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QUANTUM ELECTRODYNAMICS AS A SUPERSYMMETRIC THEORY OF LOOPS*

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ABSTRACT

It is shown that the Wilson loop is not a renormalizable operator even in free QED. A modification is suggested and proven to be renormalizable. Feynman's worldline formulation of QED is generalized to include spinning particles and matter loops, using this modified Wilson loop. QED formulated this way is shown to have a supersymmetry. A superfield formalism is found that describes it. QED is rewritten as a second quantized field theory of loop functionals.

1. INTRODUCTION

The problem of understanding Quantum Chromodynamics (QCD) in terms of color singlet fields is one of the outstanding challenges in particle physics. Much work has been done in this direction,¹⁻⁵ yet the problem remains unsolved. There are some indications that QCD is equivalent to a theory of strings² which approach a free theory as $s \rightarrow \infty$, $N_c \rightarrow \infty$.

In this paper, we study the much simpler problem of rewriting quantum electrodynamics (QED) in terms of gauge invariant operators. The main motivation is the hope that some light will be shed on the non-Abelian theory. Of course, one does not expect confinement in QED - at least not for weak couplings. So these gauge invariant operators will not create one-particle states. (In QCD, due to confinement, we expect the one particle states to be created by gauge invariant operators).

However, QED is of some interest as a theory in itself. There has been previous work aimed at rewriting QED in terms of gauge invariant operators,^{6,7,8} The basic results of this paper are as follows:

1. We first ask what the natural gauge invariant operator is, whose Green's functions are to be studied. A popular choice is the Wilson loop.*

$$W[x] = e^{ig \oint A_\mu x^\mu d\sigma}$$

However, we shall show that even in a free gauge theory, there are severe problems in defining $W[x]$ as a renormalized quantum operator.⁹⁻¹¹

2. In any case, in a theory with spinning matter fields, it is more natural to consider a modified Wilson loop.*

$$\chi[x] = tr e^{ig \int A_\mu dx^\mu + i g \int \sigma_{\mu\nu} F_{\mu\nu} d\tau}$$

The $\sigma \cdot F$ term represents the "precession" of the magnetic moment μ . In fact, this operator will be shown to be renormalizable (in an appropriate sense) if the magnetic moment is that of a Dirac particle.

* $x^\mu(\sigma)$, $0 \leq \sigma \leq T$ is a closed path: $-\mu = 1 \dots 4$; we are in Euclidean space. $[x^\mu(0) = x^\mu(T)]$

* $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$, γ_μ are the Dirac matrices. g is the electric charge. We have used " μ " both as an index and to denote magnetic moment. There should be no confusion from this.

3. $\chi[x]$ is basically the action of a spinning particle propagating in a loop. But this can also be written in terms of a path integral over Grassman variables:^{12–13}

$$\chi[x] = \int D[\psi] e^{-\int \mathcal{L} d\tau}$$

where

$$\mathcal{L} = \frac{1}{2} \dot{x}^2 + \frac{1}{2} \psi \dot{\psi} + m_0^2 + L_I$$

and

$$L_I = ig A_\mu \dot{x}^\mu + g \bar{F}_{\mu\nu} \psi^\mu \psi^\nu$$

This Lagrangian for a charged spinning particle has a supersymmetry (SUSY). In fact, one could say that it is this SUSY that is responsible for the cancellation of the non-renormalizable infinities in X . To make the SUSY more explicit, we may consider

$$\chi[x, \psi] = e^{-\int \mathcal{L} d\tau}$$

The objects we study are the n -point function of this operator.

4. We shall see that this SUSY is unbroken by quantization. The Ward identities of this symmetry will be derived.
5. For this purpose, a one-dimensional superfield formalism will be developed.
6. We will integrate over the photon and electron fields of QED and express it entirely as a theory of the operators $\chi[x, \psi]$. This will be an interacting theory of loops, exactly equivalent to QED. This^{14,15} is * similar to Feynman's action-at-a-distance formalism of electrodynamics. We have now succeeded in introducing spinning particles as well as fermion loops into this formalism.
7. There are some analogies between our theory of loops and string theories. In fact, QED seems to be equivalent to the massless projection of some supersymmetric string theory. However, we will show that our theory of loops has only a finite number of propagating modes, unlike string theories. For non-Abelian theory, there are arguments^{3,5} that there must be an infinite number of propagating modes.
8. We will see from the proof of equivalence that the theory of loops is a renormalizable theory. We will also provide a gauge and Lorentz invariant, supersymmetric regularization.

Section II and Appendix A study the question of renormalizability of the Wilson loop in free QED. Section III and Appendix B prove that the modified Wilson loop is multiplicatively renormalizable in free QED. Appendix C develops the superfield formalism to the level required. In Section IV and Appendix D we study QED with fermions and develop the expansion in powers of N_f .

* Feynman succeeded only in doing scalar electrodynamics this way, and that too without scalar self-interaction and without scalar loops. This "approximation" to QED will be of much use to us in Section II.

In Section V we compare and contrast the theory of loops with string theories. Section VI contains some inevitable speculations.

II. THE WILSON LOOP IN QUANTUM THEORY

In a pure gauge theory,

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu}^2 \quad (2.1)$$

a natural gauge invariant operator is the Wilson loop,*

$$W[x] = \text{tr} \, e^{i\oint A_\mu dx^\mu} \quad (2.2)$$

The Wilson loop plays a useful role because many other gauge invariant objects are derived from it. For example, consider a spin zero quantum field $\phi(x)$ coupled to a background electromagnetic field. The following is an example of a gauge invariant expectation value:

$$\begin{aligned} \Phi[z] &= \langle 0 | \phi^*(x) e^{i\oint_{\Gamma} A_\mu dx^\mu} \phi(x) | 0 \rangle \\ &= e^{i\oint_{\Gamma} A_\mu dx^\mu} D^A(x, x') . \end{aligned} \quad (2.3)$$

Here, z is a curve that starts at x' and ends at x .

$D^A(x, x')$ is the scalar propagator in the background A . This has a well-known²² path-integral representation:

$$\begin{aligned} D^A(x, x') &= \int_{\Omega_T(x, x')} D[y] e^{-\frac{i}{2} \int_0^T y^2 dt - \frac{m_0^2}{2} \int_0^T dt} \\ &\quad e^{i\oint_{\Gamma} A_\mu dy^\mu} \end{aligned} \quad (2.4)$$

Here, $\Omega_T(x, x')$ is the set of all curves $y(t)$ such that

$$\begin{aligned} y(0) &= x \\ y(T) &= x' \end{aligned}$$

This gives a path integral representation for $\Phi[z]$:

$$\Phi[z] = \int_{\Omega_T(x, x')} D[y] e^{-\frac{i}{2} \int y^2 dt - \frac{1}{2} m_0^2 \int dt} W[y + z] \quad (2.5)$$

Here, $y + z$ denotes the closed curve obtained by combining y and z .

In this section we study the question of renormalizability of the Wilson loop. If $x(t)$ is a differentiable curve and A_μ is a classical gauge field, there is no problem with the definition of W .

* $x(t)$ denotes a closed curve, $0 \leq t \leq T$ and $x(0) = x(T)$. The symbol $\oint$ will be used for path-ordered exponential. For Abelian classical gauge theory, there is no need for an ordering prescription.

However, if A_μ is a quantum field, there is no meaning to (2.2) without a renormalization. For example, even $\phi^2(x)$ is not well-defined in a field theory, without renormalization.

The usual resolution is to define²¹

$$\phi^2(x) = \mathcal{L}t_{x \rightarrow x'} [c_1(x, x') \phi(x) \phi(x') - c_2(x, x')] \quad (2.6)$$

If c-number functions (c_1, c_2) can be found such that the above limit exists, then this is a valid definition. But it is by no means guaranteed that such a function exists. (Typically, renormalization also might mix different operators).

In our case, there is an additional complication. We are not satisfied by W being well-defined for smooth curves. In most applications, W appears inside a path integral over x , over some range Ω :

$$\int_{\Omega} \mathcal{D}[x] e^{-\frac{1}{2} \int \dot{x}^2(\tau) d\tau} W[x] \quad (2.7)$$

This means that we must consider W as a functional on the space of all curves that can contribute to this path integral.

