CERN-TH.4373/86

COMMENT ON CHIRAL FERMIONS IN STOCHASTIC QUANTIZATION

E.R. Nissimov and S.J. Pacheva

CERN -- Geneva

ABSTRACT

The subject of this comment is twofold. First, a recently claimed²⁾ failure of stochastic quantization scheme (SQS) to describe standard chiral anomalies at finite stochastic time is shown to result from an elementary mathematical error. The consistent approach of treating chiral fermions in SQS is briefly sketched and full agreement with previous investigations³⁾ is established. Second, a serious new limitation on SQS is found, namely, its incapability to reproduce global chiral- and odd-dimensional parity-violating anomalies.

CERN-TH.4373/86

1. Generalized stochastic regularization (SR)¹⁾ explicitly preserves chiral gauge symmetries in the stochastic quantization scheme (SQS) for chiral fermions. Therefore, it is an interesting test of SQS to check if the pertinent chiral anomalies are correctly reproduced after SR is removed.

In a recent note²⁾ a negative answer to this question was claimed. This failure of SQS, implying its incapability to describe massless fermions at finite fictitious stochastic time t , was attributed in Ref.2) to the asserted non-commutativity of the zero mass limit ($m \rightarrow 0$) and the equilibrium limit ($t \rightarrow \infty$). These conclusions contradict previous considerations³⁾, where the recovery of the correct chiral anomalies within SQS [in a slightly different form from that in Ref. 2)] was explicitly demonstrated. The reason that anomalies appear within SQS is that the additional non-local terms introduced through SQS in the corresponding chiral Ward identities [cf., the r.h.s. of Eq. (8) below] yield finite non-zero contributions (i.e., anomalies) when SR is removed [see Eqs. (9) and (10) below].

In Section 2 we briefly outline the SQS formulation for chiral (e.g., left-handed) fermions $\psi_L(t, x)$ interacting with a background $U(n)$ gauge field $A_\mu(x)$ and the derivation of the correct standard (perturbative) chiral anomalies at finite t . Section 3 is devoted to criticism of results and to amendment of the approach of Ref. 2) so as to deduce the correct conclusions agreeing with those of Ref. 3) and of Section 2. On the other hand, some new serious drawbacks of SQS are pointed out in Section 4.

2. The Langevin equations for chiral fermions (in Euclidean D -dimensional space-time) are taken in the form³⁾:

$$\partial_t \psi_L^\perp = -(\mathcal{D}\mathcal{D}^*)\psi_L^\perp + \eta_L^\perp, \quad \partial_t \bar{\psi}_L^\perp = -(\mathcal{D}^*\mathcal{D})^\top \bar{\psi}_L^\perp + \bar{\eta}_L^\perp; \quad (1)$$

$$\langle \eta_L^\perp(t, x) \bar{\eta}_L^\perp(t', x') \rangle = -2i \delta_A(t-t') \mathcal{D}(1-\Pi_0) \delta_{(x-x')}^{(D)}, \quad (2)$$

with the following notations:

$$\mathcal{D}^{(*)} \equiv \mathcal{D}^{(*)}(A) = i\sigma_{\mu}^{(*)} \nabla_{\mu}(A) = i\sigma_{\mu}^{(*)} (\partial_{\mu} + iA_{\mu}(x)) ,$$

$$\gamma_{\mu} = i \begin{pmatrix} 0 & \sigma_{\mu} \\ \sigma_{\mu}^{*} & 0 \end{pmatrix} , \gamma^{(D+1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \not{D}(A) = \gamma_{\mu} \nabla_{\mu}(A) = \begin{pmatrix} 0 & \mathcal{D}(A) \\ \mathcal{D}^{*}(A) & 0 \end{pmatrix} , \quad (3)$$

$$\psi_L = \frac{1}{2} (1 + \gamma^{(D+1)}) \psi , \quad A_{\mu}(x) = A_{\mu}^a(x) T^a , \quad a=0,1,\dots,n^2-1;$$

$$\psi_L^{\perp} \equiv (1 - \Pi_0) \psi_L , \quad \bar{\psi}_L^{\perp} \equiv (1 - \Pi_0) \bar{\psi}_L \quad \text{and analogous for } \bar{\psi}_L^{(\perp)} . \quad (4)$$

