d = 11 SUPERGRAVITY WITH LOCAL SU(8) INVARIANCE

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A B S T R A C T

The transformation rules and field equations of $d = 11$ supergravity are given in a form which is manifestly covariant under local $SU(8)$ transformations. In this formulation the $SO(1,10)$ local Lorentz group is replaced by $SO(1,3) \times SU(8)$. The spinless fields are described by covariant quantities which constitute the 56 and 133 representations of $E_{7(+7)}$.

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1. - INTRODUCTION

One of the most remarkable features of extended supergravity theories in four dimensions is the appearance of "hidden symmetries". A celebrated example is $N=8$ supergravity whose equations of motion possess a nonlinearly realized E_7 symmetry and whose Lagrangian is invariant under local chiral $SU(8)$ transformations [1]. Although this theory was originally constructed from $N=1$, $d=11$ supergravity [2] by a torus reduction, the origin of these symmetries has so far not been completely understood in the framework of higher dimensions. We have recently shown that $d=11$ supergravity also possesses "hidden symmetries" [3]: it is possible to rewrite all the transformation laws and field equations in a form that is manifestly covariant under local $SU(8)$ in eleven dimensions. The fermions transform as representations of $SU(8)$ which are chiral in the four-dimensional sense notwithstanding the fact that the $SU(8)$ group is local in all eleven coordinates. The bosonic quantities pertaining to the spin-zero fields, which include the $SU(8)$ connections, constitute representations of the group E_7 . Although the new version of $d=11$ supergravity is still physically equivalent to the original one of ref. [2] (at least at the classical level), its very existence strongly points towards the emergence of a new geometrical structure which transcends the framework of Riemannian geometry. It is the purpose of this paper to explain the principles underlying this construction and to give a detailed derivation of the main results, some of which were already reported in [3]. Our work has also important implications concerning the nonlinear aspects of spontaneously compactified $d=11$ supergravity. In a subsequent paper [4], we will employ the formalism developed here to establish the consistency of the S^7 -truncation to all orders. Although our results have been inspired to a large extent by the comparison of $d=11$ supergravity

with (gauged) $N=8$ supergravity, we will attempt to make the presentation as self-contained as possible and avoid reference to $d=4$ results in this paper.

The main strategy for obtaining the new version of $d=11$ supergravity has already been outlined in [5]. The construction is based on a $11 = 4 + 7$ split of the coordinates and indices. The introduction of new symmetries is first accomplished on the fermionic fields which transform as spinors under the tangent space group $SO(1,10)$ in the original formulation. The central idea is to first restrict this group to $SO(1,3) \times SO(7)$ by a partial gauge choice and then to enlarge it to $SO(1,3) \times SU(8)$ by the introduction of new gauge degrees of freedom. It is important that, in this procedure, all physical degrees of freedom of the $d=11$ theory are retained. The identification of the proper bosonic $SU(8)$ quantities is more difficult since it necessitates a complete reformulation of the bosonic sector. Somewhat surprisingly it turns out that some of the bosonic quantities can be compactly assembled into E_7 representations; in particular the siebenbeine e_a^m are absorbed into a "fünfundsechzigbein" (e_{AB}^m, e^{mAB}) which transforms covariantly under E_7 . Finally, it turns out that the $SU(8)$ covariant bosonic equations of motion mix Bianchi identities with genuine equations of motion when analyzed in terms of the original $d=11$ formulation.

The organization of this paper is as follows. In section 2, we review the various redefinitions necessary for the construction in more detail than has been done before; we stress that their necessity is evident from the analysis of the transformation rules performed in section 3, and that we need not refer to the action. Section 4 is devoted to a discussion of the supersymmetry algebra and general coordinate transformations which

take a rather unusual form in the present setting. The integrability conditions, which arise in the SU(8) covariant formulation, are treated in section 5, where we also define the covariant SU(8) field strengths. In section 6, we derive the SU(8)-invariant version of the fermionic lagrangian directly from the original d=11 lagrangian. These results will be needed in section 7, where we give the SU(8) covariant equations of motion. The bosonic equations are derived first by varying the fermionic ones; subsequently they are compared to the original d=11 ones (which are quoted below). Because the two versions of d=11 supergravity are only equivalent at the level of the equations of motion the duality rotations performed in [1] are implicitly contained in our treatment. We have not attempted to construct an action from which the SU(8) covariant equations given in that section would follow, and the action principle underlying the new version remains to be worked out. In the final section, we discuss possible extensions of our results to other models and mention some unresolved problems. Some useful formulas are collected in the appendix.

To conclude this introduction, we briefly recall the main results of [2] in our notation and conventions. The transformation laws are given by

$$\delta E_M^A = \frac{1}{2} \bar{\epsilon} \tilde{\Gamma}^A \Psi_M \quad (1.1)$$

$$\delta A_{MNP} = - \frac{\sqrt{2}}{8} \bar{\epsilon} \tilde{\Gamma}_{[MN} \Psi_{P]} \quad (1.2)$$

$$\delta \Psi_M = D_M \epsilon + \frac{\sqrt{2}}{288} (\tilde{\Gamma}_M^{ABCD} - 8 E_M^A \tilde{\Gamma}^{BCD}) \epsilon_{FABCD} \quad (1.3)$$

where $\tilde{\Gamma}$ are d=11 Γ -matrices which obey

$$\{\tilde{\Gamma}^A, \tilde{\Gamma}^B\} = 2\delta^{AB}, \quad \Gamma^{A_1 \dots A_{11}} = -i\epsilon^{A_1 \dots A_{11}} \mathbb{1}. \quad (1.4)$$

In (1.1-3) and in the remainder of this paper, we consistently ignore higher-order fermionic terms. The lagrangian of d=11 supergravity reads

$$\begin{aligned} L = & -\frac{1}{2} E R - \frac{1}{2} E \bar{\Psi}_M \tilde{\Gamma}^{MNP} D_N \Psi_P - \frac{1}{48} E F_{MNPQ}^2 \\ & - \frac{1}{(12)^3} \frac{1}{\sqrt{2}} \epsilon^{MNPQRSTUVWXYZ} F_{MNPQ} F_{RSTU} A_{VWX} - \\ & - \frac{\sqrt{2}}{192} E F_{MNPQ} (\bar{\Psi}_R \tilde{\Gamma}^{MNPQRS} \Psi_S + 12 \bar{\Psi}^M \tilde{\Gamma}^{NP} \Psi_Q) \end{aligned} \quad (1.5)$$

The bosonic field equations, which follow from (1.5), are, up to fermionic terms,

$$R_{MN} = \frac{1}{72} g_{MN} F^2 - \frac{1}{6} F_{MPQR} F_N{}^{PQR} \quad (1.6)$$

$$E^{-1} \partial_M (E F^{MNPQ}) = \frac{i\sqrt{2}}{1152} \epsilon^{NPQRSTUVWXYZ} F_{RSTU} F_{VWXY}. \quad (1.7)$$

In contrast to the lagrangian, the field equations only contain the invariant field strength F_{MNPQ} , which is subject to the Bianchi identity

$$\partial_{[M} F_{NPQR]} = 0. \quad (1.8)$$

2. - FIELD REDEFINITIONS

The derivation of the manifestly SU(8) covariant form of d=11 supergravity is based on a 4+7 split of the d=11 dimensional coordinates z^M . Subsequently one performs a number of field redefinitions, which

coincide to a large extent with the standard ones that are needed in order to rediagonalize the kinetic terms after dimensional reduction of a supergravity theory. However, in the context of our work those redefinitions arise because we insist that the supersymmetry transformations take an $SU(8)$ covariant form. The resulting field representation is thus found without using the equations of motion (for instance, for performing duality transformations) or restricting the dependence on the $d=11$ coordinates. Not surprisingly, the new formulation makes it easier to make contact with the usual $d=4$ version of $N=8$ supergravity, which also exhibits $SU(8)$ symmetry.

In this section, we will describe the required redefinitions in some detail and collect several formulas that will be needed later. Since we intend to rewrite $d=11$ supergravity on the basis of a $4+7$ split of the coordinates and indices, we distinguish between curved and flat $d=4$ indices $\mu, \nu, \dots$ and $\alpha, \beta, \dots$, respectively, and curved and flat $d=7$ indices $m, n, \dots$ and $a, b, \dots$, respectively. Similarly, $d=11$ coordinates are written as $z^M = (x^\mu, y^m)$. This decomposition is also appropriate for describing spontaneous compactifications of the $d=11$ space time to four dimensions according to

$$M_{11} \rightarrow M_4 \times M_7 \quad (2.1)$$

From this perspective, it is desirable to eliminate the transformations which mix M_4 and M_7 , and we therefore make partial use of the local $SO(1,10)$ invariance of the $d=11$ theory to fix a gauge where the elfbein takes the form

$$E_M^A = \begin{pmatrix} e_\mu^\alpha & B_\mu^m e_m^a \\ 0 & e_m^a \end{pmatrix} . \quad (2.2)$$

It is straightforward to verify that the inverse elfbein is then given by

$$E_A^M = \begin{pmatrix} e_\alpha^\mu & -e_\alpha^\mu B_\mu^m \\ 0 & e_a^m \end{pmatrix} . \quad (2.3)$$