Unfortunately, this is a bizarre class of curves. They are functions such that,

$$||x||^2 = \int \dot{x}^2(\tau) d\tau$$

is finite. It is known that a "typical" function of this class is continuous but not differentiable.²² In fact, they are "half-differentiable" i.e.,

$$\mathcal{L}t_{t' \rightarrow t} \frac{|x(t') - x(t)|}{|t' - t|^{\frac{1}{2}}}$$

exists. Even when A_μ is a classical background field, $W[x] = e^{i\int A_\mu \dot{x}_\mu(\tau) d\tau}$ may not be well-defined²³ as a function on the space of curves $\mathcal{P}$. Rather, it is a distribution, so that

$$\int \mathcal{D}[x] e^{-\frac{1}{2} \int \dot{x}^2(\tau) d\tau} W[x] F[x]$$

exists for an appropriate class of test functionals. The class of test-functions allowed is determined by physics.

A natural choice is

$$F[x] = \exp i \int J_\mu(t) x_\mu(t) dt \quad (2.8)$$

If we differentiate with respect to J , we will generate the various expectation values

$$\langle x_{\mu_1}(t_1) \dots x_{\mu_n}(t_n) \rangle$$

On the other hand we may choose

$$J_n(t) = k_\mu^{(1)} \delta(t - t_1) + k_\mu^{(2)} \delta(t - t_2) \dots + k_\mu^{(n)} \delta(t - t_n) \quad (2.9)$$

Then we are calculating

$$\langle e^{ik_1 \cdot x(t_1)} e^{ik_n \cdot x_n(t_n)} \rangle \quad (2.10)$$

which is also very useful physically. (See section IV). The quantity

$$V(kt|x) = e^{ik \cdot x(t)}$$

will be called a "vertex functional" by analogy with string theories.

Now let us consider the case where A_μ is quantized as a free field. We may be tempted to say that $W[x]$ is a renormalizable operator if we can find a function C depending on the u.v. cut-off Λ such that

$$W_R[x] = W[x] C_\Lambda[x]$$

has finite n-point functions in the limit $\Lambda \rightarrow \infty$. But, as described earlier, we are really interested in averages of the kind*

$$S(k_i t_i p_j s_j) = \int \tilde{\mathcal{D}}[x] \tilde{\mathcal{D}}[y] \langle 0 | W[x] W[y] | 0 \rangle \sum_{i=1}^m e^{ik_i \cdot x(t_i)} \sum_{j=1}^n e^{ip_j \cdot y(s_j)} \quad (2.11)$$

Let us introduce a length a as an ultra-violet cut off. $W[x]$ will then be replaced by a "bare" Wilson loop,

$$W_a[x] = e^{ig_0(a) \int A_\mu dx^\mu} C_a[x] \quad (2.12)$$

$g_0(a)$ being a bare coupling constant. $C_a[x]$ is a sort of wave function renormalization of W . This is a C -number function.

Actually through the path integral over x , the vertex operators $e^{ik \cdot x(t)}$ might also need a wave function renormalization. Thus we are finally led to the following definition.

Def: $W[x]$ will be said to be a renormalizable operator, if there are $C_a[x]$, $Z(a)$, $g_0(a)$ such that

$$G(k_1 t_1; k_2 t_2 \dots k_p t_p) = \int \langle 0 | \prod_{\ell=1}^p \tilde{\mathcal{D}}[x_\ell] W_a[x_\ell] | 0 \rangle \prod_{\ell=1}^p \prod_{i=1}^m V(k_\ell t_\ell | x_\ell(t_\ell)) Z(a) \quad (2.13)$$

exists in the limit $a \rightarrow 0$.

* $\tilde{\mathcal{D}}[x] = e^{-\frac{1}{2} \int \dot{x}^2 dt} \mathcal{D}[x]$, so $\tilde{\mathcal{D}}[x]$ simply denotes the Wiener measure over closed curves.

This definition is somewhat different from the way the renormalizability of a composite operator is usually²¹ defined. In a scalar field theory, for example, an operator $\hat{O}(x)$ is defined through the functions

$$G(x; x_1 \dots x_n) = \langle 0 | \hat{O}(x) \phi(x_1) \dots \phi(x_n) | 0 \rangle$$

Here, $\phi(x)$ is the fundamental scalar field. Even for a multiplicatively renormalizable operator one does not find that

$$\mathcal{G}(x_1 \dots x_n) = \langle 0 | 0(x_1) \dots 0(x_n) | 0 \rangle$$

exist. The latter is more divergent than G .

However, our aim is to rewrite the entire theory in terms of Wilson loops. So it seems natural to require that the n -point function of the Wilson loop (similarly averaged) exists.

Next, we will study whether the Wilson loop is renormalizable in the sense defined above.

For practice, let us first consider smooth curves that do not intersect. The n -point functions

$$\mathcal{G}_0[x_1 \dots x_n] = \langle 0 | W[x_1] \dots W[x_n] | 0 \rangle \quad (2.14)$$

can be calculated explicitly (in the presence of an u.v. cut-off a) for pure Maxwell theory. It is just a matter of doing a Gaussian path integral over A :

$$\mathcal{G}_0[x_1 \dots x_n] = \int D[A] e^{-\frac{1}{2} \int d^2x d^2y \int_{\Lambda_n(x,y)} dx_\mu^2} \quad (2.15)$$

we get

$$\begin{aligned} \mathcal{G}_0[x_1 \dots x_n] = & \exp -\alpha_0(a) \left[\sum_i \int \frac{\dot{x}_i(t) \dot{x}_i(t')}{[x_i(t) - x_i(t')]^2 + a^2} dt dt' \right. \\ & \left. + 2 \sum_{i < j} \int \frac{\dot{x}_i(t) \dot{x}_j(t')}{[x_i(t) - x_j(t')]^2 + a^2} dt dt' \right] \end{aligned} \quad (2.16)$$

Here, we have introduced a length “ a ” to serve as a u.v. cut-off replacing the photon propagator

$$\frac{\delta_{\mu\nu}}{x^2} \rightarrow \frac{\delta_{\mu\nu}}{x^2 + a^2}$$

(More fancy cut-offs like dimensional regularization could have been used, but this would do just as well for our purposes).

Divergences arise whenever $x_i(t) \rightarrow x_j(t')$. This happens whenever

- (i) $t \rightarrow t'$
- (ii) two different curves intersect

$$x_i(t) = x_j(t'), \quad i \neq j$$

- (iii) Some curve intersects itself

$$x_i(t) = x_j(t'), \quad t \neq t'$$

The latter two cases do not arise by hypothesis.

Let us see if W can be “renormalized” for the case of smooth, non-intersecting curves. This has been studied by Polyakov and the answer seems to be that it is possible.

The infinity in $\mathcal{G}$ arises from the region $\tau \rightarrow \tau'$ in the first term of (2.5). One may calculate in the neighborhood of $\tau \sim \tau'$

$$\begin{aligned} & \alpha \int d\tau \int d\tau' \frac{\dot{x}^2(\tau)}{\dot{x}^2(\tau)(\tau' - \tau)^2 + a^2} \\ & \sim \alpha \int d\tau \frac{\dot{x}^2(\tau)}{\sqrt{\dot{x}^2(\tau)}} \frac{\pi}{a} = \frac{\pi\alpha}{a} \int \sqrt{\dot{x}^2(\tau)} d\tau \end{aligned} \quad (2.17)$$

Thus, the infinity can be removed by defining

$$W_R[x] = e^{-\frac{1}{2} \delta m(a) \int \sqrt{\dot{x}^2} d\tau} e^{i\theta \int A_\mu \dot{x}^\mu d\tau}$$

where $\delta m \sim \frac{a}{2}$ and m_0 is renormalized to cancel the above infinity. For this to work it was important that there were no infinities from the second term, which represents the interaction between two curves. This happens because we assumed that they would not intersect.

If the curves intersect, there is an infinity as can be seen by studying the neighborhood of an intersection. Let $x(\tau)$ and $y(\tau')$ be two curves intersecting at $\tau = t_i$, $\tau' = t'_i$. In this neighborhood

$$\begin{aligned} & \alpha \int d\tau d\tau' \frac{\dot{x}(\tau) \cdot \dot{y}(\tau')}{[x(\tau) - y(\tau')]^2 + a^2} \\ & \sim \alpha \int d\tau d\tau' \frac{\dot{x}(t) \cdot \dot{y}(t')}{[x(t)(\tau - t) - y(t')(t' - t')]^2 + a^2} \\ & \sim \alpha \frac{\dot{x}(t) \cdot \dot{y}(t')}{\sqrt{\dot{x}^2(t) \dot{y}^2(t) - (\dot{x}(t) \cdot \dot{y}(t'))^2}} \ln a/\epsilon. \end{aligned}$$

So within the class of smooth curves the operator W cannot be given a renormalized definition: there are infinities that cannot be absorbed by redefining W .

