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ANALYTIC RESULTS FOR THE EFFECTIVE THEORY OF THERMAL POLYAKOV LOOPS

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A B S T R A C T

Analytic results for deconfinement in the effective theory of Polyakov loops are presented: i) in the neighbourhood of the lower critical dimensionality ($D=1$); ii) at infinite number of colours, $N_c \rightarrow \infty$. The results are compared with recent Monte Carlo data on both the effective theory at finite N_c , and the full finite temperature gauge theory in (3+1) dimensions.

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1. - INTRODUCTION

One of the most powerful theoretical approaches to deconfinement in gauge theories at finite temperature comes from the universality arguments of Svetitsky and Yaffe¹⁾. The statement can be phrased in a compact way as follows: if the phase transition is continuous, and if the breakdown of the relevant symmetry is described by a unique fixed point, then one finds the critical exponents by universality.

Even if the second condition is met, the eventuality of a first-order phase transition cannot be excluded. As a matter of fact, among $SU(N_c)$ gauge theories the deconfining transition is, in (3+1) dimensions, continuous only for $SU(2)$ ²⁾. One has numerical evidence which confirms the relevance of the Ising fixed point in this case³⁾. For higher groups ($N_c = 3, 4$ and ∞) all transitions have been observed to be of first order in four dimensions⁴⁾⁻⁷⁾.

Analytic explanations of this phenomenon have consisted almost exclusively in mean field treatments of the effective action for the thermal Polyakov loop $W(x)$, the order parameter of the theory:

$$S_{\text{eff}} = \frac{1}{2} \sum_{x,j} \{ W(x) W^\dagger(x+j) + W^\dagger(x) W(x+j) \} \quad (1)$$

$W(x)$ takes its values from the group characters in the fundamental representation. Its vacuum expectation value gives the excess free energy (over the ground state energy) of fundamental, coloured, sources. The effective theory (1) can be derived either at strong coupling⁸⁾, or by a Kadanoff-inspired potential-moving scheme⁹⁾.

In this letter we shall discuss two approaches which are partly superior, partly complementary to the mean field treatment.

First, we shall demonstrate explicitly in the case of the $SU(2)$ gauge group that the lower critical dimension really is 1, as it should be when a discrete symmetry gets broken. The leading behaviour of the $D = 1+\epsilon$ dimensional theory is determined, verifying analytically that the model belongs to the Ising universality class. The critical exponent ν , when continued to the physical dimensions $D = 2$ and 3, gives a reasonable description with its leading ϵ form.

this result hints at a more general interest in the application of the $1+\epsilon$ expansion to the effective theory (1).

Second, we shall reanalyze the case when the number of colours N_c goes to infinity. Comparison of the exact $N_c = \infty$ solution¹⁰⁾ with Monte Carlo data for the effective $SU(4)$ model¹¹⁾ strongly suggests its relevance in understanding why the $SU(N_c > 4)$ models "choose" first-order transitions, although the $Z(N_c)$ symmetries could be broken continuously too. A comparison of the exact analytic expression for the magnetization $\langle W(x) \rangle / N_c$ in the effective model is made with recent Monte Carlo results for the full finite temperature gauge theory at $N_c = \infty$ ⁶⁾. The good agreement found suggests that detailed predictions for the magnetization can be made also in the presence of quarks when N_f/N_c is kept fixed, N_f being the number of flavours.

2. - THE THEORY IN $1+\epsilon$ DIMENSIONS

We have applied a Migdal-Kadanoff type real space renormalization group transformation¹²⁾ to (1) for the case of an $SU(2)$ gauge group in $D = 1+\epsilon$ dimensions. This approximate technique is known to provide critical indices associated with the breaking of discrete symmetries which are exact to leading order in ϵ ¹²⁾.

The generic form of the one-dimensional decimation at large J is parametrized in terms of the couplings as

$$J' = J \left(1 - \frac{c}{J^\beta} + \dots \right), \quad J \gg 1 \quad (2)$$

The leading term, if confirmed, reflects the marginality of J in one dimension, while the subleading term governs the flow away from the unstable fixed point $J^* = \infty$. The bond moving step of the renormalization group transformation in $D = 1+\epsilon$ dimensions introduces the opposite tendency by letting $J \rightarrow \lambda^\epsilon J$ (λ is the scale factor of the transformation). The net result of a Kadanoff-type transformation hence looks like

$$J' = \lambda^\epsilon J \left(1 - \frac{c}{\lambda^{\epsilon\beta} J^\beta} + \dots \right) \quad (3)$$

Assuming self-consistently that $J^\beta \sim 0(\epsilon^{-1})$ one determines the new location of the unstable fixed point on the J-axis:

$$J^*\beta = \frac{c}{\epsilon \ln \lambda} \quad (4)$$

The standard linearization of (3) around J^* then yields the critical exponent

$$\nu_J = \frac{1}{\beta \epsilon} \quad (5)$$

This shows that the subleading piece of the one-dimensional decimation determines the critical data to leading order in ϵ .

