial}{\partial g} + \frac{1}{2} \left[(1+\omega) \left(\frac{\partial \Delta^1}{\partial g} \right) \delta_1 + (1-\omega) \left(\frac{\partial \Delta^2}{\partial g} \right) \delta_2 + h.c. \right] , \quad (2.24)$$

the contractions indicate full fermionic propagators that originate from integrating out the fermionic content of the theory. After inserting Δ^α , expressing δ_α via functional derivatives and expanding the fermionic propagators R_ω is cast into the form (1.6), (1.18), (2.9) and reproduces 2.22). This derivation has been presented in more detail for $\omega=0$ in Ref. 4).

A particularly interesting case besides the covariant one ($\omega=0$) occurs at $\omega=1$ (or $\omega=-1$):

$$k_{lc}^{\mu_1 \dots \mu_n} = 2^{2n-1} (\sigma^{\mu_1} \bar{\sigma}^{\nu_1} \dots \sigma^{\mu_n} \bar{\sigma}^{\nu_n})_l{}^1 . \quad (2.25)$$

The index "lc" reminds us of the fact that in light cone coordinates

$$\begin{aligned} X_0 + X_3 &= X_R & X_1 + iX_2 &= X_+ \\ X_0 - X_3 &= X_L & X_1 - iX_2 &= X_- \end{aligned} \quad (2.26)$$

$k_{lc}^{\mu_1 \nu_1 \dots \mu_n \nu_n}$ represents only 2^{2n-1} non-vanishing coefficients, which is probably the minimal number and only half of what one would naively expect from

$$k^{\{\mu_1 \nu_1\} \{\mu_2 \nu_2\} \dots \{\mu_n \nu_n\}} = 2^{2n-1} \eta^{\mu_1 \nu_1} \eta^{\mu_2 \nu_2} \dots \eta^{\mu_n \nu_n} . \quad (2.27)$$

For instance, $k^{\{\mu\nu\}} = 2\eta^{\mu\nu}$ is already satisfied by taking

$$k^{LR} = 1 = -k^{+-} , \quad \text{others vanish,} \quad (2.28)$$

a choice that involves only two non-zero coefficients as opposed to four in a

covariant scheme ($k^{00} = -k^{11} = -k^{22} = -k^{33} = 1$).

For the coefficients of $K_n^*[\phi; x]$, $k^{*\mu_1 \dots \mu_n}$, the line of arguments starting from Eq. (2.4) can be repeated identically. Attempts to set K_n^* to zero, i.e. to saturate the inhomogeneities in Eqs. (2.4), (2.5) (together with their hermitian conjugates) by K_n alone, must fail because the non-vanishing imaginary part of $g^{\mu_1 \dots \mu_n}$ then creates inconsistencies in Eq. (2.12). However, we are free to select different solutions k and k^* in the sense that

$$k^{*\mu_1 \dots \mu_n} \neq (k^{\mu_1 \dots \mu_n})^* \quad \Leftrightarrow \quad (2.29)$$

$$K^*[\varphi; x] \neq K[\varphi^*; x] .$$

Since all possible coefficients can be obtained from K_{cov} (Eq.(2.17)) by adding suitable solutions $\tilde{K}$ of the homogeneous system we shall now turn our attention to Eqs. (2.7). As for Eqs. (2.5) the occurrence of L_{2n} (Eq. (2.9)) motivates a linear tree ansatz, but no inhomogeneities restrict its finer structure. A possible iterative procedure

$$\begin{aligned} \tilde{K}_0 &\rightarrow \tilde{K}_2 \rightarrow \tilde{K}_4 \rightarrow \dots \\ \tilde{K}_1 &\rightarrow \tilde{K}_3 \rightarrow \tilde{K}_5 \rightarrow \dots \end{aligned}$$

requires as a starting point and at each further step the complete solution of

$$\int \tilde{K}[\varphi; x] \square \varphi^*(x) = 0 \quad , \quad \int \frac{\delta \tilde{K}[\varphi; x]}{\delta \varphi(x)} = 0 \quad . \quad (2.30)$$