In this gauge, the local $SO(1,10)$ invariance is reduced to local $SO(1,3) \times SO(7)$. The gauge choice (2.2) is not maintained under the supersymmetry transformations (1.1-3), and we must therefore add a compensating $SO(1,10)$ rotation which restores the gauge (2.2). The compensating (infinitesimal) rotation, characterized by an antisymmetric matrix Ω_{AB} , is determined by requiring

$$\delta_{\text{SUSY}} E_m^\alpha + E_m^A \Omega_A^\alpha = 0 . \quad (2.4)$$

Substituting the supersymmetry variation (1.1) in (2.4) gives

$$\Omega_{a\alpha} = -\Omega_{\alpha a} = -\frac{1}{2} \epsilon \tilde{\Gamma}_\alpha \Psi_a . \quad (2.5)$$

We first discuss the redefinitions of the bosonic fields. One starts by Weyl-rescaling the vierbein in (2.2) according to

$$e_\mu^\alpha = \Delta^{-\frac{1}{2}} e_\mu'^\alpha , \quad (2.6)$$

where

$$\Delta(x,y) \equiv \det e_m^a(x,y) . \quad (2.7)$$

If one is interested in a nontrivial background M_7 , it is usually convenient to perform the Weyl-rescaling not with respect to the full siebenbein determinant (2.7); rather, we write

$$\begin{aligned} e_m^a(x,y) &= e_m^{\circ b}(y) S_b^a(x,y) \\ e_a^m(x,y) &= e_b^{\circ m}(y) S_a^{-1 b}(x,y) \end{aligned} \quad (2.8)$$

where $e_m^{\circ a}(y)$ is an orthonormal frame on M_7 , and define the Weyl rescaling with respect to the factor*)

$$\Delta(x,y) \equiv \det S_a^b(x,y) \quad (2.9)$$

We also need to know the various components of the spin connection after the gauge choice (2.2) and the Weyl rescaling (2.6). The d=11 spin connection is given by the well-known formula

$$\omega_M^{AB} = \frac{1}{2} E_M^C (\Omega_{ABC} - \Omega_{BCA} - \Omega_{CAB}) \quad (2.10)$$

with the anholonomy coefficients

$$\Omega_{AB}^C \equiv 2 E_{[A}^M E_{B]}^N \partial_{MN}^C \quad (2.11)$$

In the gauge (2.2), we obtain, using (2.3),

*) The substitution of the Weyl rescaled vierbein (2.6) into the d=11 action yields the standard Einstein action in d=4 (with the Riemann tensor defined in terms of e_μ^{α}). The gravitational coupling constants in d=4 and d=11 are then related by

$$\left(\frac{1}{\kappa^2}\right)_{d=4} = \left(\frac{1}{\kappa^2}\right)_{d=11} \int_{M_7} d^7 y \det e_m^{\circ a}(y)$$

$$\Omega_{\alpha\beta}^{\gamma} = 2e_{[\alpha}^{\mu} e_{\beta]}^{\nu} D_{\mu} e_{\nu}^{\gamma},$$

$$\Omega_{\alpha\beta}^c = 2e_{[\alpha}^{\mu} e_{\beta]}^{\nu} D_{\mu} B_{\nu}^m e_m^c,$$

$$\Omega_{a\beta}^{\gamma} = e_a^m e_{\beta}^{\nu} \partial_m e_{\nu}^{\gamma},$$

$$\Omega_{ab}^{\gamma} = 0$$

$$\Omega_{a\beta}^c = e_a^m e_{\beta}^{\nu} (\partial_m B_{\nu}^n e_n^c - D_{\nu} e_m^c),$$

$$\Omega_{ab}^c = 2e_{[a}^m e_{b]}^n \partial_m e_n^c, \quad (2.12)$$

where we have defined

$$D_{\mu} \equiv e_{\mu}^{\alpha} \partial_{\alpha} = e_{\mu}^{\alpha} E_{\alpha}^M \partial_M = \partial_{\mu} - B_{\mu}^m \partial_m. \quad (2.13)$$

For a nontrivial background it is straightforward to evaluate the corresponding expressions. To write them in a systematic form we first note that the decomposition (2.8) is invariant under y -dependent frame rotations

$$\delta e_m^a(y) = \varepsilon^{ab}(y) e_m^b(y),$$

$$\delta S_b^a(x,y) = \varepsilon_{bc}(y) S_c^a(x,y), \quad (2.14)$$

$$\delta S_a^{-1b}(x,y) = \varepsilon^{bc}(y) S_a^{-1c}(x,y).$$

As the anholonomy coefficients are invariant under (2.14) their expansion about a nontrivial background leads to the covariant background

derivative

$$\overset{\circ}{D}_m S_b^a = \partial_m S_b^a - \overset{\circ}{\omega}_{mbc} S_c^a \quad (2.15)$$

where

$$\overset{\circ}{\omega}_m^{ab} = \frac{1}{2} \overset{\circ}{e}_m^c (\overset{\circ}{\Omega}_{abc} - \overset{\circ}{\Omega}_{bca} - \overset{\circ}{\Omega}_{cab}). \quad (2.16)$$

In addition S_a^b transforms under the x- and y-dependent $SO(7)$ subgroup of the d=11 Lorentz group, which requires the full $\omega_m^{ab}(x,y)$ in the definition of the covariant derivative. When working in a nontrivial background it is convenient to decompose ω_m^{ab} into $\overset{\circ}{\omega}_m^{ab}$ and a remainder, so we will extend the covariant derivative (2.15) to

$$\overset{\circ}{D}_m S_b^a = \partial_m S_b^a - \overset{\circ}{\omega}_{mbc} S_c^a - \overset{\circ}{\omega}_m^{ac} S_b^c. \quad (2.17)$$

This definition will be used henceforth. Using (2.17) and (2.6) we find the following expressions for the anholonomy coefficients,

$$\begin{aligned} \Omega_{\alpha\beta}^\gamma &= \Delta^{\frac{1}{2}} \Omega_{\alpha\beta}^{\prime\gamma} - e_{[\alpha}^{\mu} \delta_{\beta]}^{\gamma} \Delta^{-\frac{1}{2}} D_{\mu} \Delta, \\ \Omega_{\alpha\beta}^c &= \Delta \Omega_{\alpha\beta}^{\prime b} S_b^c, \\ \Omega_{a\beta}^\gamma &= S_a^{-1} b_a (\Omega_{b\beta}^{\prime\gamma} - \frac{1}{2} \delta_{\beta}^{\gamma} \overset{\circ}{e}_b^m \Delta^{-1} \partial_m \Delta), \\ \Omega_{ab}^\gamma &= 0, \\ \Omega_{a\beta}^c &= \Delta^{\frac{1}{2}} S_a^{-1} b_a (\overset{\circ}{e}_b^m D_m B_{\mu}^n e_n^d S_d^c - D_{\mu} S_b^c) e_{\beta}^{\mu} + \Delta^{\frac{1}{2}} e_{\beta}^{\mu} B_{\mu}^m \overset{\circ}{\omega}_m^{ca}, \\ \Omega_{ab}^c &= -2 S_{[a}^{-1} d_{a} S_{b]}^{-1} e_{b}^e D_m S_d^c \overset{\circ}{e}_e^m + \overset{\circ}{\Omega}_{ab}^c, \end{aligned} \quad (2.18)$$

where the Ω' are to be computed with the rescaled vierbein e'_μ and the background siebenbein $\overset{\circ}{e}_m^a$, and the definition (2.13) is replaced by

$$D_\mu \equiv \partial_\mu - B_\mu^m \overset{\circ}{D}_m \quad (2.19)$$

Henceforth it will be understood that D_μ is either (2.13) or (2.19) depending on whether one considers a flat or curved background. Note that in $\overset{\circ}{D}_m B_\mu^n$ the background covariant derivative contains the Christoffel connection associated with the background metric.

All fermionic fields originate from the d=11 gravitation decomposed according to

$$\Psi_M(x,y) \rightarrow \begin{cases} \Psi_\mu(x,y) \\ \Psi_m(x,y) \end{cases} \quad (2.20)$$

where each d=11 spinor has 32 components. Accordingly the d=11 gamma matrices (denoted by $\tilde{\Gamma}$; see (1.4)) are represented as follows [1,2]

$$\tilde{\Gamma}_\alpha = \gamma_\alpha \otimes \mathbb{1} \quad \tilde{\Gamma}_a = \gamma_5 \otimes \Gamma_a \quad (2.21)$$

where $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$, and

$$\begin{aligned} \{\gamma_a, \gamma_b\} &= 2\delta_{ab} \mathbb{1}, & \{\Gamma_a, \Gamma_b\} &= 2\delta_{ab} \mathbb{1}, \\ \Gamma[a\Gamma^b \dots \Gamma^g] &= -i\epsilon^{abcdefg} \mathbb{1} \end{aligned} \quad (2.22)$$

The last formula is consistent with (1.4).