The superscript "T" in (1) means operator transposition. $\Pi_0, \bar{\Pi}_0$ in (2) and (4) are the zero-mode projectors of $\mathcal{D}^* \mathcal{D}, \mathcal{D} \mathcal{D}^*$ respectively. The Gaussian random source η_L in (1) is anticommuting chiral spinor field with two-point correlation function 2. SR is represented through the introduction of $\delta_{\Lambda}(t-t')$ in (2) obeying the following properties:

$$\lim_{\Lambda \rightarrow \infty} \delta_{\Lambda}(t) = \delta(t) , \quad \delta_{\Lambda}(-t) = \delta_{\Lambda}(t) , \quad \left. \frac{d^k}{dt^k} \delta_{\Lambda}(t) \right|_{t=0} = 0 , \quad k=0,1,\dots,L-1 , \quad (5)$$

where L is an appropriate integer. A particular choice of (5) is:

$$\delta_{\Lambda}(t) = \frac{1}{2} (L!)^{-1} \Lambda (\Lambda |t|)^L \exp\{-\Lambda |t|\} . \quad (5')$$

The projection (4) in (1) and (2) is needed to guarantee approach to equilibrium [$\mathcal{D} \mathcal{D}^*$ and $\mathcal{D}^* \mathcal{D}$ in (1) should be positive in order to yield a "drift force"]:

$$\langle F[\bar{\psi}_L^{(\perp)}(x)] \rangle_Q = \lim_{t \rightarrow \infty} \langle F[\bar{\psi}_L^{(\perp)}(t,x)] \rangle_{\mathcal{Z}} . \quad (6)$$

The subscript "Q" on the l.h.s. of (6) denotes the usual Euclidean quantum average of an arbitrary functional $F[\bar{\psi}_L^{(\perp)}(x)]$ with regularized standard weight $\exp\{-S_{\Lambda}\}$:

$$\begin{aligned}
 S_{\Lambda} &= i \int d^D x d^D x' \bar{\psi}_L^{\perp}(x) K_{\Lambda}(x, x' | A_{\mu}) \psi_L^{\perp}(x') , \\
 \lim_{\Lambda \rightarrow \infty} K_{\Lambda} &= \mathcal{D}^* , \\
 K_{\Lambda}^{-1}(x, x' | A_{\mu}) &\equiv \lim_{t \rightarrow \infty} i \langle \psi_L^{\perp}(t, x) \bar{\psi}_L^{\perp}(t, x') \rangle_{\eta} = \\
 &= \mathcal{D} (\mathcal{D}^* \mathcal{D})^{-1} \left\{ 2 \int_0^{\infty} d\tau \delta_{\Lambda}(\tau) e^{-\tau \mathcal{D}^* \mathcal{D}} (1 - \Pi_0) \right\} (x, x') = \\
 &= \mathcal{D} (\mathcal{D}^* \mathcal{D})^{-1} \left[\frac{\Lambda}{\Lambda + \mathcal{D}^* \mathcal{D}} \right]^{L+1} (1 - \Pi_0)(x, x')
 \end{aligned} \tag{7}$$

[for the choice (5')]. The subscript "η" on the r.h.s. of (6) indicates SQS average according to (2) with $\bar{\psi}_L^{\perp}(t, x)$ being solutions of (1) subject to arbitrary initial conditions.

Choosing, e.g., $\psi_L^{\perp}(t=0, x)=0$ [in Ref. 3) another choice was made $\psi_L^{\perp}(t=-\infty, x)=0$] one gets in analogy with Ref. 3):

$$\begin{aligned}
 \nabla_{\mu}^{ab} J_{\mu}^{L, b}(t, x) &\equiv \nabla_{\mu}^{ab} \langle \bar{\psi}_L^{\perp}(t, x) \gamma_{\mu}^b \sigma_{\mu}^* \psi_L^{\perp}(t, x) \rangle_{\eta} = \\
 &= i \langle \{ [\mathcal{D}^* (\mathcal{D} \mathcal{D}^*)^{-1}]^{\pi} (\bar{\eta}_L^{\perp} - \partial_t \bar{\psi}_L^{\perp}) \} (t, x) \psi_L^{\perp}(t, x) \rangle_{\eta} - \\
 &\quad - \langle \bar{\psi}_L^{\perp}(t, x) \{ [\mathcal{D}^* (\mathcal{D} \mathcal{D}^*)^{-1}] (\eta_L^{\perp} - \partial_t \psi_L^{\perp}) \} (t, x) \rangle_{\eta} = \\
 &= \lim_{\Lambda \rightarrow \infty} 2 \int_0^{\infty} d\tau \delta_{\Lambda}(\tau) \theta(t-\tau) t_{\tau} \left\{ \eta^a \left[(e^{-\tau \mathcal{D}^* \mathcal{D}} - e^{-(2t-\tau) \mathcal{D}^* \mathcal{D}}) (1 - \Pi_0)(x, x) - \right. \right. \\
 &\quad \left. \left. - (e^{-\tau \mathcal{D} \mathcal{D}^*} - e^{-(2t-\tau) \mathcal{D} \mathcal{D}^*}) (1 - \bar{\Pi}_0)(x, x) \right] \right\} = \\
 &= \lim_{\Lambda \rightarrow \infty} t_{\tau} \left[\left(\frac{\Lambda}{\Lambda + \mathcal{D}^* \mathcal{D}} \right)^{L+1} (1 - \Pi_0) - \left(\frac{\Lambda}{\Lambda + \mathcal{D} \mathcal{D}^*} \right)^{L+1} (1 - \bar{\Pi}_0) \right] (x, x) + O(e^{-t})
 \end{aligned} \tag{8}$$