In a previous publication¹³⁾, we have introduced the double character expansion for the Boltzmann factor of the theory (1), whose coefficients in the SU(2) case are given as

$$F_{ij} = \frac{1}{(2i+1)(2j+1)} \left\{ I_{j-i}(2J) [I_{i+j+2}(2J) + I_{i+j}(2J)] \right. \\ \left. - I_{i+j+1}(2J) [I_{i-j+1}(2J) + I_{j-i+1}(2J)] \right\} \quad (6)$$

In Eq. (6) the isospin indices i and j are either both integers or both half-integers. A one-dimensional decimation with scale factor $\lambda = 2$ looks in terms of these coefficients like

$$F'_{ij}(2J) = \sum_k d_k^2 F_{ik}(2J) F_{kj}(2J) \quad (7)$$

with $d_j = 2j+1$.

Using the asymptotic saddle point expansion of the modified Bessel functions, we get (with $x = 2J$):

$$F_{ij}(x) = \frac{1}{d_i d_j} \left\{ F_i^{(1)} F_j^{(1)} + \frac{1}{x^3} F_i^{(2)} F_j^{(2)} + O(x^{-4}) \right\} \quad (8a)$$

$$F_i^{(\alpha)} \equiv f_i^{(\alpha)} - f_{i+1}^{(\alpha)}, \quad \alpha = 1, 2 \quad (8b)$$

$$f_i^{(4)} = (2\pi x)^{-1/2} \exp\{x - i^2/x\} \left[1 + \frac{1}{8x} + \frac{1}{x^2} \left(\frac{9}{138} - \frac{i^2}{2} \right) + \frac{1}{2x^3} \left(\frac{75}{512} - \kappa^2 - \left(\frac{29}{24} - \kappa^2 \right) i^2 + \frac{i^4}{6} \right) \right] \quad (8c)$$

$$f_i^{(2)} = (4\pi x)^{-1/2} \exp\{x - i^2/x\} [i^2 + \kappa] \quad (8d)$$

The complete factorizability of F_{ij} is broken by the $O(x^{-3})$ term proportional to $i^2 j^2$. We are allowed to shift a $\text{const.} x^{-3}$ piece from $f_i^{(1)}$ to $f_i^{(2)}$, which is reflected by the free parameter κ in Eq. (8). The renormalization of J is found from the requirement of form-reproduction of $F_i^{(1)} F_j^{(1)}$ under the $\lambda = 2$ decimation, and we find below that it is only possible when a specific value of κ is chosen.

For the character coefficients (8) the decimation takes the following form:

$$F'_{ij}(x) = \frac{a_1}{d_i d_j} \left[F_i^{(1)} F_j^{(1)} + \frac{a_2}{x^3} (F_i^{(1)} F_j^{(2)} + F_i^{(2)} F_j^{(1)}) + \dots \right],$$

$$a_1 = \sum_k F_k^{(1)2}, \quad a_1 a_2 = \sum_k F_k^{(1)} F_k^{(2)} \quad (9)$$

We now compute the two constants to leading order in J^{-1} with the help of Poisson's resummation formula. Miraculously, their values to leading order do not depend on whether λ runs over integers or half-integers:

$$a_1 = \left(\frac{\pi}{16J} \right)^{-1/2} + O(J^{-3/2}), \quad a_2 = -\frac{1}{2\sqrt{2}} J + \frac{1}{8\sqrt{2}} + O(J^{-1}) \quad (10)$$

One now notices that the transformed coefficient (9) is only accurate to $O(J^{-2})$. Absorbing the second term in (9) into $F_i^{(1)} F_j^{(1)}$ by a coupling renormalization, one fulfils the renormalization group invariance to $O(J^{-2})$:

$$F'_{ij}(J) = F_{ij}(J') \quad (11)$$

Evaluating $F_i^{(1)} F_j^{(1)}$ at J' and expanding it around J , the parameters c and β of Eq. (2) and κ of Eq. (8) can be determined by demanding (11). The equality is ensured by choosing

$$K = \frac{1}{8}, \quad C = \frac{1}{16}, \quad \beta = 1 \quad (12)$$

These results, when used in (4) and (5) lead to

$$\mathcal{J}^* = (16 \ln 2)^{-1}, \quad \nu_{\mathcal{J}} = \epsilon^{-1} \quad (13)$$

The location of the fixed point itself clearly depends on the scale parameter, while the exponent ν is independent of it, even in this approximate framework.