A few examples are easily found:

$$\begin{aligned} &\iint dy dz \quad \varphi^*(z) \partial_\mu \partial_\nu^{-1}(z-y) \varphi^*(y) \partial_\nu \square^{-1}(y-x) \\ &\iint dy dz \quad \varphi^*(z) \partial_\mu \partial_\nu \square^{-1}(z-y) \varphi^*(y) \partial^\nu \square^{-1}(y-x) \\ &\int dy \quad (\partial_{[\mu} \varphi^2_{\nu]}) (\partial_{\nu]} \varphi^*_{\mu]) \square^{-1}(y-x) \\ &\iint dy dz \quad (\partial_{[\mu} \varphi^*(z)) \varphi(z) \square^{-1}(z-y) \varphi(y) \partial_{\nu]} \square^{-1}(y-x) \quad , \end{aligned} \quad (2.31)$$

a list which is far from being complete. Nevertheless, completeness is not essentially what we are aiming at, for the following reason: One of our main purposes is to explore the possibility of a polynomial, i.e. considerably simpler structured Nicolai map, a task that will be undertaken in the next section. Rather than to write down the most general mapping (which is beyond our scope) we shall concentrate on those homogeneous solutions $\tilde{K}$ that can substantially alter T_ω , the part of T which stems from the inhomogeneous K_ω already found (Eq. (2.22)), and lead to a breakoff of the perturbative expansion (1.19). Hence, we shall basically be interested only in those $\tilde{K}$ that interfere with K_ω in Eq. (1.21). Other solutions cannot help in cancelling part of T_ω to transform the power series (1.21) into a polynomial and will therefore be discarded in the following. A short examination shows that the relevant $\tilde{K}$ have to contain pieces of certain homogeneities:

$$\left. \begin{aligned} \tilde{K}_{2n-1} &: \mathcal{O}(y^{n+1} y^{*n}) \\ \tilde{K}_{2n-2} &: \mathcal{O}(y^{n-1} y^{*n+1}) \end{aligned} \right\} n \in \mathbb{N} \quad (2.32)$$

Because different contributions lead to new orders in (ϕ, ϕ^*) in T_n which then should be cancelled as well it is desirable to exclude them from the discussion. In the same spirit we are finally forced back to the "standard structure"

$$\begin{aligned} \tilde{K}_{2n-1}(x) &= \tilde{K}^{\mu_1 \dots \mu_n} \int dx_1 \int dy_1 \dots \int dy_n \varphi^2(x_1) \partial_{\mu_1} \bar{\phi}^{-1}(x_1 - y_1) \varphi^*(y_1) \partial_{\nu_1} \bar{\phi}^{-1}(y_1 - x_2) \varphi(x_2) \dots \partial_{\nu_n} \bar{\phi}^{-1}(y_n - x) \\ \tilde{K}_{2n-2}(x) &= \tilde{K}^{\nu_1 \mu_2 \dots \nu_n} \int dy_1 \int dx_2 \dots \int dy_n \varphi^2(y_1) \partial_{\nu_1} \bar{\phi}^{-1}(y_1 - x_2) \varphi(x_2) \partial_{\mu_2} \bar{\phi}^{-1}(x_2 - y_2) \varphi^*(y_2) \dots \partial_{\nu_n} \bar{\phi}^{-1}(y_n - x) \end{aligned} \quad (2.33)$$

and the conditions

$$\tilde{K}^{(\mu_1) \nu_1 \dots \nu_{n-1} \{ \mu_n \nu_n \}} = 4 \tilde{K}^{(\mu_1) \nu_1 \dots \nu_{n-1}} \eta^{\mu_n \nu_n} \quad (2.34)$$