But this in itself need not be an insurmountable problem. For, we are interested in whether G exists in the limit $a \rightarrow 0$ after renormalization. The situation that $\mathcal{G}$ blows up on a certain class of curves need not be a problem if this class does not contribute to the path integral.

Now the question of whether the above average exists is the same as whether the one-dimensional "field theory" defined by the action

$$S = \frac{1}{2} \sum_i \int_0^{T_i} \dot{x}_i^2(\tau) d\tau + \alpha \sum_i \int \frac{\dot{x}_i^2(\tau) x_i^2(\tau') d\tau d\tau'}{[x_i(\tau) - x_i(\tau')]^2 + a^2} + \frac{m_0^2}{2} \int_0^{T_1} d\tau + 2\alpha \sum_{i < j} \int \frac{\dot{x}_i^2(\tau) \dot{x}_j^2(\tau')}{[x_i^2(\tau) - x_j^2(\tau')]^2 + a^2}$$

leads to finite n -point $\langle 0 | e^{ik_1 x(t_1)} \dots e^{ik_n x(t_n)} | 0 \rangle$ if m_0 is appropriately renormalized in the limit $a \rightarrow 0$. A renormalization of m will, (if we go back to the definition of $\mathcal{S}_0[x_1 \dots x_n]$) be absorbed as a redefinition of W .

This seems like a very difficult issue at first. But in fact we can reduce this to a well-known problem in ordinary quantum field theory. The action S is the action-at-a-distance form of scalar electrodynamics.^{14*} In fact, by doing perturbation theory in α we can recover the Feynman rules. By expanding in α we see that Feynman diagrams of scalar QED are recovered with two important exceptions.

- (1) Only diagrams with exactly n scalar loops appear.
- (2) There are no scalar self-couplings.

Thus, the problem of renormalizability of the composite operators $W[x]$ has been reduced to the following question: Can all the divergences of this truncated scalar electrodynamics be absorbed into a mass renormalization?^{**}

First of all, since no "virtual" scalar loops appear due to exception (1), there is no charge renormalization. There is a mass renormalization due to the usual self energy diagrams.

However, in scalar electrodynamics there is also another divergence, a self interaction of the scalar field. There is no way to remove this divergence except by adding a term

$$\int \delta\lambda \sum_{i < j} d\tau_i d\tau_j \delta^4[x_i(\tau_i) - x_j(\tau_j)] d\tau_i d\tau_j$$

to the action S . But this means that the operator W are not renormalizable in the sense defined earlier. There are infinities in the n -point functions that cannot be removed by a redefinition of W . This is claim (1) of the introduction.

III. THE MODIFIED WILSON LOOP AND SUPERSYMMETRY

Since we are ultimately interested in QED with fermionic matter, it seems natural to consider*

$$X[x] = \text{tr} e^{i\oint_0^T \Lambda_\mu \dot{x}^\mu dt + g \int_0^T \sigma \cdot \mathcal{F} dt} \quad (3.1)$$

* We are still studying the renormalizability of a certain composite operator in free gauge theory. What we are doing is to rephrase the problem as the renormalizability of a certain interacting theory.

** There might also be renormalization of the source, $Z(a)$.

$$\sigma \cdot \mathcal{F} = \sigma_{\mu\nu} \mathcal{F}_{\mu\nu}; \quad \sigma_{\mu\nu} = \frac{1}{2} [\gamma_\mu, \gamma_\nu]$$

instead of the Wilson loop. This is the amplitude for a spinning particle with $g = 2$, to propagate around a loop.

χ is as good as the Wilson loop for heavy particles because the magnetic moment term is suppressed for such particles. Also, in the limit of very large curves, this term has little effect, the magnetic moment coupling falls off faster.**

However, the $\sigma \cdot \mathcal{F}$ does have an effect on the ultraviolet properties of the theory. In fact, it improves the u.v. properties so that χ is renormalizable whereas W is not. This is proved in Appendix B. Thus, even in a pure gauge theory, we should use χ rather than W to get gauge invariant amplitudes.

The improved renormalization properties suggests that there is a hidden symmetry in the problem. We shall see next that this is so. There is a hidden supersymmetry in the Green's function of χ .

There is a way to describe a spinning particle using Grassman variables rather than spin matrices. Consider the Lagrangian¹³

$$\mathcal{L} = \frac{1}{2} \dot{x}^\mu \dot{x}_\mu + \frac{i}{2} \dot{\psi}^\mu \frac{d\psi^\mu}{dt} + ig A_\mu \dot{x}^\mu + \frac{g}{2} \psi^\mu \psi^\nu F_{\mu\nu} + m_0^2 \quad (3.2)$$

Here ψ^μ is a real Grassman variable. This is invariant under

$$\delta x^\mu = -i\epsilon \psi^\mu \\ \delta \psi^\mu = \epsilon \dot{x}^\mu$$

as can be verified by straightforward computation. ϵ is a real Grassman number. To see that this supersymmetric quantum mechanics really describes a spinning particle, let us canonically quantize $\mathcal{L}$. If $g = 0$,

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \dot{x}^\mu \\ \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\mu} = i\psi^\mu$$

So that the canonical commutation relations are

$$[x^\mu, \dot{x}^\nu] = i\delta_{\mu\nu} \\ \{\psi^\mu, \psi^\nu\} = \delta^{\mu\nu}$$

A representation of this algebra is obtained on Dirac wave functions $q_\alpha(x)$:

$$\dot{x}^\mu \rightarrow \frac{1}{i} \frac{\partial}{\partial x^\mu} \\ \psi^\mu \rightarrow \frac{1}{\sqrt{2}} \gamma_\mu$$

** We thank B. Svetitsky for discussions on this point.

γ_μ being the Hermitian Dirac matrices. The Hamiltonian is:^{*}

$$H = +\frac{1}{2}(-\nabla^2 + m_0^2)\delta_{\alpha\beta}$$

For the interacting theories, we get similarly

$$\begin{aligned} H &= \frac{1}{2}((i\gamma \cdot D)^2 + m_0^2) \\ &= \frac{1}{2}[-D^2 - \sigma \cdot F + m_0^2] \end{aligned}$$

D being the covariant derivative:

$$D_\mu = \partial_\mu - igA_\mu$$

Now we can write χ in terms of $\mathcal{L}$:

$$\chi[x] = \int D[\psi] e^{\int_0^T \mathcal{L} dt}$$

To see this, we use the following expression for e^{iHt} for small^{**} T :

$$\langle x|\beta|e^{-HT}|x,\alpha\rangle = \frac{e^{-\frac{(x-x')^2}{4T}}}{16\pi^2 T^2} \times \Phi(x, x') + O(T)$$

where

$$\Phi(x'x) = e^{i\vartheta} \int_0^{x'} dx'' A_\mu + \vartheta \int_0^T \sigma \cdot F dt$$

the integral being along the straight line.

$$x(t) = x' + \frac{(x-x')}{T}t.$$

It is more reasonable, in view of SUSY to consider

$$\chi[x, \psi] = e^{-\int_0^T \mathcal{L}_I dt}$$

without averaging over ψ . This makes the consequences of SUSY explicit.

$$\xi^\mu(t) := [x^\mu(t), \psi^\mu(t)]$$

is a closed curve in a space with four real and four Grassman coordinates. Let Q be the generator of SUSY. Consider the n -point function

$$\int \chi[\xi_1] \dots \chi[\xi_n] e^{-\frac{1}{2} \int x^2} D[A] = \mathcal{G}_0[\xi_1 \dots \xi_n]$$

^{*} $\alpha\beta$ are Dirac indices.

^{**} $O(T)$ denotes relative order in T . Remaining terms are $\sim T \times$ first term. See appendix D.

These Green functions satisfy the Ward identities of SUSY:

$$Q_i \mathcal{G}_0 = 0; \quad i = 1 \dots n$$

i.e.,

$$\int_0^{T_i} dt_i \left[-i\psi^\mu(t_i) \frac{\delta}{\delta x_\mu(t_i)} + \dot{x}^\mu(t_i) \frac{\delta}{\delta \dot{\psi}^\mu(t_i)} \right] \mathcal{G}_0[\xi_1 \dots \xi_n] = 0$$

These survive quantization because dimensional regularization (for example) preserves this symmetry. We will actually use a different regularization which also preserves it.