[for the choice (5')].

Now from (9) by a straightforward computation, using, e.g., Seeley expansion of the heat kernels $\exp \{-\tau \mathcal{D}^* \mathcal{D}\}$, $\exp \{-\tau \mathcal{D} \mathcal{D}^*\}$ [e.g., Ref. 4)] one easily gets the standard covariant form of the (non-Abelian) chiral anomaly plus a correction for finite t vanishing in the equilibrium limit:

$$\nabla_\mu^{ab} J_\mu^{L,b}(t,x) = tz \left[\text{Tr}^a (\bar{\Pi}_0 - \Pi_0)(x,x) \right] - \mathcal{A}^a(x) + \\ + tz \left\{ \text{Tr}^a \left[e^{-2t \mathcal{D} \mathcal{D}^*} (1 - \bar{\Pi}_0) - e^{-2t \mathcal{D}^* \mathcal{D}} (1 - \Pi_0) \right] (x,x) \right\} ; \quad (10)$$

$$\mathcal{A}^a(x) \equiv \left[\left(\frac{D}{2} \right)! (4\pi)^{D/2} \right]^{-1} \varepsilon_{\mu_1 \dots \mu_D} tz \left[\text{Tr}^a F_{\mu_1 \mu_2} \dots F_{\mu_{D-1} \mu_D} \right] .$$

Let us particularly stress that SQS with SR (1) and (2) manifestly preserves chiral gauge symmetries in the SQS averages, the r.h.s. of (6), for any t . However, this last property is not in conflict with the property of SQS with SR to correctly reproduce the standard chiral anomalies (9) and (10). Indeed, on the one hand chiral anomalies result from the non-existence of consistent chiral gauge invariant definition of the chiral fermion effective action⁵⁾ which in our case reads [cf. (7)]:

$$S_\Lambda^{\text{eff}} = - \ln \det [-i K_\Lambda] \quad (11)$$

[recall that K_Λ (7) is not an operator in one and the same space, but maps left-handed into right-handed spinors and, therefore, there is no unambiguous definition of $\det(-i K_\Lambda)$]. On the other hand, the object S_Λ^{eff} (11) cannot be obtained within SQS as an appropriate SQS average according to (6). Hence, there is no clash between chiral gauge non-invariance of (11) and manifest chiral gauge symmetry of the stochastic averages (6).

3. In Ref. 2) the following SQS Langevin equations were chosen:

$$\begin{aligned}\partial_t \psi &= - [\not{A}^2(A) + m^2] \psi + (-i \not{A}(A) + m) \eta_1 + \eta_2, \\ \partial_t \bar{\psi} &= - [\not{A}^2(A) + m^2] \bar{\psi} + (-i \not{A}(A) + m) \bar{\eta}_1 + \bar{\eta}_2; \\ \langle \eta_a(t, x) \bar{\eta}_b(t', x') \rangle &= \delta_{ab} \delta_\Lambda(t-t') \delta^{(D)}(x-x'), \quad a, b = 1, 2,\end{aligned}\quad (12)$$

where notations (3) are employed. Using (12) with initial conditions $\psi(t=0, x)=0$ one gets in complete analogy with Ref. 3) and Section 2:

$$\begin{aligned}\partial_\mu J_\mu^{(D+1)}(t, x) &\equiv \partial_\mu \langle \bar{\psi}(t, x) (-i \gamma_\mu) \gamma^{(D+1)} \psi(t, x) \rangle_\eta = \\ &= \lim_{\Lambda \rightarrow \infty} (-4) \int_0^\infty d\tau \delta_\Lambda(\tau) \theta(t-\tau) t_2 \left\{ \gamma^{(D+1)} \left[\left(1 - \frac{m}{m+i\not{A}}\right) \times \right. \right. \\ &\quad \left. \left. \times \left(e^{-\tau(\not{A}^2+m^2)} - e^{-(2t-\tau)(\not{A}^2+m^2)} \right) \right] (x, x) \right\}.\end{aligned}\quad (13)$$