$$\begin{aligned} \tilde{K}^{(\mu_1) \nu_1 \dots \nu_n} &= \tilde{K}^{(\mu_1) \nu_1 \dots \nu_{n-1}} \eta^{\mu_n \nu_n} + \tilde{K}_{(1)}^{(\mu_1) \nu_1 \dots \nu_n} \\ &= \tilde{K}^{(\mu_1) \nu_1 \dots \nu_{n-2}} \eta^{\mu_{n-1} \nu_{n-1}} \eta^{\mu_n \nu_n} + \tilde{K}_{(2)}^{(\mu_1) \nu_1 \dots \nu_n} \\ &\quad \vdots \\ &= \tilde{K}^{(\mu_1) \nu_1} \eta^{\mu_2 \nu_2} \dots \eta^{\mu_n \nu_n} + \tilde{K}_{(n-1)}^{(\mu_1) \nu_1 \dots \nu_n} \\ &= \tilde{K}_{(n)}^{\mu_1 \nu_1 \dots \mu_n \nu_n} \quad (\text{odd case only}) \quad (2.35) \end{aligned}$$

$$\tilde{k}_{(n-l)}^{\mu_1 \nu_1 \dots \nu_n} \Lambda_{\mu_{l+1} \nu_{l+1} \dots \mu_n \nu_n}(x) = 0, \quad (2.36)$$

$$l = \begin{cases} 0, 1, \dots, n-1 & \text{odd case} \\ 1, 2, \dots, n-1 & \text{even case} \end{cases}$$

the index μ_1 is absent in the even case. Compatibility again demands the relation (2.14) between $\tilde{k}_{(n-l)}$ and $\tilde{k}_{(n-l-1)}$. The last equality in Eq. (2.35) tells us that $\tilde{k}_{2n-1}$ of (2.33) has already been included in (2.12), indeed

$$\tilde{k}^{\mu_1 \nu_1 \dots \mu_n \nu_n} = \tilde{k}_{(n)}^{\mu_1 \nu_1 \dots \mu_n \nu_n} = \overline{k}_{(n)}^{\mu_1 \nu_1 \dots \mu_n \nu_n} \quad (2.37)$$

The even case, on the contrary, gives rise to new solutions (we omit the tildes):

$$k^s \text{ arbitrary} \quad (2.38)$$

$$k^{\mu\nu} = 4k^s \gamma^{\mu\nu} + \overline{k}^{\mu\nu} \quad \text{etc.}$$

which, however, cannot be Lorentz-covariant. Thus, our investigation of the homogeneous equations (2.7) exhibited the option of including even ordered K_n into the construction (1.21) of T . This may be relevant for achieving polynomiality of the latter.

4. POLYNOMIALITY ?

The importance of a polynomial Nicolai map

$$(T\varphi)(x) = \sum_{n=0}^N g^n T_n[\varphi; x] \quad (3.1)$$

lies in its intimate connection to a stochastic process^{4,5)}. In such a "stochastic" case T^{-1} , the relevant transformation to calculate vacuum expectation values via Eq. (1.2a), is generated by a Langevin (integro-) differential equation involving only a finite number of non-localities, if any. Moreover, such a mapping could carry us beyond perturbation theory. Due to the behaviour under the rescaling $g\phi \rightarrow \phi$ of any theory considered here polynomiality in the coupling constant g or in the fields ϕ are equivalent (and differ only by one in power). Taking into account the results of the preceding section we now have at our disposal the tools to decide the question if in the complete set of Nicolai maps there exists a subclass of polynomial ones, at least for the Wess-Zumino model. In this section we will set out to demonstrate that for chiral theories in 4 dimensions the answer

is no. An important implication of this result for certain super-Yang-Mills theories (SYM) will be presented thereafter.

To start with, we note the conditions for polynomiality:

$$\exists N \in \mathbb{N} \quad \forall n > N : T_n[\varphi; x] = 0. \quad (3.2)$$

Recalling the recursion relation (1.20) this reads

$$\exists N \in \mathbb{N} \quad \forall s \in \mathbb{N}_0 : \sum_{\ell=0}^N \int dy K_{s+\ell}(y) \frac{\delta T_{N-\ell}(x)}{\delta \varphi(y)} = 0. \quad (3.3)$$