Now, we can actually calculate $\mathcal{G}_0$ by integrating out the photon explicitly:

$$\mathcal{G}_0[\xi_1 \dots \xi_n] = \langle e^{i\vartheta} \int A_\mu(z) J_\mu(z) dz \rangle$$

where

$$\begin{aligned} J_\mu(x) &= \sum_i \int_0^{T_i} dt \dot{x}_\mu^{(i)}(t) \delta(x - x_i(t)) \\ &+ \sum_i \int_0^{T_i} dt \psi^{\mu^{(i)}}(t) \psi^{\nu^{(i)}}(t) \partial_\nu \delta[x - x_i(t)] \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{G}_0 &= e^{-\vartheta^2 \int \frac{J_\mu^{(i)}(t) J_\nu^{(j)}(t')}{(x-x')^2 + a^2} dx dx'} \\ &= \exp[-S_I[\xi]] \end{aligned}$$

where

$$\begin{aligned} S_I[\xi] &= -\sum_{ij} g^2 \int_0^{T_i} \int_0^{T_j} \left[\frac{\dot{x}_\mu^{(i)}(t) \dot{x}_\nu^{(j)}(t')}{[x_i(t) - x_j(t')]^2 + a^2} \right. \\ &+ 2\dot{x}_\mu^{(i)}(t) \psi_j^\mu(t') \psi_j^\nu(t') \partial_\nu \frac{1}{[x_i(t) - x_j(t')]^2 + a^2} \\ &\left. + \psi_i^\mu(t) \psi_i^\nu(t') \psi_j^\mu(t') \psi_j^\nu(t') \partial_\nu \partial_{\sigma'} \frac{1}{[x_i(t) - x_j(t')^2] + a^2} \right] \end{aligned}$$

The expressions are beginning to get complicated. It would be nice to have a short-hand notation for this. Such a one is provided by the "one-dimensional" superfield formalism^{*} of Appendix C. Then

$$\begin{aligned} S[\xi] &= \frac{1}{2} \sum_i \int \xi_i^\mu D^3 \xi_i^\mu dr + m_0^2 \int dt \\ &+ \alpha \sum_{ij} \int \frac{D\xi_i^\mu(z) D\xi_j^\mu(z')}{[\xi_i(z) - \xi_j(z')]^2} dz dz' \end{aligned}$$

^{*} Ref. 24 contains a similar formalism for left-handed superfields in two dimensions.

We conclude by pointing out that $\mathcal{G}_0$ has the following form:

$$\begin{aligned}\mathcal{G}_0 &= \langle 0 | \chi[\xi_1] \dots \chi[\xi_f] | 0 \rangle \\ &= \pi_{ij} e^{-\alpha D[\xi_i, \xi_j]}\end{aligned}$$

Compare with a free field theory of a field with expectation value zero,

$$\langle \phi(x) \phi(y) \rangle = D(x, y)$$

then

$$\langle e^{i\phi(x_1)} \dots e^{i\phi(x_n)} \rangle = \pi_{ij} e^{-\alpha D(x_i, x_j)}$$

So we are beginning to see the connection of QED to a free field theory. However, this is a free field theory in an infinite dimensional space. In the next section we elaborate on this connection and also extend it to the interacting theory.

IV. Expansion in Powers of N_f

Consider QED with N_f "flavors" of electrons. If one looks at the action of string theory (in section 5) that is claimed to be equivalent to QED, one sees that N_f acts like a coupling constant. This suggests that we should look for an expansion of QED in terms of N_f and not α . Such an expansion is interesting in itself.

N_f is simply a parameter that counts the number of electron loops. The zeroth order in N_f is the sum over all Feynman diagrams of QED without electron loops. The first-order is the sum over all diagrams with exactly one electron loop and so on. For example, the β -fn of QED vanishes to zeroth order, but not to the 1st order in N_f . Such an approximation to the β -fn has been studied before.¹⁸

If we "integrate-out" the photon field in QED paying no attention to the dressing of the photon Green's function, we get the zeroth orders approximation to QED. This is the action-at-a-distance QED discussed¹⁴ in Section III. By a systematic expansion in powers of N_f we will find this idea of Feynman to reproduce QED exactly, not merely in the $N_f \rightarrow 0$ limit. Using supersymmetry, we will also be able to extend his ideas to spin $\frac{1}{2}$ particles. Actually, the case of fermionic QED is more self-contained, since all interactions are given by the gauge principle. In scalar QED, there is also the $\lambda\phi^4$ interaction (necessary for renormalizability) which has to be represented as a contact interaction of the world lines.

The idea of the limit $N_f \rightarrow 0$ is a well-known technique in the statistical mechanics.²⁰ There it is used to approximate the average of an exponential by the exponential of the average:

$$\langle e^{N_f z} \rangle \approx e^{N_f \langle z \rangle} + O(N_f)$$

The approximation of ignoring electron loops in QED is somewhat analogous to the large N_c limit in QCD. In the large N_c limit, quark loops disappear. But the analogy should not be carried too far, since there is no restriction in QED on the topology of the Feynman graphs.

Let us consider a typical two-point function:

1.

$$\Delta'(x, y) = \int e^{-\frac{1}{2} \int \bar{\psi} \psi} e^{\int \bar{\psi}(\gamma \cdot D + m_0) \psi} \frac{1}{m_0} \bar{q}(x) q(y) \mathcal{D}[A] \mathcal{D}[q]$$

In Appendix D we have shown that the functional integral over q can be converted to an infinite series of worldline integrals:

2.

$$\begin{aligned}\Delta'_A &= \int \mathcal{D}[q] e^{\int \bar{q}(\gamma \cdot D + m_0) q} \bar{q}(x) q(y) \\ &= \sum_0^{\infty} \frac{N_f^\ell}{2^\ell \ell!} \int_0^\infty \int \pi_{k=1}^\ell D_{T_k}[z] \mathcal{D}_{T_k}[\eta] \frac{dT_k}{T_k} e^{-\sum_k \int_0^{T_k} \mathcal{L}[x_k, m_k] dt} \\ &\quad \int_0^\infty dt \int_{x(0)=x}^{x(T)=y} \mathcal{D}[x] \mathcal{D}[\psi] e^{-\int_0^T \mathcal{L}[x, \psi] dt}\end{aligned}$$

The point of writing the fermion path integral this way is that the photon integral is a mere Gaussian. In fact, 2 may be written as:

$$\begin{aligned}\Delta_A(x, y) &= \sum_0^{\infty} \left(\frac{N_f}{2} \right)^\ell \frac{1}{\ell!} \int \pi_1^\ell \bar{\mathcal{D}}_e[\eta_k] \bar{\mathcal{D}}[\xi] \\ &\quad e^{-\sum_{k=1}^\ell i g \int A_\mu[\eta_k] dt d\theta} \\ &\quad e^{-i g \int A_\mu[\xi] dx d\theta}\end{aligned}$$

Here, the measures have been redefined:

$$\begin{aligned}\bar{\mathcal{D}}_e[\eta_k] &= \int_e^\infty \frac{dT_k}{T_k} e^{-m_0^2 T_k} \int \mathcal{D}[\eta_k] \\ &\quad \exp + \frac{1}{2} \int D\eta_k D^2 \eta_k dt d\theta \\ \bar{\mathcal{D}}[\xi] &= \int_0^\infty dt e^{-m_0^2 T} \int \mathcal{D}_T[\xi] e^{-\frac{1}{2} \int \xi D^2 \xi dt d\theta}\end{aligned}$$

Now we just do the integration over A_μ using Appendix C. We have

$$\Delta'(x, y) = \sum_{\ell=0}^{\infty} \left(\frac{N_f}{2} \right)^\ell \frac{1}{\ell!} \int \pi_{k=1}^\ell \mathcal{D}_e[\eta_k] \bar{\mathcal{D}}[\xi] e^{\mathcal{S}_e[\eta_k, \xi]}$$

where the $K.E.$ terms have been absorbed into the measure.

Thus, we have the required action-at-a-distance form of fermionic electrodynamics. Note that the zeroth term in N_f is what occurred in Appendix B. Note also the manifest supersymmetry.