Again we give the various field redefinitions without too much

motivation. First the fields (2.20) are reexpressed with flat $d=11$ indices; subsequently one takes linear combinations of Ψ_α and Ψ_a , performs a Weyl rescaling and a chiral redefinition to have the conventional $d=4$ parity assignments^{*}). All this is summarized in the formulae

$$\begin{aligned}\phi'_\mu &= e_\mu^\alpha \Delta^{-\frac{1}{4}} (i\gamma_5)^{-\frac{1}{2}} (\Psi_\alpha - \frac{1}{2} \gamma_5 \gamma_\alpha \Gamma^a \Psi_a) \\ \phi'_a &\equiv \Delta^{-\frac{1}{4}} (i\gamma_5)^{-\frac{1}{2}} \Psi_a\end{aligned}\tag{2.23}$$

The corresponding redefinition for the supersymmetry transformation parameter is

$$\varepsilon' \equiv \Delta^{\frac{1}{4}} (i\gamma_5)^{-\frac{1}{2}} \varepsilon .\tag{2.24}$$

An important observation is that the $SO(7)$ tangent space group for Majorana fermions can be extended to $SU(8)$ by introducing chiral transformations. This is so because one can construct 63 8×8 matrices, namely $(i\Gamma^a)_{AB}$, $(\Gamma^{ab})_{AB}$ and $\gamma_5 (\Gamma^{abc})_{AB}$, which generate the $SU(8)$ algebra and are consistent with the Majorana condition for spinors. Here and henceforth we have adopted a representation where the $d=7$ charge conjugation matrix equals the identity; A and B denote spin-7 indices, and we use a notation where

$$\Gamma^{a_1 \dots a_n} \equiv \frac{1}{n!} \sum_{\text{perm}} (-)^p \Gamma^{a_1} \Gamma^{a_2} \dots \Gamma^{a_n}\tag{2.25}$$

For this reason it is convenient to introduce chiral projections and

^{*}) These linear combinations coincide with those needed to diagonalize the spin-3/2 and -1/2 kinetic terms in the reduction from $d=11$ to $d=4$. Here they are motivated by $SU(8)$ covariance, as will be discussed in section 3.

promote spin-7 indices (at least for the fermions) to chiral SU(8) indices^{*}). Denoting positive (negative) chirality components with upper (lower) indices, we define

$$\begin{aligned}\psi_{\mu}^A &\equiv \frac{1}{2}(1+\gamma_5) \psi_{\mu A}^{\prime}, & \psi_{\mu A} &\equiv \frac{1}{2}(1-\gamma_5) \psi_{\mu A}^{\prime}, \\ \varepsilon^A &\equiv \frac{1}{2}(1+\gamma_5) \varepsilon_A^{\prime}, & \varepsilon_A &= \frac{1}{2}(1-\gamma_5) \varepsilon_A^{\prime}.\end{aligned}\tag{2.26}$$

Also the spin- $\frac{1}{2}$ fields $\psi_{aA}^{\prime}$ can be written in SU(8) covariant form by converting the d=7 vector index to spinor indices. It turns out that $\psi_{aA}^{\prime}$ is equivalent to an antisymmetric trispinor, and we define^{**})

$$\begin{aligned}\chi^{ABC} &\equiv (1+\gamma_5) \frac{3\sqrt{2}}{4} i \Gamma_{[AB}^a \psi_{aC]}^{\prime} \\ \chi_{ABC} &\equiv (1-\gamma_5) \frac{3\sqrt{2}}{4} i \Gamma_{[AB}^a \psi_{aC]}^{\prime}.\end{aligned}\tag{2.27}$$

Thus the spinors ψ_{μ}^A , ε^A and χ^{ABC} now may be assigned to the 8 and 56 representations of chiral SU(8).

The redefinitions discussed in this section may seem rather ad hoc, but as we shall see shortly they are essential in order to derive a full set of transformation rules covariant under chiral SU(8). We should repeat that we have not made any assumptions regarding the y-dependence of the fields or the nature of the dimensionally compactified vacuum configuration. So we are still dealing with the full d=11 supergravity theory.

^{*}) Chiral SU(8) indices must not be confused with SO(1,10) flat indices. In our discussion it should be clear from the context which is meant.

^{**}) Thus $\bar{\chi}_{ABC} = \frac{3\sqrt{2}}{4} i \bar{\psi}_{a[A}^{\prime} \Gamma_{BC]}^a (1-\gamma^5)$.

3. - TRANSFORMATION LAWS

In the previous section we have assigned the fermion fields to representations of chiral $SU(8)$. To obtain a similar result for the boson fields is a much harder task. In the original version of $d=11$ supergravity there seems no room for proper $SU(8)$ rotations, which should mix the fields originating from the elfbein E_M^A and the three-index gauge field A_{MNP} . Somehow these fields must be combined in new quantities on which $SU(8)$ can be covariantly realized. In the absence of a proper understanding of a possible underlying geometrical structure we will have to identify these quantities through a systematic analysis of the supersymmetry transformation laws.

There are different ways to proceed. The fermionic field redefinitions introduced in section 2 can be motivated by requiring that certain bosonic terms appearing in the fermionic supersymmetry transformation can be combined to $SU(8)$ representations. This was the procedure followed in [5]. Subsequently one can then compute the supersymmetry variation for bosonic fields and write them in $SU(8)$ covariant form. An alternative approach is to start from the variation of certain bosonic fields and identify the fermionic field redefinition by requiring $SU(8)$ covariance. The disadvantage of this construction is that the proper identification of the fermionic fields is postponed to a later stage, which makes the calculations very cumbersome. As it is not our intention to give a deductive derivation, we will simply start from the supersymmetry variations of certain boson fields, express them in terms of the $SU(8)$ covariant spinor fields defined in (2.26-27) and analyze the $SU(8)$ covariance.

For the supersymmetry transformation of the bosonic fields we must take into account the compensating $SO(1,10)$ rotation (2.5) to preserve the gauge condition (2.2) (for the fermions this rotation can be ignored as long as one is not interested in higher-order fermionic terms). In this way one easily derives for the spin-2 field

$$\begin{aligned}\delta e_{\mu}^{\alpha} &= \delta_{\text{SUSY}} e_{\mu}^{\alpha} + E_{\mu}^A \Omega_A^{\alpha} = \\ &= \frac{1}{2} \bar{\varepsilon} \gamma^{\alpha} \Psi^{\beta} e_{\mu\beta} + \frac{1}{2} \bar{\varepsilon} \gamma^{\alpha} \gamma_5 \gamma^{\beta} \Gamma_a^{\Psi} e_{\mu\beta} .\end{aligned}\quad (3.2)$$

Substituting the redefined quantities ϕ'_{μ} and ε' (cf. (2.23-24)) into this equation and discarding a local $SO(1,3)$ rotation proportional to $\bar{\varepsilon} \gamma_5 \gamma_{\alpha\beta} \Gamma_a^{\Psi}$, it is straightforward to show that

$$\delta e_{\mu}^{\alpha} = \delta(\Delta^{\frac{1}{2}} e_{\mu}^{\alpha}) = \frac{1}{2} \bar{\varepsilon}' \gamma^{\alpha} \phi'_{\mu} = \frac{1}{2} \bar{\varepsilon}' \gamma^{\alpha} \phi_{\mu A} + \text{h.c.} \quad (3.3)$$

For the spin-1 field, a similar analysis yields the result

$$\delta B_{\mu}^m = \delta(E_{\mu}^a e_a^m) = \frac{1}{2} i \Delta^{-\frac{1}{2}} e_a^m \{ \bar{\varepsilon}' \Gamma^a \phi'_{\mu} + \bar{\varepsilon}' \gamma_{\mu}^{\alpha} (\delta^{ab} + \frac{1}{2} \Gamma^a \Gamma^b) \phi'_{\alpha} \} \quad (3.4)$$

where, of course, $\gamma_{\mu}^{\alpha} \equiv e_{\mu}^{\alpha} \gamma_{\alpha}$. Now using (A.12) we get

$$\delta B_{\mu}^m = \frac{\sqrt{2}}{8} i \Delta^{-\frac{1}{2}} \Gamma_{AB}^m (2\sqrt{2} \bar{\varepsilon}' \phi_{\mu}^B + \bar{\varepsilon}' \gamma_{\mu}^{\alpha} \chi^{ABC}) + \text{h.c.} \quad (3.5)$$

where we have used the chiral notation for the fermions. Clearly B_{μ}^m must be inert under $SU(8)$, whereas the combination $2\sqrt{2} \bar{\varepsilon}' \phi_{\mu}^B + \bar{\varepsilon}' \gamma_{\mu}^{\alpha} \chi^{ABC}$ appearing on the right-hand side of (3.5) transforms according to the 28 representation of $SU(8)$. Therefore to have $SU(8)$ covariance of the transformation (3.5) one is forced to conclude that

$$e_{AB}^m \equiv i \Delta^{-\frac{1}{2}} e_a^m \Gamma_{AB}^a \quad (3.6)$$

is an SU(8) tensor. As it stands, however, this expression is not SU(8) covariant because e_{AB}^m is real. To circumvent this difficulty, we may view (3.6) as an SU(8) tensor taken in a special gauge. This gauge choice can be undone by redefining the spinor fields ψ_μ^A , χ^{ABC} and ε^A by means of a local (i.e. x- and y-independent) SU(8) transformation Φ_B^A . This induces the replacements $\psi_\mu^A \rightarrow \Phi_B^A \psi_\mu^B$, $\chi^{ABC} \rightarrow \Phi_D^A \Phi_E^B \Phi_F^C \chi^{DEF}$, $\varepsilon^A \rightarrow \Phi_B^A \varepsilon^B$ in (3.6), which then gives rise to a modification of e_{AB}^m , namely

$$e_{AB}^m \equiv i \Delta^{-\frac{1}{2}} e_a^m \Gamma_{CD}^a \Phi_A^C \Phi_B^D \quad (3.7)$$