Expressing $T_{N-\ell}$ in terms of K_n it yields a series of homogeneous relations for K_n . Let us study the case $s=0$ (i.e. $T_{N+1}=0$) as a necessary condition in the context of the 4-dimensional massless Wess-Zumino model. Here it translates into restrictions on the "standard structure" coefficients $k^{\mu_1 \nu_1 \dots \nu_n}$ and $k^{\nu_1 \dots \nu_n}$. We analyze the first three non-trivial conditions in more detail from Eq. (1.22):

$$N=1 : K_1(x) = \int K_0(y) \frac{\delta K_0(x)}{\delta \varphi(y)}$$

$$N=2 : 2K_2(x) = 2 \int K_1(y) \frac{\delta K_0(x)}{\delta \varphi(y)} + \int K_0(y) \frac{\delta K_1(x)}{\delta \varphi(y)} \\ - \iint K_0(z) \frac{\delta}{\delta \varphi(z)} \left(K_0(y) \frac{\delta K_0(x)}{\delta \varphi(y)} \right)$$

$$N=3 : 6K_3(x) = 6 \int K_2(y) \frac{\delta K_0(x)}{\delta \varphi(y)} + 3 \int K_1(y) \frac{\delta K_1(x)}{\delta \varphi(y)} + 2 \int K_0(y) \frac{\delta K_2(x)}{\delta \varphi(y)} \\ - 3 \iint K_1(z) \frac{\delta}{\delta \varphi(z)} \left(K_0(y) \frac{\delta K_0(x)}{\delta \varphi(y)} \right) - 2 \iint K_0(z) \frac{\delta}{\delta \varphi(z)} \left(K_1(y) \frac{\delta K_0(x)}{\delta \varphi(y)} \right) \\ - \iint K_0(z) \frac{\delta}{\delta \varphi(z)} \left(K_0(y) \frac{\delta K_1(x)}{\delta \varphi(y)} \right) + \iiint K_0(w) \frac{\delta}{\delta \varphi(w)} \left(K_0(z) \frac{\delta}{\delta \varphi(z)} \left(K_0(y) \frac{\delta K_0(x)}{\delta \varphi(y)} \right) \right)$$

(3.4)

implies

$$N=1 : k^{\mu\nu} = 2k^{*\mu}k^\nu$$

$$N=2 : k^{\mu\nu\rho} = k^\mu k^{\nu\rho} + 2k^{*\mu\nu}k^\rho - 2k^\mu k^{*\nu}k^\rho$$

$$0 = k^{*\mu} \cdot (k^{\nu\rho} - 2k^{*\nu}k^\rho) + (\mu \leftrightarrow \nu)$$

$$\begin{aligned}
 N=3: k^{\mu\nu\rho\lambda} &= k^{\mu\nu}k^{\rho\lambda} - 2k^{\mu\nu}k^{*\rho}k^\lambda - \frac{2}{3}k^{*\mu}k^\nu k^{\rho\lambda} + \frac{4}{3}k^{*\mu}k^\nu k^{*\rho}k^\lambda \\
 &\quad + 2k^{*\mu\nu\rho}k^\lambda + \frac{2}{3}k^{*\mu}k^{\nu\rho\lambda} - \frac{4}{3}k^{*\mu}k^{*\nu\rho}k^\lambda \\
 0 &= 2(k^{\mu\nu\lambda} - k^\mu k^{\nu\lambda} - 2k^{*\mu\nu}k^\lambda + 2k^\mu k^{*\nu}k^\lambda) \cdot k^{*\rho} \\
 &\quad + (3k^{*\mu\nu} - 2k^\mu k^{*\nu}) \cdot (k^{\rho\lambda} - 2k^{*\rho}k^\lambda) \\
 0 &= k^\mu (k^{\nu\rho\lambda} - k^\nu k^{\rho\lambda} - 2k^{*\nu\rho}k^\lambda + 2k^\nu k^{*\rho}k^\lambda) + (\mu \leftrightarrow \nu)
 \end{aligned} \tag{3.5}$$

together with hermitian conjugate identities. Apparently a factorization of branches occurs which always leads us back to conditions of lower order, a phenomenon that persists at higher orders. After excluding various cases, e.g.