V. QUANTUM FIELD THEORY OF LOOPS

We have accomplished our task. We have shown that given

$$\mathcal{G}[\xi_1, \dots, \xi_n] = \langle 0 | \chi[\xi_1] \dots \chi[\xi_n] | 0 \rangle$$

We can calculate any object in QED by a "mere" worldline integral $e.g.$

$$\Delta'(x, y) = \int_0^\infty dt e^{-m_0^2 x} \int_{\xi(x)=x}^{\xi(0)=x} \tilde{D}[\xi] \mathcal{G}[\xi]$$

Furthermore, we have expressed $\mathcal{G}$ in terms of $\mathcal{G}_0$:

$$\begin{aligned} \mathcal{G} &= \int D[A] e^{-\frac{1}{2} \int F^2} [\text{Det}(i\gamma \cdot D + m)]^{N_f} \chi[\xi_1] \dots \chi[\xi_n] \\ \mathcal{G}_0 &= \int D[A] e^{-\frac{1}{2} \int F^2} \chi[\xi_1] \dots \chi[\xi_n] \end{aligned}$$

In fact, $\mathcal{G}_0$ is known explicitly:

$$\mathcal{G}_0 = e^{-\sum_{i,j} S_1[\xi_i, \xi_j]} = e^{-\alpha \sum_{i,j} D[\xi_i, \xi_j]}$$

where

$$D[\xi, \xi'] = \int dt d\theta D\xi^\mu(t, \theta) \frac{1}{(\xi - \xi')^2 + a^2} D\xi^\mu(t', \theta') dt' d\theta'$$

The $\mathcal{G}$ is known as an infinite series in $\mathcal{G}_0$:

$$\mathcal{G} = \sum_{k=0}^{\infty} \left(\frac{N_f}{2} \right)^k \frac{1}{k!} \int_e \prod_{h=1}^k \tilde{D}[\eta_h] \mathcal{G}_0[\xi_1 \dots \xi_n; \eta_1 \dots \eta_k] .$$

Let us now see if we can rewrite these to from a quantum field theory of loops. An object that creates a loop will be an operator-valued functional $\phi[\xi]$. Just as $\phi(x)$ in scalar field theory creates a particle at x , $\phi[\xi]$ creates a closed string ξ .

Suppose we postulate a "propagator" for ϕ :

$$\begin{aligned} \langle 0 | \phi[\xi] | 0 \rangle &= 0 \\ \langle 0 | \phi[\xi] \phi[\xi'] | 0 \rangle &= D[\xi, \xi'] \end{aligned}$$

D being as defined earlier. Then, we can verify that

$$\langle 0 | e^{i\vartheta \phi[\xi_1]} \dots e^{i\vartheta \phi[\xi_n]} | 0 \rangle = \mathcal{G}_0[\xi_1 \dots \xi_n]$$

Proof: By Wick's theorem

$$\langle 0 | e^{i\vartheta \int J[\xi] \phi[\xi] \tilde{D}[\xi] | 0 \rangle} = e^{-\alpha \int J[\xi] D[\xi, \xi'] J[\xi'] \tilde{D}[\xi] \tilde{D}[\xi']}$$

Now choose

$$J[\xi] = \sum_{i=1}^n \delta[\xi - \xi_i]$$

These formal manipulations can be given a precise meaning by approximating a path integral by a multiple integral and taking the limit. Note that reversing the orientation of the loop changes the sign of ϕ

$$\int \tilde{D}[\eta] e^{i\vartheta \phi[\eta]} = \int \tilde{D}[\eta] \cos g\phi[\eta]$$

Thus,

$$\begin{aligned} \mathcal{G}[\xi_1 \dots \xi_n] &= \sum_0^{\infty} \frac{1}{k!} \int D[\eta_1] \dots D[\eta_k] \\ &\langle 0 | N_f : \cos g\phi[\eta_1] : \dots N_f : \cos g\phi[\eta_k] : \\ &: e^{i\vartheta \phi[\xi_1]} \dots e^{i\vartheta \phi[\xi_n]} | 0 \rangle \\ &= \langle 0 | e^{-S_1[\phi]} e^{i\vartheta \phi[\xi_1]} \dots e^{i\vartheta \phi[\xi_n]} | 0 \rangle \end{aligned}$$

where

$$S_1 = N_f \int : \cos g\phi[\xi] : \tilde{D}[\xi]$$

This looks like the Gell-Mann-Low formula in field theory.

Thus, if we have a field theory in loop space with the propagator

$$\langle 0 | \phi[\xi] \phi[\xi'] | 0 \rangle = D[\xi, \xi']$$

and interaction action

$$S_1 = N_f \int \cos g\phi[\xi] : \tilde{D}[\xi]$$

we reproduce the series for QED.

Now, it seems that we have more than we started with. $\phi[\xi]$ being a functional has an infinite number of fields in it. But we are claiming that this is equivalent to a field theory with a finite number of fields. How is this possible?

In fact only a finite number of modes of ϕ actually propagate. To see this, let us "diagonalize" the propagator D .

$$\begin{aligned} D[\xi, \xi'] &= \int D\xi^\mu \frac{1}{(\xi - \xi')^2 + a^2} D\xi^\nu \\ &= \int dq dr d\theta d\theta' d\tau D_{\mu\nu}^{(\alpha)}(q) V_{\mu q r \theta}^*[\xi] V_{\nu q r' \theta'}[\xi'] \end{aligned}$$

(This is not diagonal in μ, ν and r, r' . But these indices can also be diagonalized by a linear and a Laplace transform, respectively.)

The "vertex function" is defined by

$$V_{\mu q r \theta}[\xi] = D_{\xi(\tau, \theta)}^{\mu} e^{i\vartheta \xi(\tau, \theta)}$$

These functions span a subspace in the space $\phi[\xi]$ labelled by a $A_\mu(q)$. Thus only the modes of $\phi[\xi]$ that can be written as

$$\phi[\xi] \sim \int dq A_\mu(q, \tau) V_{\mu q r \theta}[\xi]$$

actually propagate. This is merely a finite number of modes in space-time.

$D[\xi, \xi']$ cannot be inverted to yield a "kinetic energy" operator, because it has mostly zero eigenvalues. There is no K such that

$$\int K[\xi, \eta] D[\eta, \xi'] D[\eta] = \delta[\xi, \xi']$$

However, suppose $P[\xi, \xi']$ is the projection operator to the "photon" modes. Then, we can verify that

$$\begin{aligned} P[\xi, \xi'] &= \int \delta_{\mu\nu} V_{\mu\alpha\sigma}[\xi] V_{\nu\sigma\alpha}^*[\xi'] d\tau d\tau' d\theta d\theta' \\ &= \int d\tau d\theta' D\xi^\mu(\tau, \theta) D\xi^\mu[\tau', \theta'] \delta^4[\xi(\tau, \theta) - \xi(\tau', \theta')] d\tau d\theta \end{aligned}$$

Consider the operator

$$\hat{K} = \frac{1}{2} \int d\sigma \int_{-\epsilon}^{+\epsilon} d\epsilon \frac{\delta^2}{\delta x(\sigma + \epsilon) \delta x(\sigma - \epsilon)} + \text{SUSY completion}$$

Then

$$\hat{K} D[\xi, \xi'] = P[\xi, \xi'] .$$

Thus, within the subspace of propagating modes, $\hat{K}$ is the " $K \cdot E$ " operator.

Thus, we find the theory in loop space to be *not* a string theory. There are only a finite number of propagating modes, whereas in string theory there are an infinite number.

VI. CONCLUSIONS

We have shown that the correct operator to study in a gauge theory is the modified Wilson loop $\chi[x]$ and not the usual Wilson loop $W[x]$. In this paper, only the Abelian theory was considered. It will be shown in a forthcoming paper that $\chi[x]$ and not $W[x]$ is renormalizable in non-Abelian gauge theories also.

A formulation of non-Abelian gauge theories in terms of $\chi[x]$ will be quite interesting. The field $A_\mu(x)$ of QCD does not appear to have a direct relation to observable particles. But, we conjecture that $\chi[x]$ creates linear combinations of one-particle states (i.e., "glueballs"). Also, we believe that the projection to an operator that creates a single eigenstate is the average

$$\int \tilde{D}[x] \chi[x] V_\alpha[x]$$

$V_\alpha[x]$ being vortex functionals similar to the ones encountered in the text. Of course, nothing in this paper confirms these conjectures. We intend to follow up on these issues in later publications.

The study of QED has led us to the correct choice of loop variables. It also has shown us how to include fermionic matter fields. Even though the pure gauge sector

of QCD is much more involved, the interaction of gluons to quarks is not too different from that in QED. So we expect Appendix D to go through without major changes. A functional integral representation for the quark Green function in the presence of an external gluon field can be derived using the Lagrangian of Reference 13.

It is conceivable that our way of formulating QED will lead to a better understanding of its non-perturbative features.

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APPENDIX A

Action-at-a-distance electrodynamics for scalar particles.