Because of the degrees of freedom provided by Φ_B^A , e_m^{AB} can now transform covariantly under SU(8), so that (3.5) acquires the SU(8) covariant form

$$\delta B_\mu^m = \frac{\sqrt{2}}{8} e_{AB}^m (2\sqrt{2} \varepsilon^A \psi_\mu^B + \varepsilon_C \gamma_\mu^{ABC} \chi^{ABC}) + \text{h.c.} \quad (3.8)$$

In order to avoid the introduction of new degrees of freedom, we let Φ be subject to local SU(8) transformations

$$\Phi_B^A \rightarrow \Phi_C^A U^C_B, \quad (3.9)$$

so that by going to the special gauge $\Phi = 1$, we recover (3.6). After extracting Φ from the fermion fields and the supersymmetry parameter ε , these quantities and (3.7) will transform covariantly under the local SU(8) induced by (3.9) according to their index structure; in accordance with our conventions, the complex conjugate tensor of (3.7) has upper indices, i.e. $e^{mAB} \equiv (e_{AB}^m)^*$. The tensor e_{AB}^m and its complex conjugate replace the siebenbein e_a^m . Instead of the usual relation between vielbein

and metric, one now has the SU(8) covariant "Clifford property"

$$e_{AB}^m e^{nBC} + e_{AB}^n e^{mBC} = \frac{1}{4} \delta_A^C e_{BD}^m e^{nDB} \quad (3.10)$$

At this point the introduction of the "compensating" fields Φ_B^A and the local SU(8) may seem rather ad hoc. Its significance will become more obvious at a later stage, where we find a complete reformulation of d=11 supergravity with local SU(8) invariance, in which e_{AB}^m is one of the basic fields. A first indication that this approach will be successful follows from considering the supersymmetry transformation of e_{AB}^m (we want to express the theory in terms of e_{AB}^m and no longer in terms of the siebenbein e_m^a); it is given by

$$\delta e_{AB}^m = i \delta(\Delta^{-\frac{1}{2}} e_a^m \Gamma_{AB}^a) = -i \Delta^{-\frac{1}{2}} (\frac{1}{4} \bar{\epsilon} \gamma_5 \Gamma^b \Psi_b \Gamma_{AB}^m + \frac{1}{4} \bar{\epsilon} \gamma_5 \Gamma^m \Psi_a \Gamma_{AB}^a) \quad (3.11)$$

in terms of unredifined quantities ϵ and Ψ_a . On the other hand, a little algebra leads to

$$\begin{aligned} & -\sqrt{2}(\bar{\epsilon} [A \chi_{BCD}] + \frac{1}{24} \epsilon_{ABCDEFGH} \bar{\epsilon}^E \chi^{FGH}) e^{mCD} = \\ & = -\frac{1}{2} i \Delta^{-\frac{1}{2}} \{ 4 e^{am} \bar{\epsilon} [A (1-\gamma_5) \Psi_{aB}] - \bar{\epsilon} [A (1-\gamma_5) (\Gamma^a \Gamma^m \Psi_a)_B] \\ & \quad - (\bar{\epsilon} (1-\gamma_5) \Gamma^m \Gamma^a) [A \Psi_{aB}] + \bar{\epsilon} \gamma_5 \Gamma^m \Psi_a \Gamma_{AB}^a \\ & \quad + \frac{1}{2} \bar{\epsilon} (1+\gamma_5) \Gamma^a \Psi_a \Gamma_{AB}^m + 2(\bar{\epsilon} (1+\gamma_5)) \Gamma^{(a)} [A (\Gamma^m) \Psi_a)_B] \} \end{aligned} \quad (3.12)$$

where we have used (2.23), (2.24), (2.27) and (A.2). By means of the completeness relation (A.17), this can be cast into the form

$$\begin{aligned}
 (3.12) = & -i\Delta^{-\frac{1}{2}} \left\{ \frac{3}{8} \bar{\epsilon} \Gamma^{[ab} \Psi^m] \Gamma_{AB}^{ab} + \frac{1}{4} \bar{\epsilon} \gamma_5 \Gamma^{ab} \Psi_b \Gamma_{AB}^{ma} + \frac{1}{4} \bar{\epsilon} \gamma_5 \Gamma^a \Psi_a \Gamma_{AB}^m \right. \\
 & \left. + \frac{1}{4} (\bar{\epsilon} \gamma_5 \Gamma^m \Psi_a + \bar{\epsilon} \gamma_5 \Gamma_a \Psi^m) \Gamma_{AB}^a \right\} \quad (3.12')
 \end{aligned}$$

This does not yet coincide with (3.11) but the mismatch can be removed by a local SU(8) rotation

$$\delta_{\text{SU}(8)} e_{AB}^m = 2\Lambda^C_{[A} e_{B]C}^m \quad (3.13)$$

with parameter

$$\Lambda_A^B \equiv \frac{1}{8} \bar{\epsilon} \gamma_5 \Gamma_{ab} \Psi^b \Gamma_A^a - \frac{1}{8} \bar{\epsilon} \gamma_5 \Gamma_a \Psi_b \Gamma_{AB}^{ab} - \frac{1}{16} \bar{\epsilon} \Gamma_{ab} \Psi_c \Gamma_{AB}^{abc} . \quad (3.14)$$

Discarding this SU(8) rotation, we arrive at the manifestly SU(8) covariant result

$$\delta e_{AB}^m = -\sqrt{2} \Sigma_{ABCD} e^{mCD} , \quad (3.15)$$

where

$$\Sigma_{ABCD} \equiv \bar{\epsilon}_{[A} \chi_{BCD]} + \frac{1}{24} \epsilon_{ABCDEFGH} \bar{\epsilon}^E \chi^{FGH} \quad (3.16)$$

is complex selfdual.

In order to identify other SU(8) covariant bosonic quantities it is convenient to first evaluate the transformation laws for ϕ_μ^A and χ^{ABC} . The starting point for this analysis is the supersymmetry variation of the d=11 gravitino field written in tangent-space indices. From that one finds

$$\begin{aligned} \delta(e_\mu^\alpha \Psi_\alpha) &= (\partial_\mu - \frac{1}{4} \omega_\mu^{AB} \tilde{\Gamma}_{AB} - B_\mu^m \partial_m) \epsilon \\ &+ \frac{\sqrt{2}}{288} e_\mu^\alpha (\tilde{\Gamma}^{ABCD} - 8\delta_\alpha^A \tilde{\Gamma}^{BCD}) F_{ABCD} \epsilon \end{aligned} \quad (3.17)$$

$$\begin{aligned} \delta\Psi_a &= e_a^m (\partial_m - \frac{1}{4} \omega_m^{AB} \tilde{\Gamma}_{AB}) \epsilon \\ &+ \frac{\sqrt{2}}{288} (\tilde{\Gamma}_a^{ABCD} - 8\delta_a^A \tilde{\Gamma}^{BCD}) F_{ABCD} \epsilon \end{aligned} \quad (3.18)$$

(note that A, B, C, .. denote d=11 tangent space indices in (3.17-18))

where

$$\begin{aligned} \omega_\mu^{AB} &\equiv \frac{1}{2} e_\mu^\alpha (\Omega_{AB\alpha} - \Omega_{B\alpha A} - \Omega_{\alpha AB}) , \\ \omega_m^{AB} &\equiv \frac{1}{2} e_m^a (\Omega_{ABa} - \Omega_{BaA} - \Omega_{aAB}) . \end{aligned} \quad (3.19)$$

Note that ω_μ^{AB} does not coincide with ω_M^{AB} in (2.10) for $M=\mu$ while it does for $M=m$. The higher-order fermionic terms have been neglected throughout. After a little calculation, we obtain

$$\begin{aligned} \delta(e_\mu^\alpha (\Psi_\alpha - \frac{1}{2} \gamma_5 \gamma_\alpha \Gamma^a \Psi_a)) &= \\ &= (\partial_\mu - B_\mu^m \partial_m - \frac{1}{4} (\omega_\mu^{\alpha\beta} - e_\mu^\alpha \Omega_{\beta a}^a) \gamma_{\alpha\beta}) \epsilon \\ &+ e_\mu^\alpha \Omega_{\alpha a}^\beta \gamma_\beta \gamma_5 \Gamma^a \epsilon + \frac{1}{4} e_\mu^\alpha \Omega_{\alpha a}^a \epsilon \\ &+ (-\frac{1}{4} \omega_\mu^{ab} \Gamma_{ab} - \frac{\sqrt{2}}{24} e_\mu^\alpha F_{\alpha abc} \Gamma^{abc} \gamma_5 + \frac{\sqrt{2}}{24} e_\mu^\alpha \epsilon^{\alpha\beta\gamma\delta} F_{\beta\gamma\delta a} \Gamma^a) \epsilon \\ &+ e_\mu^\alpha \gamma_\beta \gamma_\alpha (\frac{1}{8} \gamma_5 \Gamma^a \omega_a^{\beta\gamma} + \frac{\sqrt{2}}{32} \Gamma_{ab} F^{\beta\gamma ab}) \epsilon \\ &+ e_\mu^\alpha \gamma_\alpha (-\frac{1}{8} \gamma_5 \omega_a^{bc} \Gamma^a \Gamma_{bc} - \frac{\sqrt{2}}{192} F_{abcd} \Gamma^{abcd} + \frac{\sqrt{2}}{24} \epsilon^{\beta\gamma\delta\epsilon} F_{\beta\gamma\delta\epsilon}) \epsilon , \end{aligned} \quad (3.20)$$