$$k^{\nu_1 \mu_2 \dots \nu_n} = 0 \quad \forall n, \quad k^{\mu\nu\rho\lambda} = k^{\mu\nu}k^{\rho\lambda}, \quad k^{*\mu\nu}k^{\rho\lambda} = 0, \tag{3.6}$$

by invoking Eq. (2.11), we finally end up with

$$k^{\mu\nu} = 2k^{*\mu}k^\nu, \quad k^{*\mu\nu} = 2k^\mu k^{*\nu} \tag{3.7}$$

or, with the help of Eq. (2.15),

$$k^\mu k^{*\nu} + k^\nu k^{*\mu} = 2\eta^{\mu\nu}. \tag{3.8}$$

Here we meet the definition of the Dirac-Weyl algebra which in 4 dimensions is certainly not realizable by complex numbers k^μ . Dimensional reduction of the whole scheme yields the same condition (3.8) for the 2-dimensional $N=2$ extended chiral model; its stochasticity is revealed by the light cone solution

$$k_+ = 1 = k_-^*, \quad k_- = 0 = k_+^*. \tag{3.9}$$

On the other hand, for the most simple, the $N=1$ Wess-Zumino model in 2 dimensions we find (due to Majorana instead of Weyl fermions)

$$k^\mu k^\nu = \eta^{\mu\nu} \tag{3.10}$$

which cannot be satisfied by real coefficients k^μ . Obviously, stochasticity in chiral models just requires their fermions to sit in one-dimensional representations of the Lorentz group. It has to be remarked that the dimensionality of the

irreducible spinor representations depends on the underlying number field. In other words, it is the non-triviality of the fundamental Lorentz group representations over the complex numbers in 4 dimensions and over the real numbers in 2 dimensions (hiding behind Eqs. (3.8) and (3.10), respectively) which renders a polynomial Nicolai map for the corresponding theories impossible. By supersymmetry this is related to the bosonic field content: Stochastic chiral models can only exist if the bosonic fields form an element of a division algebra of the same dimensionality as space-time. This result extends to chiral theories containing several, even massive Wess-Zumino multiplets, because the space-time structure is always decoupled from the internal one.

Following this point of view it is tempting to make the quaternionic nature of 4-dimensional Yang-Mills fields responsible for the stochasticity of N=1 SYM (unfortunately there exists no SYM theory in 8 dimensions). But for gauge theories the situation is more complicated in various respects. First of all, any mapping has to depend on the gauge fixing chosen. Secondly, space-time and internal indices are partially identified (as in 2-dimensional N=2 Wess-Zumino models) which puts derivatives ∂_μ and the gauge field A_μ on the same footing and necessitates the inclusion of terms like $\square^{-1} \partial_\mu A_\nu$ together with $\square^{-1} A_\mu \times A_\nu$ in our tree ansatz, i.e. point-splitting at branch endings. This entanglement is reflected by the fact that the d'Alembertian appearing in chiral models is replaced by

$$\left[\eta_{\mu\nu} - (G_\mu \partial_\nu + G_\nu \partial_\mu) (G^\rho \partial_\rho)^{-1} + (G^\lambda G_\lambda) (G^\rho \partial_\rho)^{-2} \partial_\mu \partial_\nu \right] \square^{-1} \quad (3.11)$$

in the gauges $G(A_\mu) = G^\mu A_\mu$, introducing non-localities on top of $\square^{-1}$.

From the work of de Alfaro, Fubini, Furlan and Veneziano we know about quadratic^{+) Nicolai maps in 4-dimensional⁶⁾ and cubic^{+) ones in 6-dimensional⁷⁾ N=1 SYM in light cone gauges, connected to the quaternions and octonions, respectively. Thus there is no need to apply our techniques to these cases. Only for the 10-dimensional model that reduces to the 4-dimensional N=4 SYM⁸⁾ the polynomiality question is still open. However, our no-go result for the Wess-Zumino model implies a negative answer here, too: An inspection of the truncation $A_\mu = 0$ (supersymmetrically consistent together with sending its partners to zero) reveals that it forces a potentially polynomial Nicolai transformation to pass over into a polynomial one for a purely chiral model. The latter consists of Wess-Zumino multiplets Φ_i^a ; $i=1,2,3$; $a=1,\dots,r$, coupled by the superpotential}}

$$V(\varphi_i^a) = \frac{1}{3} g \varepsilon^{ijk} f_{abc} \varphi_i^a \varphi_j^b \varphi_k^c \quad (3.12)$$