Our argument demonstrates without any explicit input about the order parameter, that it is a discrete symmetry which is broken at the transition. The index ν of Eq. (13) agrees exactly with the thermal index of the Ising model in $D = 1+\epsilon$ dimensions¹²⁾. When continued to higher dimensions it gives accidentally also the exact exponent in $D = 2$, and a reasonable value $\frac{1}{2}$ in three dimensions instead of $\sim 2/3$ as it is computed from the ϵ -expansion in $4-\epsilon$ dimensions. From this we conjecture that a more refined $1+\epsilon$ expansion might give more insight into the phase transition in the model (1) for $SU(2)$, also away from the lower critical dimension.

3. - THE LARGE N_c LIMIT

Interestingly, in the large- N_c limit the effective theory (1) can be solved exactly in all dimensions D . To see this, add to the action (1) a $Z(N_c)$ symmetry breaking term, and define

$$S(\mathcal{J}, h) = \frac{1}{2} \mathcal{J} \sum_{x,j} \{W(x)W^\dagger(x+j) + W^\dagger(x)W(x+j)\} + h N_c \sum_x \{W(x) + W^\dagger(x)\} \quad (14)$$

Writing

$$S(\mathcal{J}, 0) = \frac{1}{2} \mathcal{J} \sum_{x,j} \{ \langle W(x) \rangle W^\dagger(x+j) + W(x) \langle W^\dagger(x+j) \rangle - \langle W(x) \rangle \langle W^\dagger(x+j) \rangle \\ + (W(x) - \langle W(x) \rangle)(W^\dagger(x+j) - \langle W^\dagger(x+j) \rangle) + h.c. \} \quad (15)$$

we note that the last term does not contribute because of large- N_c factorization. Using translational invariance [$\langle W(x) \rangle = \langle W \rangle$], the action (15) can therefore be written

$$S(\mathcal{J}, 0) = \mathcal{J}D \sum_x \{ \langle W \rangle W^\dagger(x) + \langle W^\dagger \rangle W(x) - |\langle W \rangle|^2 \} \quad (16)$$

which shows that

$$S(\mathcal{J}, h) = S(0, h + \mathcal{J}D \langle W \rangle / N_c) - \mathcal{J}D \sum_x |\langle W \rangle|^2 \quad (17)$$

This last equation leads to non-trivial self-consistency conditions on $\langle W \rangle$.

We define

$$\frac{1}{N_c} \langle W \rangle_0 \equiv \frac{1}{Z_0} \int dW \left[\frac{1}{N_c} W \right] e^{S(0, h_{\text{eff}})} \quad (18)$$

which at $N_c = \infty$ is a known function of h_{eff} ¹⁴⁾:

$$f(h_{\text{eff}}) \equiv \frac{1}{N_c} \langle W \rangle_0 = \begin{cases} h_{\text{eff}} & , h_{\text{eff}} \leq \frac{1}{2} \\ 1 - \frac{1}{4h_{\text{eff}}} & , h_{\text{eff}} \geq \frac{1}{2} \end{cases} \quad (19)$$

and then solve the self-consistency equation

$$h_{\text{eff}} = h + \mathcal{J}D f(h_{\text{eff}}) \quad (20)$$

using Eq. (19).

The general D-dimensional solution becomes essentially identical to that of the two-dimensional mixed fundamental-adjoint action at $N_c = \infty$ ¹⁵⁾. Consider first the case $h = 0$. The self-consistency Eq. (20) in this case clearly leads to two solutions:

$$\frac{1}{N_c} \langle W \rangle = \begin{cases} 0 & , \mathcal{J} < 1/D \\ \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{1}{\mathcal{J}D}} & , \mathcal{J} > 1/D \end{cases} \quad (21)$$

which means that the free energy density (per degree of freedom)