where the tensors F are the various components of the $d=11$ field strength F_{ABCD} written with flat indices, and we made use of $\Omega_{ab}^{\alpha} = 0$. At this point we discover why it is important that we have chosen the particular linear combination of the fermion fields in (2.23). Precisely for this choice, ω_{μ}^{ab} , $F_{\alpha abc}$ and $F_{\alpha\beta\gamma a}$ combine into the expression shown in (3.20), which is decomposed into the generators of chiral $SU(8)$, namely $i\Gamma^a$, Γ^{ab} and $\gamma_5\Gamma^{abc}$. Therefore this part of the transformation rule is covariant under chiral $SU(8)$ provided ω_{μ}^{ab} , $F_{\alpha abc}$ and $F_{\alpha\beta\gamma a}$ are assigned to the 63 (adjoint) representation of $SU(8)$. The same phenomenon takes place for the terms containing $\omega_a^{\alpha\beta}$ and $F_{\alpha\beta ab}$, which constitute the 28 representation of $SU(8)$. Of course, at this point we have not yet taken into account the Weyl rescaling, so that this conclusion is somewhat premature but the pattern is clear.

After Weyl rescaling and chiral redefinition, the result (3.20) takes the form (in chiral notation)

$$\begin{aligned}
 \delta\psi_{\mu}^A = & \left\{ \partial_{\mu} - B_{\mu}^m \overset{\circ}{D}_m - \frac{1}{4} \omega_{\mu}^{\alpha\beta} \gamma_{\alpha\beta} - \frac{1}{4} \gamma'_{\mu} \gamma'^{\nu} \overset{\circ}{D}_m B_{\nu}^m \right\} \varepsilon^A \\
 & + \frac{1}{2} B_{\mu B}^A \varepsilon^B + \gamma^{\alpha\beta} \gamma_{\mu} G_{\alpha\beta}^{-AB} \varepsilon_B \\
 & + i\Delta^{-\frac{1}{2}} \left\{ \frac{1}{2} \Gamma_{AB}^a S_a^{-1b} (\overset{\circ}{D}_b - \frac{1}{4} \Delta^{-1} \overset{\circ}{D}_b \Delta) + \frac{1}{8} \Gamma_{AB}^{abc} S_a^{-1d} (S^{-1} \overset{\circ}{D}_d S)_{bc} - \right. \\
 & \left. - \frac{1}{4} \Gamma_{AB}^a S_a^{-1c} (S^{-1} \overset{\circ}{D}_c S)_a^b + \frac{1}{8} \sqrt{2} \Gamma_{AB}^{abcd} F_{abcd} \right. \\
 & \left. + \frac{1}{8} \sqrt{2} f \delta_{AB} \right\} \gamma'_{\mu} \varepsilon_B
 \end{aligned} \tag{3.21}$$

where ω_{μ}^{ab} follows from (3.19) after replacing $\Omega_{\alpha\beta}^{\gamma}$ by $\Omega_{\alpha\beta}'^{\gamma}$ and $\gamma'_{\mu} = e_{\mu}^{\alpha} \gamma_{\alpha}$.

Note that the background derivative in the last bracket acts also on the

rescaled vierbein in γ'_μ . The new quantities in (3.21) are

$$f = -\frac{i}{24} \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta\gamma\delta} \quad (3.22)$$

$$\begin{aligned} B_{\mu B}^A = & -\frac{1}{2} \{ e_a^m D_m B_{\mu}^n e_{nb} - (S^{-1} D_\mu S)^{ab} \} \Gamma_{AB}^{ab} \\ & - \frac{\sqrt{2}}{12} e_\mu'^\alpha \Delta^{-\frac{1}{2}} F_{\alpha abc} \Gamma_{AB}^{abc} + \\ & + \frac{\sqrt{2}}{12} e_\mu'^\alpha \Delta^{-\frac{1}{2}} \epsilon^{\alpha\beta\gamma\delta} F_{\beta\gamma\delta a} \Gamma_{AB}^a \end{aligned} \quad (3.23)$$

$$\begin{aligned} G_{\alpha\beta AB}^+ = & \{ (-\frac{i}{16} \Delta^{\frac{1}{2}} \Omega_{\alpha\beta a} + \frac{i}{8} \Delta^{-\frac{1}{2}} e_a^m e_{[\alpha}^\mu \partial_m e_{\beta]}^\nu) \Gamma_{AB}^a \\ & + \frac{\sqrt{2}}{32} i \Delta^{-\frac{1}{2}} F_{\alpha\beta ab} \Gamma_{AB}^{ab} \}^+ \end{aligned} \quad (3.24)$$

where we have used the shorthand notation

$$X_{\alpha\beta}^\pm \equiv \frac{1}{2} X_{\alpha\beta} \pm \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} X_{\gamma\delta} \quad (3.25)$$

Note that $G_{\alpha\beta AB}^- \equiv (G_{\alpha\beta}^{+ AB})^*$ is selfdual in our convention. As selfduality or antiselfduality follows from the position of the indices AB, we will not always indicate the duality phase in subsequent formulae.

The important feature of (3.21), as alluded to above, is that the first term in parentheses and the terms proportional to $B_{\mu B}^A$ and $G_{\alpha\beta}^{+ AB}$ take an SU(8) covariant form, because $B_{\mu B}^A$ and $G_{\alpha\beta}^{+ AB}$ take their values in the 63 and 28 representation of SU(8). The term $B_{\mu B}^A$ thus belongs to the Lie algebra of SU(8) and has the structure of a connection field associated with local SU(8) (note that $B_{\mu B}^A$ is antihermitean: $(B_{\mu B}^A)^* = -B_{\mu A}^B$). What remains is to understand the SU(8) properties of the

last term in parenthesis. Before doing that let us first present the analogous result for the spin- $\frac{1}{2}$ fields. After similar steps as those leading to (3.21) we find

$$\begin{aligned} \delta \chi^{ABC} &= 3\sqrt{2} \gamma^{\alpha\beta} G_{\alpha\beta}^{-} [AB_{\epsilon} C] - \sqrt{2} \gamma^{\mu} A_{\mu}^{ABCD} \epsilon_D \\ &+ \Delta^{-\frac{1}{2}} \left\{ \frac{3}{\sqrt{2}} i \left[i \Gamma_{[AB}^a \delta_{C]D} S^{-1}{}_a^b (D_b^{\circ} - \frac{1}{4} \Delta^{-1} D_b^{\circ} \Delta) \right. \right. \\ &\quad + \frac{1}{4} \Gamma_{[AB}^a \Gamma_{C]D}^{bc} (S^{-1}{}_c^d (S^{-1}{}_d^{\circ} S)_{ba} + \\ &\quad \left. \left. + S^{-1}{}_a^d (S^{-1}{}_d^{\circ} S)_{bc} + S^{-1}{}_c^d (S^{-1}{}_d^{\circ} S)_{ab} \right) \right] \\ &\quad \left. - \frac{1}{4} f \Gamma_{[AB}^a \Gamma_{C]D}^a + \frac{1}{96} \Gamma_{[AB}^a (\Gamma_a^{bcde} - 8 \delta_a^b \Gamma^{cde})_{C]D} F_{bcde} \right\} \quad (3.26) \end{aligned}$$

where*)

$$\begin{aligned} A_{\mu}^{ABCD} &\equiv -\frac{\sqrt{2}}{48} \Delta^{-\frac{1}{2}} e_{\mu}^{\alpha} \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta\gamma\delta} \Gamma_{[AB}^b \Gamma_{CD]}^{ab} \\ &+ \frac{\sqrt{2}}{8} \Delta^{-\frac{1}{2}} e_{\mu}^{\alpha} F_{abc\alpha} \Gamma_{[AB}^a \Gamma_{CD]}^{bc} - \\ &+ \frac{3}{4} \{ e_a^m D_m^{\circ} B_{\mu}^n e_{nb} - (S^{-1}{}_D^{\circ} S)_{ab} \} \Gamma_{[AB}^a \Gamma_{CD]}^b. \quad (3.27) \end{aligned}$$

Here we note the same features as before: The first term contains the same SU(8) covariant tensor $G_{\alpha\beta}^{AB}$ as defined in (3.24); the second term yields the tensor A_{μ}^{ABCD} which is fully antisymmetric and selfdual in the indices ABCD:

$$A_{\mu}^{ABCD} = \frac{1}{24} \epsilon^{ABCDEFGH} A_{\mu EFGH} \quad (3.28)$$

*) We have redefined A_{μ} and A_m in [3] by a factor $(-4/3)$. Note also that the last term in (3.27) disagrees with the corresponding one in eq. (5.16) of ref. [1] even in the torus reduction.