We want to show that

$$S = \sum_i \frac{1}{2} \int \dot{x}_i^2(r) dr + \alpha \sum_i \int \frac{dr dr' \dot{x}_i(r) \dot{x}_i(r')}{[x_i(r) - x_i(r')]^2 + a^2} + m_0^2(a) \sum_i T_i + 2\alpha \sum_{i < j} \int \frac{\dot{x}_i(r) x_j(r') dr dr'}{[x_i(r) - x_j(r')]^2 + a^2} \quad (A.1)$$

describes scalar QED except that the following two kinds of Feynman diagrams are absent:

1. Scalar loops
2. Scalar self-couplings

We will prove this in the presence of an u.v. cut-off. So u.v. infinities are regulated.

One proof is, of course, in Feynman's papers¹⁴. We give a more updated one. A different approach is mentioned in Appendix B.

Consider a scalar $2n$ -pt function:

$$G_n = \langle 0 | T \phi^*(x_1) \phi(y_1) \phi^*(x_2) \phi(y_2) \dots \phi^*(x_n) \phi(y_n) | 0 \rangle \quad (A.2)$$

in a theory with action

$$S = \frac{1}{2} \int A_\mu(x) k_{\mu\nu}^{(a)}(x, x') A_\nu dx dx' + \frac{1}{2} \int (D\phi)^2(x) dx + \frac{1}{2} \int m^2 |\phi|^2 dx \quad (A.3)$$

$k_a(x, x')$ is the photon inverse propagator with a cut-off a . In terms of path integrals

$$G = \int D[A] e^{-\frac{1}{2} \int A K A dx dx'} e^{-\frac{1}{2} \int D\phi^* D\phi dx - \frac{1}{2} \int m^2 |\phi|^2 dx} \phi^*(x_1) \phi(y_1) \dots \phi^*(x_n) \phi(y_n) \quad (A.4)$$

The functional integral over ϕ can be done:

$$G = \int D[A] e^{-\frac{1}{2} \int A K A dx dx'} (\text{Det} [-D^2 + m^2]) \sum \{ D_A(x_{i_1} - y_{j_1}) D_A(x_{i_2} - y_{j_2}) D_A(x_{i_n} - y_{j_n}) \} \quad (A.5)$$

Now the process of ignoring scalar loops is the same as dropping the $[\text{Det}(-D^2 + m^2)]$. (We will call this process "quenching"). Then

$$G_n^{\text{Quo}} = \int \mathcal{D}[A] e^{-\frac{1}{2} \int A K A dx dx'} \int \prod_i dT_i \int_{\Omega} \prod_{i=1}^n \mathcal{D}[x_i] e^{-\frac{1}{2} \sum_{i=1}^n \int_0^{T_i} \dot{x}_i^2(t) dt - m^2 \int_0^{T_i} dt} e^{ie \sum_i \int A_\mu \dot{x}_i^\mu(t) dt} \quad (\text{A.6})$$

Where the well-known path integral representation for the scalar n -pt function in the presence of an external gauge field has been used. Ω is the set of all paths such that

$$\begin{aligned} \{x_i(0)\} &= \{x_i\} \\ \{x_i(T_i)\} &= \{y_i\} \end{aligned} \quad (\text{A.7})$$

Now do the path integral over A to get

$$G = \int \prod_{i=1}^n dT_i e^{-m^2 \frac{T_i^2}{2}} \int_{\Omega} \mathcal{D}[x_i] e^{-S[x]} \quad (\text{A.8})$$

S being the action defined in (A.1). Thus, the action S yields the n -pt of truncated scalar QED when integrated over the region Ω .

Now, we know from the renormalization theory of SQED that

1. If scalar loops are ignored there is no coupling constant renormalization. In fact, there is no correction at all to the photon propagator.
2. There is a mass renormalization.
3. There is a wave function renormalization $Z^{\pi}(a)$
4. There is also a divergence associated to the four pt function. So

$$G_n = Z^{\pi}(a) G(\mu_0(a); x_1 \dots x_n)$$

is *not* finite; one has to add a $\lambda\phi^4$ term to the action to recover renormalizability. In the worldline formalism this is

$$\delta\lambda(a) \int d\tau d\tau' \frac{a^2}{[(x-x')^2 + a^2]^3}$$

The quantity in the integral leads to a δ function as $a \rightarrow 0$.

In the text we considered path integral over closed curves whereas above we considered a path integral over open curves. The difference will show up in the Feynman rules as a modification of the propagator, and of the range of integration over the

proper time parameters. Now integrate over all t 's between 2π and 0. To derive the new propagator let us calculate directly from the path integral:

$$\langle 0 | e^{ik[x(t)-x(t')]} | 0 \rangle = \int_{x(t)=x(r+2\pi)} \mathcal{D}[x] e^{-\frac{1}{2} \int_0^{2\pi} \dot{x}^2(r) dr} e^{ik[x(r)-x(r')]} \quad (\text{A.9})$$

This can be done by Fourier analysis

$$x(t) = a_0 + \sum_{n \neq 0} a_n \cos 2\pi n \frac{t}{T} + \sum_{n \neq 0} b_n \sin \frac{2\pi n t}{T}$$

We will spare the reader the agony of detail.

The answer is

$$\langle 0 | e^{ik[x(t)-x(t')]} | 0 \rangle = e^{-k^2 f(t'-t)}$$

where

$$f(t) = \sum_{n \neq 0} \frac{e^{in t}}{n^2}$$

Note that as $(t' - t) \rightarrow 0$, $f(t' - t) \sim |t' - t|$. Thus, near the region $t' \sim t$ or $k^2 \rightarrow \infty$, these propagators reduce to the earlier one. So the change in boundary condition does not change the degree of divergence of Feynman diagrams.

APPENDIX B

Proof of Renormalizability of $\chi[x]$ in Free QED

We have

$$\chi[x] = \text{tr} \, e^{i\theta} \int A_\mu \dot{x}^\mu dt + g \int \sigma_{\mu\nu} F_{\mu\nu} dt \quad (B1)$$

where $x(\tau)$, $x(0) = x(T)$ is a closed curve and

$$\sigma_{\mu\nu} = \frac{1}{2} [\gamma_\mu, \gamma_\nu]$$

We will be interested in the expectation values

$$G(k) = \int D[x] e^{i k_1 \cdot \dot{x}(t_1)} D[x] e^{i k_n \cdot \dot{x}(t_n)} \prod_{i=1}^k D[x_i^k] e^{-\frac{1}{2} \int_0^T \sum_i \dot{x}^2(t) dt - m_0^2(a) T} \langle 0 | \chi[x_1] \dots \chi[x_k] | 0 \rangle \quad (B2)$$

The task is to show that all the divergences in this that occur as the u.v. cut-off $a \rightarrow 0$ can be absorbed into $m_0(a)$. There is also in general a "Wave fn" renormalization $z(a)$ of $e^{i k \cdot x}$.

$$\begin{aligned} & \langle 0 | \chi[x_1] \dots \chi[x_k] | 0 \rangle \\ &= \int D[A] e^{-\frac{1}{2} \int A_\mu(\dot{x}) K_{\mu\nu}^{(a)}(x, x') A_\nu(x') dx dx'} \\ & \chi[x_1] \dots \chi[x_k] \end{aligned}$$

$K^{(a)}$ is the photon kinetic energy operator with a u.v. cut-off a . Consider the simplest case $k = 1$, $n = 0$.

$$G(T) = \int D[x] e^{-\frac{1}{2} \int_0^T \dot{x}^2(t) dt - m_0^2 T} \langle 0 | \chi[x] | 0 \rangle.$$

or

$$G(T) = \int D[A] e^{-\frac{1}{2} \int F^2 dx} G_A(T)$$

where

$$G_A(T) = \int D[x] e^{-\frac{1}{2} \int_0^T \dot{x}^2(t) dt - m_0^2 T} \text{tr} \, e^{i\theta} \int A_\mu \dot{x}^\mu(t) dt + g \int \sigma \cdot F dt$$

Now, we will see in Appendix D that

$$G_A(T) = \int \text{tr} \langle x | e^{-HT} | x \rangle dx$$

with

$$H = -D^2 + \sigma \cdot F + m_0^2.$$

It is convenient to study instead of $G(T)$, its Laplace transform.

$$\begin{aligned} \bar{G}(\mu) &= \int_0^\infty G(T) e^{-\mu^2 T} dT \\ &= \int D[A] e^{-\frac{1}{2} \int F^2 dx} \\ & \int dx \langle \text{tr} \frac{1}{-D^2 + g\sigma \cdot F + m^2} 1x \rangle \end{aligned}$$

where we have put $m^2 = m_0^2 + \mu^2$.

$$\text{tr} \frac{1}{-D^2 + \sigma \cdot F + m^2} = \frac{1}{m} \text{tr} \frac{1}{i\gamma \cdot D + m}.$$

Thus, we have related $\bar{G}(\mu)$ to the propagator of fermionic QED. If we expand the right hand side in powers of g , we will recover all the Feynman diagrams of QED, except for those involving "dressing" of the photon Green's function. Thus, the u.v. divergences in $G(T)$ are the same as those in QED. But we know that in QED all divergences can be removed by mass and wave function renormalization. (There is no charge renormalization, because the photon Green's functions are not "dressed". A similar argument applies to more complicated cases.