$$\mathcal{F}(\mathcal{J}) = \begin{cases} \mathcal{J}D |\langle W \rangle|^2 / N_c^2 - h_{\text{eff}}^2 & , h_{\text{eff}} \leq \frac{1}{2} \\ \mathcal{J}D |\langle W \rangle|^2 / N_c^2 - 2h_{\text{eff}} + \frac{1}{2} \ln(2h_{\text{eff}}) + \frac{3}{4} & , h_{\text{eff}} \geq \frac{1}{2} \end{cases} \quad (22)$$

actually reads

$$F(J) = \begin{cases} 0, & J < 1/D \\ J D \langle W \rangle^2 / N_c^2 - 2h_{\text{eff}} + \frac{1}{2} \ln(2h_{\text{eff}}) + \frac{3}{4}, & J > 1/D \end{cases} \quad (23)$$

where for $J > 1/D$ the non-trivial solution in (21) should be used. At the transition point $J = 1/D$ the free energies are equal ($=0$), and by taking left and right limits of $\partial F / \partial J$ one finds:

$$\left. \frac{\partial F}{\partial (JD)} \right|_{JD=1^+} = -\frac{1}{4}, \quad \left. \frac{\partial F}{\partial (JD)} \right|_{JD=1^-} = 0 \quad (24)$$

which shows that for $J > 1/D$ the non-trivial solution in (21) is energetically favoured. The first-order transition at $J = 1/D$ in which $\langle W \rangle / N_c$ jumps discontinuously from 0 to $\frac{1}{2}$ corresponds to $J_1 = J_c$ in our earlier terminology¹³⁾.

The full phase diagram in the J - h plane is shown in Fig. 1. It is surprisingly simple, but it contains several intriguing aspects. First of all, $\langle W \rangle / N_c = 0$ only on the double line indicated at $h = 0$. When, still at $h = 0$, J reaches $J_c = 1/D$ we have the first-order transition discussed above. For $0 < h < \frac{1}{2}$ we find a third-order phase transition along the line

$$J_c = 1/D - 2h/D \quad (25)$$

and, finally, for $h > \frac{1}{2}$ there is no phase transition.

The same phase diagram was found in Ref. 10) from a somewhat different point of view. There the starting point was the $N_c = \infty$ reduced and twisted Wilson action, upon which was performed the Kadanoff-type bond movings of Ref. 9) in order to obtain an effective theory for $W(x)$. The resulting effective action is not the one of Eq. (1); however, to leading order in $1/N_c$ the solutions of the two actions are identical. This shows that the result is independent of whether one first starts with the full gauge theory, performs the Kadanoff-type transformations of Ref. 9) and then takes the $N_c \rightarrow \infty$ limit, or starts with the $N_c = \infty$ reduced model and then performs the transformations^{*)}.

*) Note incidentally that bond moving at $N_c = \infty$ preserves the free energy density because of factorization.

We now turn to an evaluation of the physical significance of the large- N_c results. Again, consider first the pure gauge theory corresponding to $h = 0$. In Fig. 2, we compare the exact $N_c = \infty$ solution (21) with Monte Carlo results for the SU(4) effective action¹¹⁾. Even on the fine-grained scale used in the Figure it is striking to see how well the SU(4) data are described by the exact $N_c = \infty$ solution. (The agreement is within 7%). The large- N_c solution appears to fare equally well if we compare it with recent Monte Carlo data⁶⁾ for the full (reduced, twisted) $N_c = \infty$ gauge theory in four dimensions (i.e., $D = 3$). The comparison is shown in Fig. 3 using the strong coupling relation⁸⁾ $J = 2\beta^{N_c}$ (a), and the weak coupling relation $J = 2\beta/N_\tau$ (b), respectively. The agreement is best on the $N_\tau = 1$ lattice (a), as compared with the $N_\tau = 4$ lattice (b), simply because for small N_τ the transition is pushed well into the strong coupling regime.

If fermions are added to the theory, they correspond to lowest order in a hopping parameter expansion to the addition of a magnetic field $H = 2N_f(2K)^{N_\tau}$ to the action¹⁶⁾ (N_f = number of flavours, and K = hopping parameter). In a topological $1/N_c$ expansion¹⁷⁾ in which $N_c \rightarrow \infty$ and $N_f \rightarrow \infty$, their ratio $\xi = N_f/N_\tau$ being kept fixed, we then identify the $h = 0$ part of the phase diagram as modelling the effect of quarks on the $N_c = \infty$ transition. From Eq. (14) it follows that the relation is $h = 2\xi(2K)^{N_\tau}$.