This follows from the Γ -matrix identities given in the appendix. Hence A_{μ}^{ABCD} can be assigned to the 70 representation of $SU(8)$. Observe that our conventions are such that $A_{\mu ABCD} \equiv (A_{\mu}^{ABCD})^*$.

It remains to unravel the $SU(8)$ structure of the last terms in (3.21) and (3.26). In view of our earlier discussion on local $SU(8)$ one expects the appearance of a quantity B_{mB}^A assigned to the adjoint representation of $SU(8)$, which together with $B_{\mu B}^A$ can play the role of an $SU(8)$ connection field for x - and y -dependent $SU(8)$ transformations. In order to find an expression for B_{mB}^A we first return to the definitions of

$B_{\mu B}^A$ and A_{μ}^{ABCD} , and note that they are both expressed in terms of $D_{\mu} B_{\mu}^n$, $S^{-1} D_{\mu} S$, $F_{\alpha abc}$ and $F_{\alpha\beta\gamma a}$. Therefore, $B_{\mu B}^A$ and A_{μ}^{ABCD} are not completely independent, and this fact is expressed by an equation

$$D_{\mu} e_{AB}^m + \frac{1}{2} \partial_n B_{\mu}^n e_{AB}^m + \partial_n B_{\mu}^m e_{AB}^n + B_{\mu}^C [A_{\mu}^m e_{AB}^m]_C + A_{\mu ABCD} e^{mCD} = 0 \quad (3.29)$$

where e_{AB}^m is defined by (3.6). Provided that e_{AB}^m is a covariant $SU(8)$ tensor and $B_{\mu B}^A$ is the $SU(8)$ connection, (3.29), which generalizes the usual vielbein postulate of Riemannian geometry to complex geometry, is an $SU(8)$ covariant equation. However, as discussed before, (3.29) may be interpreted as a covariant equation taken in a special gauge. Hence (3.6), (3.23) and (3.27) define e_{AB}^m , $B_{\mu B}^A$, and A_{μ}^{ABCD} in a special gauge and to reinstate the extra $SU(8)$ gauge degrees of freedom one performs a compensating gauge transformation as in (3.7). Therefore (3.29) holds true for e_{AB}^m as defined by (3.7) provided that one replaces $B_{\mu B}^A$ and A_{μ}^{ABCD} by

$$B_{\mu B}^A \rightarrow \Phi_C^A (B_{\mu D}^C + 2\delta_D^C D_{\mu}^D) \Phi_B^D.$$

$$A_{\mu}^{ABCD} \rightarrow \Phi_E^A \Phi_F^B \Phi_G^C \Phi_H^D A_{\mu}^{EFGH}, \quad (3.30)$$

(since $\Phi_B^A \in \text{SU}(8)$ we have $\Phi_B^A \Phi_C^B = \delta_C^A$). Substituting the SU(8) transformation (3.9) into the new B_{μ} shows that it acts as an SU(8) gauge field

$$B_{\mu B}^A \rightarrow U_C^A (B_{\mu D}^C + 2\delta_D^C D_{\mu}^D) U_B^D \quad (3.31)$$

With these redefinitions (3.29) has thus become covariant under local SU(8) transformations. Surprisingly enough (3.29) is also covariant with respect to rigid $E_{7(+7)}$ transformations if e_{AB}^m and e^{mAB} are assigned to the 56-dimensional fundamental representation and $B_{\mu B}^A$ and A_{μ}^{ABCD} to the 133-dimensional adjoint representation of that gauge.

We now conjecture that there should be a d=7 dimensional counterpart of (3.29), so we write

$$D_m^n e_{AB}^n + B_m^C [A e_B^n]_C + A_{mABCD} e^{nCD} = 0. \quad (3.32)$$

The new quantities B_m and A_m are obviously related by this equation but not yet fully determined. We assume them to depend on $S^{-1} D_m S$, F_{abcd} and f , such that the corresponding terms in $\delta\phi_{\mu}^A$ and $\delta\chi^{ABC}$ can be expressed in an SU(8) covariant way in terms of e^m , B_m and A_m . As it turns out this determines uniquely the form of B_m and A_m , which is given by

$$B_{mB}^A = \frac{1}{2} (S^{-1} D_m S)_{ab} \Gamma_{AB}^{ab} + \frac{i\sqrt{2}}{14} f e_m^a \Gamma_{AB}^a - \frac{\sqrt{2}}{48} e_m^a F_{abcd} \Gamma_{AB}^{bcd}, \quad (3.33)$$

$$A_m^{ABCD} = -\frac{3}{4}(S^{-1} D_m S)_{ab} \Gamma_{[AB}^a \Gamma_{CD]}^b + \frac{i\sqrt{2}}{56} e_{ma} f \Gamma_{[AB}^{ab} \Gamma_{CD]}^b - \frac{\sqrt{2}}{32} e_m^a F_{abcd} \Gamma_{[AB}^b \Gamma_{CD]}^{cd} , \quad (3.34)$$

where A_m^{ABCD} is selfdual, i.e.

$$A_m^{ABCD} = \frac{1}{24} \epsilon^{ABCDEFGH} A_{mEFGH} \quad (3.35)$$

Again (3.32-34) should be viewed as the result in the special SU(8) gauge (3.6), but by exploiting the compensating SU(8) transformation Φ_B^A the result will become fully SU(8) covariant with B_{mB}^A playing the role of an SU(8) gauge potential. Furthermore (3.32) exhibits covariance under the aforementioned $E_7(+7)$.

Using the tensors B_{mB}^A and A_m^{ABCD} all terms in $\delta\psi_\mu^A$ and $\delta\chi^{ABC}$ can now be written in SU(8) covariant form. Introducing a modified spin connection

$$\hat{\omega}_\mu^{\alpha\beta} = \omega_\mu^{\alpha\beta} + e_\mu^{[\alpha} e^{,\nu\beta]} \overset{\circ}{D}_m B_{\nu}^m \quad (3.36)$$

and SU(8) x SO(1,3) covariant derivatives

$$D_\mu \epsilon^A \equiv D_\mu \epsilon^A - \frac{1}{4} \hat{\omega}_\mu^{\alpha\beta} \gamma_{\alpha\beta} \epsilon^A + \frac{1}{2} B_{\mu B}^A \epsilon^B$$

$$D_m \epsilon^A \equiv \overset{\circ}{D}_m \epsilon^A + \frac{1}{2} B_{mB}^A \epsilon^B \quad (3.37)$$

we can write the SU(8) covariant supersymmetry transformations as follows

$$\delta e_\mu^\alpha = \frac{1}{2} \bar{\epsilon} \gamma^\alpha \phi_{\mu A} + \text{h.c.} , \quad (3.38)$$

$$\begin{aligned} \delta \phi_{\mu}^A = & \left(D_{\mu}(\hat{\omega}, B) - \frac{1}{4} \dot{D}_m B_{\mu}^m \right) \varepsilon^A + \gamma^{\alpha\beta} \gamma_{\mu} G_{\alpha\beta}^{-AB} \varepsilon_B + \\ & + \frac{1}{2} e^{mAB} \dot{D}_m (\gamma_{\mu} \varepsilon_B) - \frac{1}{4} e_{CD}^m A_m^{ABCD} \gamma_{\mu} \varepsilon_B, \end{aligned} \quad (3.39)$$

$$\delta B_{\mu}^m = \frac{\sqrt{2}}{8} e_{AB}^m \left(2\sqrt{2} \varepsilon_{\mu}^A \psi^B + \varepsilon_C \gamma_{\mu} \chi^{ABC} \right) + \text{h.c.}, \quad (3.40)$$

$$\begin{aligned} \delta \chi^{ABC} = & 3\sqrt{2} G_{\alpha\beta}^{-[AB} \gamma^{\alpha\beta} \varepsilon^C] - \sqrt{2} \gamma^{\mu} A_{\mu}^{ABCD} \varepsilon_D, \\ & + \frac{3}{\sqrt{2}} e^{m[AB} \dot{D}_m \varepsilon^C] - \frac{3}{2\sqrt{2}} e_{DE}^m A_m^{DE[AB} \varepsilon^C] - \sqrt{2} A_m^{ABCD} e_{DE}^m \varepsilon^E, \end{aligned} \quad (3.41)$$

$$\delta e_{AB}^m = -\sqrt{2} \Sigma_{ABCD} e^{mCD}, \quad (3.42)$$

where we have now dropped all primes. Furthermore we note the identity

$$e^{mAB} G_{\alpha\beta AB}^{+} + e_{AB}^m G_{\alpha\beta}^{-AB} = -e_{[\alpha}^{\mu} e_{\beta]}^{\nu} D_{\mu} B_{\nu}^m + \frac{1}{8} e_{AB}^m e^{nAB} e_{[\alpha}^{\mu} \partial_n e_{\mu\beta]}, \quad (3.43)$$

which follows from the definitions above.

4. - COORDINATE TRANSFORMATIONS AND THE SUPERSYMMETRY ALGEBRA

In the previous section we have insisted on writing the supersymmetry transformations in an SU(8) covariant form. This forced us to express the fields of d=11 supergravity into new fields transforming as proper SU(8) tensors. However, the latter transform in an unusual way under reparametrizations of the extra coordinates y^m . Since these reparametrizations are part of the supersymmetry algebra, which is the main subject of this section, we first determine the effect of coordinate

transformations on the redefined fields that emerge from the elfbein E_M^A .