Eq. (13) is an improved version of Eq. (8) in Ref. 2 since here only mathematically well-defined objects [heat kernels of elliptic positive operators $(m^2 + \nabla^2)$] are used instead of formal sums over eigenvalues of ∇ [whose spectrum is in fact continuous in general] as in Ref. 2). We stress that in obtaining (9) and (13), no additional regularization ["proper-time" cut-off, Pauli-Villars, etc.] besides SR was needed. After removing SR in (13) and taking t large enough [to simplify the subsequent expressions] one finds:

$$\begin{aligned}\partial_\mu J_\mu^{(D+1)}(t, x) &= -2 \not{A}^0(x) + 2 t_2 \left[\gamma^{(D+1)} e^{-2t(\not{A}^2+m^2)} (x, x) \right] + \\ &+ 2m^2 \int_0^\infty d\tau t_2 \left\{ \gamma^{(D+1)} \left[e^{-\tau(\not{A}^2+m^2)} - e^{-(2t+\tau)(\not{A}^2+m^2)} \right] (x, x) \right\}.\end{aligned}\quad (14)$$

Now it is evident from (14) that, contrary to the claim in Ref. 2), no difficulties do exist with the order of the limits $m \rightarrow 0$ and $t \rightarrow \infty$. Indeed, let us take, for instance, first $m \rightarrow 0$ with $t < \infty$ in (14) [this limit was claimed in Ref. 2) to annihilate the axial anomaly]:

$$\partial_\mu J_\mu^{(D+1)}(t, x) = -2 \not{A}^0(x) + 2 t_2 \left[\gamma^{(D+1)} \Pi_0^\not{A}(x, x) \right] + 2 t_2 \left[\gamma^{(D+1)} \left(e^{-2t\not{A}^2} - \Pi_0^\not{A} \right) (x, x) \right] \quad (15)$$

where $\Pi_0^\not{D}(x, x')$ denotes the kernel of the zero-mode projector of $\not{D}$. Equation (15) is the correct version of the mathematically senseless Eq. (12) of Ref. 2):

$$\partial_\mu J_\mu^{(D+1)}(t, x) = -2 \lim_{m \rightarrow 0} \sum_E \psi_E^*(x) \not{D}^{(D+1)} \psi_E(x) [1 - e^{-2t(E^2 + m^2)}] \quad (*)$$

$[\not{D}\psi_E(x) = E\psi_E(x)]$. Rewriting the r.h.s. of this last equation in terms of operator kernels:

$$-2 \lim_{m \rightarrow 0} t \not{D}^{(D+1)} (\not{D}^{(D)}(0) - e^{-2t(\not{D}^2 + m^2)})(x, x)$$

one clearly sees that it is ill-defined.

The first two terms on the r.h.s. of (15) are immediately recognized as the standard non-integrated axial anomaly, whereas the third term represents a correction for finite t which vanishes for $t \rightarrow \infty$. Upon integration of (15) over x the last term gives zero for any t and, therefore, the standard index theorem $n_+ - n_- = \int d^D x \not{D}^0(x)$ is recovered for any finite t [$n_\pm$ denoting the number of chiral zero-modes].

Thus we have shown that the variant of SQS (12) for massless Dirac fermions chosen in Ref. 2), when carefully treated, also correctly reproduces the standard chiral anomalies ^{*}).

4. Finally, let us briefly demonstrate that SQS does in fact exhibit some new difficulties, but these occur for global (non-perturbative) chiral anomalies ⁶⁾ as well as for massless fermions in odd D .

Indeed, according to Ref. 6) under homotopically non-trivial gauge transformations, $U(x) \in SU(2)$ in $D=4$:

^{*}) Let us note that, even starting from the formal expression (*), upon integration over x one gets:

$$0 = \int d^D x \partial_\mu J_\mu^{(D+1)}(t, x) = -2 \lim_{m \rightarrow 0} (n_+ - n_-) (1 - e^{-2tm^2}) = 0 \quad (**)$$

for $t < \infty$. Equation (**) should replace the incorrect Eq. (13) of Ref. 2) where on the l.h.s., instead of zero as in (**), the authors put $\int d^D x \partial_\mu J_\mu^{(D+1)}(t, x) = n_+ - n_-$???. Equation (**) is of course a trivial identity ^{μ} which does not contradict the index theorem, i.e., the integrated anomaly.