^{+) These specifications refer to the highest occurring power of the gauge field.}

where ϵ^{ijk} denotes the Levi-Civita symbol and f_{abc} the structure constants of the gauge group, r being the dimension of its adjoint representation. Hence, as this model does not allow for a polynomial mapping, $N=4$ SYM cannot, either. On the contrary, its (perturbative) Nicolai transformation must show at least the complexity we have encountered in the Wess-Zumino model. Inevitably, the same result holds true for 6-dimensional $N=2$ and 10-dimensional $N=1$ SYM, as those theories go into $N=4$ by dimensional reduction. To be complete, for the models found to be stochastic^{6,7)} the argument just given fails simply because the truncated, chiral theories are just collections of free Wess-Zumino models. Finally we should note that our treatment was based on perturbation theory so that in 10 dimensions stochastic variables, if they exist, are forced to be entirely non-perturbative.

5. CONCLUSIONS

Starting from the definition of the Nicolai transformation we developed the concept of its generator associated with an infinitesimal change of parameters. In contradistinction to earlier works^{4,9)} where the existence of a Nicolai map was proved by its explicit construction for flat space supersymmetric models we here derived necessary and sufficient conditions for any bosonic theory to admit such a mapping, without employing supersymmetry at all.

In the following these conditions were analyzed in the context of the massless 4-dimensional Wess-Zumino model. We were not only able to rederive and extend earlier results⁴⁾ but also to characterize the complete set of Nicolai maps by a system of homogeneous integral equations. The latter were then exploited to write down the most general Nicolai transformation that might be polynomial in the coupling constants and, hence, in the fields.

However, this expression could not satisfy the conditions necessary for a breakoff of the perturbative tree expansion thus rendering all Nicolai maps of 4-dimensional chiral (and 2-dimensional real) theories non-polynomial. Without repeating the analysis for the more intricate case of gauge fields this result was already strong enough to exclude a polynomial transformation also for the 4-dimensional $N=4$, 6-dimensional $N=2$ and 10-dimensional $N=1$ super-Yang-Mills theories. The relevant polynomiality criterion turned out to be the triviality of the Lorentz group representation of the fermions or, equivalently, the existence of an internal division algebra structure of the space-time dimensionality in the bosonic sector. Accordingly, stochasticity in supersymmetric field theories remains confined to the 2-dimensional $N=2$ chiral and 4-dimensional $N=1$ (eventually coupled to free chiral multiplets) and $N=2$ and 6-dimensional $N=1$ SYM theories, its extension to ten dimension being impossible.

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FIGURE CAPTIONS

- Fig. 1: A generic term in the expression (1.2) of T_n in terms of K_ℓ , $\ell < n$. The bubbles (n_i) represent $K_{n_i-1}(x_i)$, arrows symbolize functional derivatives taken at the points they emerge from.
- Fig. 2: Graphical representation of $L_{2\ell}$ (cf. Eqs. (2.6), (2.9)). Solid lines denote free Weyl propagators ($\partial_\mu \square^{-1}$, $\bar{\partial}_\mu \square^{-1}$) and undotted (dotted) wavy ones bosonic fields ϕ (ϕ^*); the cross marks the open point x , the other $2\ell-1$ points are integrated over; a spin trace along the loop is understood.
- Fig. 3: Short-circuiting of a linear tree ansatz for K_{2n-1} reproduces the structure of Eq. (2.5f). Solid lines symbolize (yet unspecified) kernels connecting points that are integrated over (except for the crossed end point); the indexed end points carry bosonic fields.
- Fig. 4: Diagrammatical notation for K_{2n-1} of Eq. (2.10). Solid lines depict propagators $\partial_{\mu_i} \square^{-1}$ or $\bar{\partial}_{\mu_i} \square^{-1}$, undotted (dotted) wavy lines indicate fields ϕ (ϕ^*); vertex points are integrated over except for the crossed end point.

- Figure 1 -

- Figure 2 -

$$K_{2n-1} :$$

$$\frac{\delta K_{2n-1}}{\delta \varphi} :$$

- Figure 3 -

- Figure 4 -