We start form the d=11 transformation

$$\delta E_M^A = \partial_M \xi^M E_N^A + \xi^M \partial_N E_M^A. \quad (4.1)$$

In the gauge (2.2) these transformations must be modified with a compensating $SO(1,10)$ transformations in order to preserve the gauge. This compensating solution is proportional to $\Omega_{\alpha a} = \partial_m \xi^\nu e_\nu^\alpha e_a^m$, where we decompose the parameters ξ^M according to $\xi^M(z) = (\xi^\mu(x,y), \xi^m(x,y))$. The next step is to covariantize the d=4 general coordinate transformation by including a d=7 transformation with $\xi^m = -B_\nu^m \xi^\nu$. One then finds the following results

$$\begin{aligned} \delta e_\mu^\alpha &= D_\mu \xi^\nu e_\nu^\alpha + \xi^\nu D_\nu e_\mu^\alpha + \xi^m \partial_m e_\mu^\alpha, \\ \delta B_\mu^m &= -2\xi^\nu D_{[\mu} B_{\nu]}^m - g_{\mu\nu} g^{mn} \partial_n \xi^\nu + D_\mu \xi^m + \xi^n \partial_n B_\mu^m, \\ \delta e_m^a &= \xi^\nu (D_\nu e_m^a - \partial_m B_\nu^n e_n^a) + \partial_m \xi^n e_n^a + \xi^n \partial_n e_m^a, \end{aligned} \quad (4.2)$$

where $g_{\mu\nu}$ and g^{mn} are the metrics formed from the vierbein e_μ^α and the siebenbein e_a^m in the usual manner, and D_μ is defined in (2.13). It is in principle straightforward to work out the transformation laws for B_μ^m , the rescaled vierbein e_μ^α and the quantity e_{AB}^m in a nontrivial background given by $e_m^a(y)$. In fact for B_μ^m one can readily check that (4.2) takes the same form, with derivatives ∂_m replaced by background derivatives $\overset{\circ}{D}_m$. However, to simplify matters we will restrict ourselves to a trivial background in this section. In terms of the fields introduced in the previous section the transformation laws then read

$$\begin{aligned}
\delta e_{\mu}^{\prime\alpha} &= D_{\mu} \xi^{\nu} e_{\nu}^{\prime\alpha} + \xi^{\nu} D_{\nu} e_{\mu}^{\prime\alpha} - \frac{1}{2} e_{\mu}^{\prime\alpha} (\xi^{\nu} \partial_{\nu} B_{\mu}^m - \partial_{\mu} \xi^m) + \xi^m \partial_{\mu} e_{\mu}^{\prime\alpha} \\
\delta B_{\mu}^m &= -2\xi^{\nu} D_{[\mu} B_{\nu]}^m - \frac{1}{8} (e_{AB}^m e^{nAB}) e_{\mu}^{\prime\alpha} e_{\nu}^{\prime\alpha} \partial_n \xi^{\nu} + D_{\mu} \xi^m + \xi^n \partial_n B_{\mu}^m \\
\delta e_{AB}^m &= \xi^{\nu} D_{\nu} e_{AB}^m + e_{AB}^n \xi^{\nu} \partial_n B_{\nu}^m + \xi^n \partial_n e_{AB}^m - e_{AB}^n \partial_n \xi^m \\
&\quad + \frac{1}{2} e_{AB}^m (\xi^{\nu} \partial_n B_{\nu}^n - \partial_n \xi^n)
\end{aligned} \tag{4.3}$$

We now turn to the commutator of two supersymmetry transformations acting on $e_{\mu}^{\prime\alpha}$, B_{μ}^m and e_{AB}^m . The starting point of this calculation are the transformation laws (3.36) evaluated in a trivial background. Just as in the previous section we consistently ignore higher-order spinor terms. For $e_{\mu}^{\prime\alpha}$, a straightforward calculation gives

$$\begin{aligned}
(\delta_1 \delta_2 - \delta_2 \delta_1) e_{\mu}^{\prime\alpha} &= D_{\mu} (\xi^{\nu} e_{\nu}^{\prime\alpha}) - \frac{1}{2} \partial_m B_{\mu}^m \xi^{\nu} e_{\nu}^{\prime\alpha} \\
&\quad + \frac{1}{2} e_{\mu}^{\prime\alpha} \partial_m \xi^m + \xi^m \partial_m e_{\mu}^{\prime\alpha} + \varepsilon^{\alpha\beta} e_{\mu}^{\prime\beta},
\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
\xi^{\mu} &= \frac{1}{2} e_{\alpha}^{\prime\mu} \bar{\varepsilon}_{2A}^{\alpha} \gamma^A \varepsilon_1^A + \text{h.c.} \\
\xi^m &= \frac{1}{2} e_{AB}^m \bar{\varepsilon}_2^A \varepsilon_1^B + \text{h.c.} \\
\varepsilon_{\alpha\beta} &= \{ \bar{\varepsilon}_2^A \gamma_{[\alpha} \gamma^{\gamma\delta} \gamma_{\beta]} \varepsilon_1^B G_{\gamma\delta AB}^+ + \frac{1}{2} e_{AB}^m (\bar{\varepsilon}_2^A \gamma_{\alpha\beta} \bar{D}_m \varepsilon_1^B) \} + \text{h.c.}
\end{aligned} \tag{4.5}$$

and we have made use of (3.32). Because of the identity

$$D_{[\mu} e_{\nu]}^{\prime\alpha} = -\frac{1}{2} e_{[\mu}^{\prime\alpha} \overset{\circ}{D}_{\nu]} B_{\nu]}^m, \tag{4.6}$$

which follows directly from the definition (3.37) of D_μ , the first term in (4.3) can be written as

$$\begin{aligned} D_\mu (\xi^\nu e_\nu^\alpha) - \omega_\mu^{\alpha\beta} \xi^\nu e_\nu^\beta &= D_\mu \xi^\nu e_\nu^\alpha + \xi^\nu D_\nu e_\mu^\alpha - (\xi^\nu \omega_\nu^{\alpha\beta}) e_\mu^\beta \\ &\quad - \frac{1}{2} e_\mu^\alpha \partial_m B_m^\nu \xi^\nu + \frac{1}{2} e_\nu^\alpha \partial_m B_m^\nu \xi^\nu \end{aligned} \quad (4.7)$$

Comparison with (4.3) shows that the commutator of two supersymmetry transformations on e_μ^α takes the form of a general coordinate transformation with parameters ξ^μ and ξ^m given by (4.4) and a local (d=4) Lorentz transformation with parameter $\xi^\nu \hat{\omega}_\nu^{\alpha\beta} + \varepsilon^{\alpha\beta}$, where $\varepsilon^{\alpha\beta}$ is also given in (4.3).

The calculation of the commutator in B_μ^m is more involved. After some recombination of terms one finds

$$\begin{aligned} (\delta_1 \delta_2 - \delta_2 \delta_1) B_\mu^m &= \frac{1}{2} (e_{AB}^m D_\mu (\bar{\varepsilon}_2^A \varepsilon_1^B) + \text{h.c.}) \\ &\quad + \frac{1}{2} (e_{AB}^m B_{\mu C}^A \bar{\varepsilon}_2^C \varepsilon_1^B + \text{h.c.}) \\ &\quad - \frac{1}{2} \partial_n B_\mu^n (e_{AB}^m \bar{\varepsilon}_2^A \varepsilon_1^B + \text{h.c.}) \\ &\quad + \frac{3}{2} \sqrt{2} (\bar{\varepsilon}_2^A \varepsilon_{1B}^A A_{\mu}^{ABCD} e_{CD}^m + \text{h.c.}) \\ &\quad + (\bar{\varepsilon}_2^C \gamma^\alpha \varepsilon_{1C} + \text{h.c.}) (e_{AB}^m G_{\mu\alpha}^{AB} + e^{mAB} G_{\mu\alpha AB}) \\ &\quad + \frac{1}{16} (e_{AB}^m e^{nAB}) (e_\mu^\alpha \hat{\partial}_n (\bar{\varepsilon}_2^C \gamma^\alpha \varepsilon_1^C) + \text{h.c.}) \end{aligned} \quad (4.8)$$

where we have used (3.10) and (3.32) to cancel certain terms proportional to G and A_m . From (3.29) one easily verifies that the first four terms in (4.7) can be written as $D_\mu \xi^m + \partial_n B_\mu^m \xi^n$, with ξ^m as defined in (4.4). The last two terms in (4.7) can be written as

$$2\xi^\nu (e_{AB}^m G_{\mu\nu}^{AB} + e^{mAB} G_{\mu\nu AB}) + \frac{1}{8}(e_{AB}^m e^{nAB}) e_\mu^{\prime\alpha} e_\nu^{\prime\alpha} \partial_n \xi^\nu$$

$$+ \frac{1}{4}(e_{AB}^m e^{nAB}) \xi^\nu e_{[\mu}^{\prime\alpha} \partial_n e_{\nu]}^{\prime\alpha}, \quad (4.9)$$

with ξ^μ as defined in (4.4) and $G_{\mu\nu} \equiv e_\mu^{\prime\alpha} e_\nu^{\prime\beta} G_{\alpha\beta}$. By virtue of (3.43) the vierbein derivatives in (4.8) cancel, and we are left with

$$(\delta_1 \delta_2 - \delta_2 \delta_1) B_\mu^m = -2\xi^\nu D_{[\mu} B_{\nu]}^m + \frac{1}{8}(e_{AB}^m e^{nAB}) e_\mu^{\prime\alpha} e_\nu^{\prime\alpha} \partial_n \xi^\nu$$

$$+ D_\mu \xi^m + \partial_n B_\mu^m \xi^n \quad (4.10)$$

which is the same general coordinate transformation that appeared in the commutator on $e_\mu^{\prime\alpha}$ (cf. (4.3)).