Even in the general case $h = 0$ we can find the exact expression for the magnetization from Eqs. (19) and (20):

$$\frac{1}{N_c} \langle W \rangle = \begin{cases} h/(1-3D) , & h_{\text{eff}} \leq \frac{1}{2} \\ \frac{1}{2} \left(1 - \frac{h}{3D} \right) + \frac{1}{2} \sqrt{\left(\frac{3D-h}{3D} \right)^2 - \left(\frac{1}{3D} - \frac{4h}{3D} \right)} \end{cases} \quad (26)$$

In particular, at the third-order phase transition point $J_c = 1/D - 2h/D$ we have $\langle W \rangle/N_c = 0.5$ independent of the dimension D . To illustrate, we show in Fig. 4 the detailed prediction for $\langle W \rangle/N_c$ as a function of J , here with $h = 0.1$. The results are shown for $D = 3$, corresponding to the four-dimensional gauge theory with quarks.

At the phase transition it is a continuous $[U(1)]$ symmetry which gets broken. This is perhaps most easily seen by considering the density of eigenvalues $\rho(\alpha)$ of the $W(x)$ matrix:

$$\rho(\alpha) = \begin{cases} \frac{1}{2\pi} (1 + 2h_{\text{eff}} \cos \alpha), & \alpha_{\text{max}} = \pi; h_{\text{eff}} \leq \frac{1}{2} \\ \frac{2}{\pi} h_{\text{eff}} \cos\left(\frac{\alpha}{2}\right) \left[\frac{1}{2h_{\text{eff}}} - \sin^2\left(\frac{\alpha}{2}\right) \right]^{1/2}, & \alpha_{\text{max}} = 2\sin^{-1}\left(\sqrt{\frac{1}{2h_{\text{eff}}}}\right); h_{\text{eff}} \geq \frac{1}{2} \end{cases} \quad (27)$$

We see that for $J < 1/D$ ($h_{\text{eff}} = 0$) $\rho(\alpha) = (2\pi)^{-1}$, which is explicitly $U(1)$ invariant, while for $J > 1/D$ the symmetry is broken.

The phase transition where this $U(1)$ symmetry is spontaneously broken occurs in all physical dimensions $D > 1$. In view of the Mermin-Wagner theorem it is very difficult to see how this phase transition can be described by a $U(1)$ valued "spin" theory. We should also point out that the full finite temperature gauge theory at $N_c = \infty$ presumably also will have a phase transition in all dimensions $D+1 > 2$.

Perhaps the strongest point to emphasize from this large- N_c analysis is the fact that the $SU(4)$ Monte Carlo results are so well fitted by the exact $N_c = \infty$ prediction. This clearly suggests that the mechanism which drives the transitions first order for all $SU(N_c > 4)$ gauge groups is the same as that of the $N_c = \infty$ theory.

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FIGURE CAPTIONS

- Fig. 1 The phase diagram in the h - J plane, as discussed in the text. There is a first-order phase transition at $(h=0) J = 1/D$.
- Fig. 2 Comparison of the exact $N_c = \infty$ solution with Monte Carlo results [Ref. 11)] for the $SU(4)$ model. Crosses and dots are from ordered and disordered starts, respectively.
- Fig. 3 Comparison of the exact $N_c = \infty$ solution with Monte Carlo results [Ref. 6)] for the full (reduced, twisted) $N_c = \infty$ gauge theory in four dimensions. The plot in (a) corresponds to $N_\tau = 1$, and the plot in (b) corresponds to $N_\tau = 4$. Triangles and circles are from strong and weak coupling starts, respectively.
- Fig. 4 Prediction for $\langle W \rangle / N_c$ for $h = 0.1$. There is a third-order phase transition at $J_c = 0.8/D \approx 0.266$. This corresponds to a large- N_c theory with fermions, as discussed in the text. Note the shift of the crossover when fermions are added, a phenomenon which has also been observed at finite N_c .

- Figure 1 -

Figure 2 is a graph showing the relationship between $\frac{1}{4} \langle W \rangle$ (vertical axis) and J (horizontal axis). The vertical axis has ticks at 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, and 0.7. The horizontal axis has ticks at 0.25, 0.30, 0.35, and 0.40. The graph shows a curve that starts at $J \approx 0.25$ and increases, approaching a value of approximately 0.7 as J increases. There are two sets of data points: one set of 'x' marks and one set of dots. The 'x' marks are located above the curve, and the dots are located below the curve. A vertical line is drawn at $J \approx 0.33$, separating the two sets of data points.

- Figure 2 -

- Figure 3 -

- Figure 4 -