APPENDIX C

1-D Superfields
Define

$$\xi^\mu(\tau, \theta) = x^\mu + i\theta\psi^\mu$$

Under SUSY,

$$\begin{aligned} x^\mu &\rightarrow x^\mu - i\epsilon\psi^\mu \\ \psi^\mu &\rightarrow \psi^\mu + \epsilon\dot{x}^\mu \end{aligned}$$

so

$$\begin{aligned} \theta &\rightarrow \theta - \epsilon \\ \tau &\rightarrow \tau + i\theta\epsilon \end{aligned}$$

Thus,

$$Q = \frac{\partial}{\partial\theta} + \frac{i\theta\partial}{\partial t}; \left\{ \frac{\partial}{\partial\theta}, \theta \right\} = 1$$

Then

$$Q^2 = i\frac{\partial}{\partial t}$$

This is the SUSY algebra. Put

$$\begin{aligned} D &= \frac{\partial}{\partial\theta} - i\theta\frac{\partial}{\partial t} \\ \{Q, D\} &= 0 \end{aligned}$$

Then

$$\begin{aligned} D\xi^\mu &= i\psi^\mu - i\theta\dot{x}^\mu \\ D^2\xi^\mu &= -i\dot{x}^\mu + \theta\dot{\psi}^\mu \end{aligned}$$

Thus

$$D\xi^\mu D^2\xi^\mu = i\theta\psi^\mu\dot{\psi}^\mu - \theta\dot{x}^\mu\dot{x}^\mu + \dot{x}^\mu\psi^\mu$$

So

$$\mathcal{L}_0 = -\frac{1}{2} \int d\theta D\xi^\mu D^2\xi^\mu$$

Superfield Interactions

$$\begin{aligned} \delta A_\mu &= \partial_\nu A_\mu \delta x^\nu = -i\epsilon[\partial_\nu A_\mu, \psi^\nu] \\ &:= -i\epsilon B_\mu \\ B_\mu &= \partial_\nu A_\mu \psi^\nu \end{aligned}$$

Thus, A_μ and B_μ form the component of a superfield $A_\mu(x + i\theta\psi) = A_\mu(x) + i\theta B_\mu$. Call $A_\mu(x + i\theta\psi) = \phi_\mu$ to avoid confusion. Thus,

$$\begin{aligned} \phi_\mu &= A_\mu(x) + i\theta\psi^\nu \partial_\nu A_\mu \\ D\xi^\mu &= \left(\frac{\partial}{\partial\theta} - i\theta\partial/\partial t \right) \xi^\mu \\ &= i\psi^\mu - i\theta\dot{x}^\mu \end{aligned}$$

So

$$\phi_\mu D\xi^\mu = -\theta\psi^\mu\psi^\nu\partial_\nu A_\mu - i\theta A_\mu\dot{x}^\mu + iA_\mu(x)\phi^\mu$$

The propagator of ϕ_n is

$$\langle \phi_\mu(\tau, \theta), \phi_\nu(\tau', \theta') \rangle = \frac{1}{[\xi(\tau, \theta) - \xi(\tau', \theta')]^2} \delta_{\mu\nu}.$$

So the effect of integrating out A is

$$S = \frac{1}{2} \int dt d\theta D\xi^\mu D^2\xi^\mu + g^2 \int D\xi^\mu(\tau, \theta) D\xi^\mu(\tau', \theta') \frac{d\tau d\tau' d\theta d\theta'}{[\xi(\tau, \theta) - \xi(\tau', \theta')]^2}$$

APPENDIX D

Path Integral Representation of Electron Amplitudes
We will be interested in evaluating functional integrals such as

$$\int D[q] D[A] e^{-S[q,A]} \bar{q}(x) q(y)$$

or

$$\int D[q] D[A] e^{-S[q,A]} \bar{q}(x) (1 + \gamma_5) \gamma_\mu q(y)$$

Here,

$$S = \frac{1}{4} \int F^2 + \int \bar{q}_a(x) [\gamma \cdot D + m_0] q dx$$

In this section, let us not condense the integration over A . Let it be an external field. Our task is to express these functional integrals over q as path integrals over world lines. We will show that the calculation of any amplitude can be reduced to that of evaluating an integral over world lines such as

$$\int \tilde{D}[\xi_1] \tilde{D}[\xi_2] \bar{\chi}[\xi_1] \chi[\xi_2]$$

ξ_1, ξ_2 are open curves ($x^\mu(\sigma), \psi^\mu(\sigma)$) and the integral is over some appropriate class of such curves.

Let us consider for example

$$\int D[q] e^{\int \bar{q}(\gamma \cdot D + m) q} d^4x \bar{q}(x) q(y) = |\text{Det}(\gamma \cdot D + m)|^{N'} [-\text{tr } S(x, y)]$$

Ignore the Det for the moment.

S_A is the fermion propagator in the presence of the gauge fixed A :

$$\begin{aligned} (\gamma \cdot D + m_0) S_A(x, x') &= \delta(x, x') \\ S_A(x, x') &= \langle 0 | q(x) \bar{q}(x') | 0 \rangle_A \end{aligned}$$

In fact,

$$S_A(x, x') = (-\gamma \cdot D + m_0) \Delta_A(x, x')$$

where Δ satisfies

$$[-(\gamma \cdot D)^2 + m_0^2] \Delta_A(x, x') = \delta(x, x')$$

Now

$$\text{tr } S_A(x, y) = \text{tr} [-\gamma \cdot D + m_0] \Delta_A(x, y')$$

Here, the " $\gamma \cdot D$ " term cannot contribute to the trace. Δ_A always involves an even number of γ matrices, so $\text{tr } \gamma \cdot D \Delta_A$ would involve an odd number. So the trace vanishes. Thus,

$$\text{tr } S_A(x, x') = m_0 \text{tr } \Delta_A(x, x')$$

The point of the above exercise is to express everything in terms of the "square" of the Dirac operator, Δ_A . This has closer connection with path integral presentations. This, however, is not essential, and we will see in fact that $S_A(x, x')$ can be given a path integral representation directly. But we will deal with Δ_A for the moment.

Clearly any number of products of q 's and $\bar{q}$ has a v.e.v. (in the presence of an internal gauge field) which can be written in terms of Δ_A and its derivatives.

Now we claim that

$$\Delta_A(x, x') = \int_0^\infty dT \int_{\Omega_x} D[x] e^{-\frac{1}{2} \int_0^T \dot{x}^2 - m_0^2 \int_0^T dt} \times e^{i \int_0^T A_\mu \dot{x}^\mu + \int_0^T \sigma F dt}$$

Here, Ω_T is the set of the worldlines such that

$$\begin{aligned} x(0) &= x \\ x(T) &= x' \end{aligned}$$

To see this, note first that

$$\text{tr } \Delta_A(x, x') = \text{tr} \langle x' | \frac{1}{H} | x \rangle = \int_0^\infty \langle x' | e^{-HT} | x \rangle dT$$

with

$$\begin{aligned} \hat{H} &= -(\gamma \cdot D)^2 + m_0^2 \\ &= -D^2 - g \sigma \cdot F + m_0^2 \end{aligned}$$

Now, for the "propagator" $\langle x' | e^{-\hat{H}t} | x \rangle$ we find the path integral representation by interpolating states $|x(t_i)\rangle$ at intermediate times t_i . The expression for $\langle x' | e^{-\hat{H}(t_i - t_{i-1})} | x_{i-1} \rangle$ is known²⁸ for small $t_i - t_{i-1}$:

$$e^{-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})} - m_0^2(t_i - t_{i-1})} \left[1 + i g \int_{t_{i-1}}^{t_i} A_\mu \dot{x}^\mu dt + g \int_{t_{i-1}}^{t_i} \sigma \cdot F dt \right]$$

Thus,

$$\begin{aligned} \langle x' | e^{-\hat{H}t} | x \rangle &= \int \pi_i dx_i \prod_{i=1}^N e^{-\frac{1}{2} \int_{t_{i-1}}^{t_i} \dot{x}^2 dt - \frac{m_0^2}{2} \int_{t_{i-1}}^{t_i} dt} \\ &\quad \prod_i e^{i g \int_{t_{i-1}}^{t_i} A_\mu \dot{x}^\mu dt + g \int_{t_{i-1}}^{t_i} \sigma \cdot F dt} \end{aligned}$$

The limit $N \rightarrow \infty, (t_i - t_{i-1}) \rightarrow 0$ such that

$$N(t_i - t_{i-1}) \rightarrow T$$

leads to the claimed path integral representation.