$$\det[-i\not{D}(A)]^{1/2} = \det[-i\not{D}^*(A)] = -\det[-i\not{D}^*(A^U)] . \quad (16)$$

On the other hand, for each SU(2) gauge invariant quantity $\mathcal{F}[A_\mu]$ SQS gives:

$$\langle \mathcal{F}[A_\mu(x)] \rangle_Q = \left\{ \int \mathcal{D}A_\mu e^{-S_{\text{eff}}[A]} \right\}^{-1} \int \mathcal{D}A_\mu e^{-S_{\text{eff}}[A]} \mathcal{F}[A_\mu] = \quad (17.a)$$

$$= \lim_{t \rightarrow \infty} \langle \mathcal{F}[A_\mu(t, x)] \rangle_\eta , \quad (17.b)$$

where $A_\mu(t, x)$ satisfies the Langevin equation:

$$\partial_t A_\mu^a = -\delta S_{\text{eff}} / \delta A_\mu^a + \eta_\mu^a , \quad (18)$$

$$S_{\text{eff}}[A] = S_{YM}[A] - \ln \det[-i\not{D}^*(A)] . \quad (19)$$

Clearly, due to (16), i.e., the gauge non-invariance of S_{eff} (19), the r.h.s. of (17.a) acquires the well-known uncertainty $0/0^6$, whereas (18) is gauge-covariant and, therefore, the SQS average (17.b) does not detect the global chiral anomaly (16) at finite stochastic time [further details will appear in a next note⁷⁾].

Next, as discussed in the second Ref. d) SQS yields both a gauge- and parity-covariant expression for the induced massless fermion current in odd D:

$$J_\mu^a(x) = \int_0^\infty d\tau t_2 \left[\gamma_\mu^a \not{D}(A) e^{-\tau \not{D}^2(A)}(x, x) \right] = (2i)^{-1} \delta / \delta A_\mu^a(x) \ln \det[\not{D}^2(A)] .$$

Hence, SQS fails to reproduce the corresponding parity-violating anomalies in odd D⁸⁾ [when the latter cannot be cancelled by appropriate counterterms⁹⁾]:

$$J_\mu^a(x) = -i \delta / \delta A_\mu^a(x) \left\{ \frac{1}{2} \ln \det[\not{D}^2(A)] - i\pi/2 \eta_\not{D}[A] \right\} ,$$

where $\eta_\not{D}[A]$ is the parity-odd spectral asymmetry measuring η -invariant¹⁰⁾ of $\not{D}(A)$. This fact poses serious restrictions on the applicability of SQS to fermions in odd D.

The reason for the failure just found of SQS to reproduce anomalies of discrete symmetries is easy to understand. SR preserves these discrete symmetries. However, unlike the case of anomalies of continuous symmetries, there are no (modified by SQS) Ward identities for the corresponding discrete symmetries, so that the mechanism for arising of the standard chiral anomalies, described in Section 2, does not operate here.

ACKNOWLEDGEMENTS

We are very indebted to A.M. Polyakov for illuminating discussions concerning Section 4.

REFERENCES

- 1) J. Breit, S. Gupta and A. Zaks, Nucl. Phys. B233 (1984) 61.
- 2) J. Alfaro and M.B. Gavela, Phys. Lett. 158B (1985) 473.
- 3) E.S. Egorian, E.R. Nissimov and S.J. Pacheva, Sofia preprint INRNE-Nov-1984; Lett. Math. Phys. 11 (1986), no. 2/3; Sofia preprint INRNE-Feb-1985 (to appear in Theor. Math. Phys.).
- 4) V.N. Romanov and A.S. Schwarz, Theor. Math. Phys. 41 (1979) 190.
- 5) L. Alvarez-Gaume and E. Witten, Nucl. Phys. B234 (1984) 269.
- 6) E. Witten, Phys. Lett. 117B (1982) 324.
- 7) E.R. Nissimov and S.J. Pacheva, CERN preprint TH-4374 (1986) (to appear in Phys. Lett. B).
- 8) A.M. Polyakov, unpublished (see Ref. 9);
J. Lott, Phys. Lett. 145B (1984) 179;
L. Alvarez-Gaume, S. Della Pietra and G. Moore, Ann. Phys. 163 (1985) 288.
- 9) E.R. Nissimov and S.J. Pacheva, Phys. Lett. 155B (1985) 76; 157B (1985) 407.
- 10) M. Atiyah, V. Patodi and I. Singer, Math. Proc. Camb. Phil. Soc. 77 (1975) 43; 78 (1975) 405; 79 (1976) 71.