Next, we write the same path integral using Grassman variables rather than Dirac matrices. We claim

$$\begin{aligned}\chi[x] &= \text{tr} e^{ig \int_0^T A_\mu \dot{x}^\mu dt + g \int_0^T \sigma \cdot F dt - \frac{1}{2} \int x^2 dt} \\ &= \int_\Omega D[\psi] e^{-\frac{1}{2} \int x^2 dt} e^{\frac{1}{2} \int \psi \dot{\psi} dt + g \int \psi^\mu \psi^\nu F_{\mu\nu} + ig \int A_\mu \dot{x}^\mu dt}\end{aligned}$$

Ω being an appropriate range. Clearly, it is sufficient to show that

$$\text{tr} e^{g \int_0^T \sigma \cdot F dt} = \int_\Omega D[\psi] e^{\frac{1}{2} \int_0^T \psi \dot{\psi} dt} e^{g \int \psi^\mu \psi^\nu F_{\mu\nu} dt}$$

Expanding both sides, we need to show that

$$\begin{aligned}& \sum_{k=0}^{\infty} \frac{g^k}{k!} \int_0^T dt_k \int_0^T dt_{k-1} \dots \int_0^T dt_1 \theta(t_k - t_{k-1}) \theta(t_{k-1} - t_{k-2}) \dots \theta(t_2 - t_1) \\ & \quad \text{tr} \sigma_{\mu_k \nu_k} F_{\mu_k \nu_k}(t_k) \dots \sigma_{\mu_1 \nu_1} F_{\mu_1 \nu_1}(t_1) \\ &= \sum_{k=0}^{\infty} \frac{g^k}{k!} \int_0^T dt_k \dots \int_0^T dt_1 \int_\Omega D[\psi] e^{\frac{1}{2} \int_0^T \psi \dot{\psi} dt} \\ & \quad \psi^{\mu_k} \psi^{\nu_k} F_{\mu_k \nu_k}(t_k) \dots \psi^{\mu_1} \psi^{\nu_1} F_{\mu_1 \nu_1}(t_1)\end{aligned}$$

To evaluate the R. H. S. we need

$$\langle \psi^\mu(t') \psi^\nu(t) \rangle = \int_\Omega D[\psi] e^{\frac{1}{2} \int \psi \dot{\psi} dt} \psi_\mu(t') \psi_\nu(t)$$

By point splitting,

$$\langle \psi^\mu(t') \psi^\nu(t) \rangle = \delta_{\mu\nu} \theta(t' - t)$$

This is basically a definition of the path integral. Different boundary conditions on the Green function of d/dt lead to different definition of the path integral. Once we know the above formula, the theorem follows using the trace formula for γ matrices. Thus we have the integral representation

$$\begin{aligned}\text{tr} \Delta_A(x, x') &= \int_\Omega D[x] D[\psi] e^{-\int \mathcal{L} dt} \\ \Omega &= \{x, \psi | x(0) = x; x(T) = x'; \psi(0) = \psi(T)\}\end{aligned}$$

Path Integral Representation of the Determinant

Now let us consider the problem of writing the $[\text{Det}(\gamma \cdot D + m_0)]^{N_f}$ as a path integral. This is quite complicated. We proceed in steps.

1. Reduce to the Square of the Dirac Operators

$$\begin{aligned}\text{Det}(\gamma \cdot D + m_0) &= \text{Det} \gamma_5 (\gamma \cdot D + m_0) \gamma_5 \\ &= \text{Det}(-\gamma \cdot D + m_0)\end{aligned}$$

So

$$[\text{Det}(\gamma \cdot D + m_0)]^{N_f} = [\text{Det}(\gamma \cdot D + m_0)(-\gamma \cdot D + m_0)]^{N_f/2}$$

i.e.

$$[\text{Det}(\gamma \cdot D + m_0)]^{N_f} = [\text{Det}(-\gamma \cdot D)^2 + m_0^2]^{\frac{N_f}{2}}$$

2. Path Integral

$$\begin{aligned}[\text{Det}(-\gamma \cdot D)^2 + m_0^2]^{N_f/2} &= \exp \frac{N_f}{2} \text{tr} \text{Det}(-\gamma \cdot D)^2 + m_0^2 \\ &= \exp \frac{N_f}{2} \text{tr} \ln [-(\gamma \cdot D)^2 + m_0^2]\end{aligned}$$

Now consider the $\text{tr} \ln$:

$$\text{tr} \ln [(\gamma \cdot D)^2 + m_0^2] = \int_0^\infty \frac{dT}{T} \text{tr} e^{-T[(\gamma \cdot D)^2 + m_0^2]}$$

But the limit $T \rightarrow 0$ generally leads to a u.v. singularity. We will assume that the Det is defined with a u.v. cut-off. A convenient cut-off is to do the integral up to $T = \epsilon$. Using earlier results,

$$\begin{aligned}\text{tr} \ln [(-\gamma \cdot D)^2 + m_0^2] &= \int_0^\infty \frac{dT}{T} \text{tr} e^{-T\hat{H}} \\ &= \int_\epsilon^\infty \frac{dT}{T} \int_{\Omega_T} D[x] D[\psi] e^{-\int \mathcal{L} dt}\end{aligned}$$

where Ω_T is the set of closed curves:

$$\begin{aligned}x(0) &= x(T) \\ \psi(0) &= \psi(T)\end{aligned}$$

3. Expansion in powers of N_f

What we need is $\exp \frac{N_f}{2} \text{tr} \ln H$ not $\text{Tr} \ln H$. We can get it by expanding the exponential:

$$\begin{aligned}\text{Det}(\gamma \cdot D + M_0) &= \sum_0^\infty \left(\frac{N_f}{2}\right)^k \frac{1}{k!} (\text{tr} \ln H)^k \\ &= \sum_0^\infty \left(\frac{N_f}{2}\right)^k \frac{1}{k!} \int_\epsilon^\infty \prod_{\ell=1}^k \frac{dT_\ell}{T_\ell} \int D_{\Omega_\ell}[x_\ell] e^{-\int_0^{T_\ell} \mathcal{L}[x_\ell] dt_\ell}\end{aligned}$$

This means that we are integrating over k closed loops. Here, as before,

$$\mathcal{L}[x] = \frac{1}{2} \dot{x}^2 + \frac{i}{2} \psi \dot{\psi} + g^{\mu\nu} \psi^\mu F_{\mu\nu} + ig A_\mu \dot{x}^\mu + m_0^2$$

It is not difficult to figure out what the expansion in N_f means. The power of N_f merely counts the number of electron loops. This expansion makes sense even though $N_f \geq 1$.

4. **Worldline form of amplitudes** Now we are ready to find any integral over q 's. Suppose we want

$$\int \mathcal{D}[q] e^{-\int \bar{q}(\gamma \cdot D + M_0) q dx} \bar{q}(x) q(y) = [\text{Det}(\gamma \cdot D + M_0)]^{N_f} S_A(x, y)$$

We will find this as a formal power series in N_f . The zeroth order in N_f contains no closed loops. Thus, the above object becomes

$$\begin{aligned} & \int_0^\infty dT e^{-m_0^2 T} \int \mathcal{D}[x] \mathcal{D}[\psi] e^{-\int_0^T L[x, \psi] dt} \\ & + \frac{N_f}{2} \int_0^\infty \frac{dT_1}{\epsilon} e^{-m_0^2 T_1} \int \mathcal{D}[x] \mathcal{D}[\psi] \mathcal{D}[y_1] \mathcal{D}[\eta_1] \\ & e^{-\int_0^{T_1} L[x, \psi] dt - \int_0^{T_1} L[y_1, \eta_1] dt_1} + \dots \end{aligned}$$

Next, y denotes closed loops of various periods. The first term is represented by the "tree-level" contribution and the next are the "one loop" contribution and so on.