New features are present in the commutator in e_{AB}^m . Using that A_μ^{ABCD} and $e_{[AB}^m e_{CD]}^n$ are selfdual in ABCD (cf. (3.28) and appendix), one first derives

$$(\delta_1 \delta_2 - \delta_2 \delta_1) e_{AB}^m = e^{mCD} A_{\mu ABCD} \xi^\mu$$

$$- 3e^{mCD} e_{[AB}^n D_n (\bar{\epsilon}_{2C} \epsilon_{1D]})$$

$$- 3e_{[AB}^m e_{CD]}^n D_n (\bar{\epsilon}_2^C \epsilon_1^D)$$

$$\begin{aligned}
& + 3 e^{mCD} \{ e^{nEF} A_{nEF} [AB \bar{\epsilon}_{2C} \epsilon_{1D}] \\
& + \frac{1}{24} \epsilon_{ABCDGHIJ} e_{EF}^n A_n^{EFGH} \bar{\epsilon}_2^I \epsilon_1^J \} \\
& + 4 e^{mCD} \{ e^{nEF} A_{nE} [ABC \bar{\epsilon}_{2D}] \epsilon_{1F} \\
& + \frac{1}{24} \epsilon_{ABCDGHIJ} \epsilon_{EF}^n A_n^{EGHI} \bar{\epsilon}_2^J \epsilon_1^F \} \quad (4.11)
\end{aligned}$$

with ξ^μ as defined in (4.4). Using the selfduality of A_m^{ABCD} we can rewrite the above result such that no explicit ϵ -tensors appear. After a tedious calculation, exploiting properties of products of selfdual tensors, this result can be cast in the form

$$\begin{aligned}
(\delta_1 \delta_2 - \delta_2 \delta_1) e_{AB}^m &= e^{mCD} A_{\mu ABCD} \xi^\mu \\
&+ e_{AB}^n \partial_n \xi^m + \frac{1}{2} e_{AB}^m \partial_n \xi^n - \xi^n \partial_n e_{AB}^m \\
&- 2 \Lambda_{[A}^C e_{B]C}^m, \quad (4.12)
\end{aligned}$$

with ξ^μ and ξ^m as given in (4.4), and Λ_A^B characterizing an infinitesimal SU(8) transformation

$$\Lambda_A^B = -e_{AC}^n (D_n (\bar{\epsilon}_2^B \epsilon_1^C) - A_n^{BCDE} \bar{\epsilon}_{2D} \epsilon_{1E}) - (\text{h.c.}; \text{traceless}) \quad (4.13)$$

By means of (3.29) it is easy to verify that the first term in (4.12) generates a general coordinate transformation with parameter ξ^μ according to the definition (4.2), with an extra SU(8) transformation with parameter $-\xi^\mu \Lambda_{\mu A}^B$.

In conclusion we have now shown that the supersymmetry transformations close consistently on the fields e_μ^α , B_μ^m and e_{AB}^m and yield $d=4$ and $d=7$ general coordinate transformations with parameters defined in (4.4), local Lorentz transformations with parameters $-\xi^\mu \hat{\omega}_\mu^{\alpha\beta} + \varepsilon^{\alpha\beta}$, where $\varepsilon^{\alpha\beta}$ has also been defined in (4.4), and local $SU(8)$ transformations with parameter $-\xi^\mu B_{\mu A}^B + \Lambda_A^B$, where Λ_A^B has been given in (4.13). Apart from verifying the consistency of the new formulation constructed in the previous section, this result illustrates the importance of restrictions such as (3.28), (3.29), (3.32) and (3.35).

5. - COVARIANT FIELD STRENGTHS AND INTEGRABILITY CONDITIONS

The generalized vielbein postulate, which is expressed through (3.29) and (3.32), implies certain integrability conditions on the $SU(8)$ covariant field strengths which will be the subject of this section. There are further relations of this type, which also contain equations of motion and will be dealt with in section 7. We first recall the definition of the $SU(8)$ covariant derivatives, which in a trivial background give (for simplicity we will assume a flat background throughout this and the following sections)

$$D_\mu e_{AB}^m \equiv D_\mu e_{AB}^m + B_\mu^C [A e_B^m]_C ,$$

$$D_n e_{AB}^m \equiv \partial_n e_{AB}^m + B_n^C [A e_B^m]_C . \quad (5.1)$$

With the definition (5.1), (3.29) and (3.32) now read

$$D_\mu e_{AB}^m + \frac{1}{2} \partial_n B_\mu^m e_{AB}^n + \partial_n B_\mu^n e_{AB}^m + A_{\mu ABCD} e^{mCD} = 0 \quad (5.2)$$

$$D_n e^m_{AB} + A_{nABCD} e^{mCD} = 0 \quad (5.3)$$

The integrability conditions are obtained by acting on these equations with another SU(8) covariant derivative. They are

$$\left(F_{mn} + \frac{2}{3}(A_m \cdot A_n - A_n \cdot A_m) \right)^C_{[A} e^P_{B]C} + (D_m A_n - D_n A_m)_{ABCD} e^{PCD} = 0 \quad (5.4)$$

$$\begin{aligned} & \left(F_{m\mu} + \frac{2}{3}(A_m \cdot A_\mu - A_\mu \cdot A_m) \right)^C_{[A} e^P_{B]C} + \partial_m \partial_n B^P_{\mu} e^n_{AB} + \partial_m \partial_n B^n_{\mu} e^P_{AB} + \\ & + (D_m A_\mu - D_\mu A_m + \partial_m B^n_{\mu} A_n)_{ABCD} e^{PCD} = 0, \end{aligned} \quad (5.5)$$

$$\begin{aligned} & \left(F_{\mu\nu} + \frac{2}{3}(A_\mu \cdot A_\nu - A_\nu \cdot A_\mu) \right)^C_{[A} e^m_{B]C} + \\ & + 2 \partial_n (D_\mu B^m_\nu - D_\nu B^m_\mu) e^n_{AB} + \partial_n (D_\mu B^n_\nu - D_\nu B^n_\mu) e^m_{AB} \\ & + (D_\mu A_\nu - D_\nu A_\mu + (D_\mu B^n_\nu - D_\nu B^n_\mu) A_n)_{ABCD} e^{mCD} = 0. \end{aligned} \quad (5.6)$$

In deriving this result, we have made use of the E₇ Lie algebra relation [2]

$$\begin{aligned} & A_m^{ABEF} A_{nCDEF} - A_n^{ABEF} A_{mCDEF} = \\ & = \frac{2}{3} \delta_{[C}^A \{ A_m^B \}^{EFG} A_{nD]EFG} - A_n^B \{ A_m^A \}^{EFG} A_{mD]EFG} \\ & \equiv \frac{2}{3} \delta_{[C}^A (A_m \cdot A_n - A_n \cdot A_m)^B_{D]} \end{aligned} \quad (5.7)$$

The SU(8) covariant field strengths in (5.4-6) are given by

$$F_{mn}^A \equiv \partial_m B_n^A - \partial_n B_m^A + \frac{1}{2} [B_m, B_n]^A_B \quad (5.8)$$

$$F_{\mu m}^A \equiv D_{\mu} B_{mB}^A - \partial_m B_{\mu B}^A + \frac{1}{2} [B_{\mu}, B_m]^A_B - \partial_m B_{\mu}^n B_{nB}^A, \quad (5.9)$$

$$F_{\mu\nu}^A \equiv D_{\mu} B_{\nu B}^A - D_{\nu} B_{\mu B}^A + \frac{1}{2} [B_{\mu}, B_{\nu}]^A_B + 2D_{[\mu} B_{\nu]}^m B_{mB}^A. \quad (5.10)$$

The somewhat unusual terms in (5.9) and (5.10) are required for covariance under local SU(8)-transformations

$$\delta B_{\mu B}^A = -2 D_{\mu} \Lambda_B^A - [B_{\mu}, \Lambda]^A_B,$$

$$\delta B_{mB}^A = -2 \partial_m \Lambda_B^A - [B_m, \Lambda]^A_B, \quad (5.11)$$

because

$$[D_{\mu}, D_{\nu}] = -2D_{[\mu} B_{\nu]}^m \partial_m,$$

$$[D_{\mu}, \partial_m] = \partial_m B_{\mu}^n \partial_n, \quad (5.12)$$

which do not vanish. It is rather curious that combinations like

$$H_{\mu\nu} = F_{\mu\nu} + \frac{2}{3} (A_{\mu} \cdot A_{\nu} - A_{\nu} A_{\mu}),$$

$$K_{\mu\nu} = D_{\mu} A_{\nu} - D_{\nu} A_{\mu} + 2D_{[\mu} B_{\nu]}^m A_m, \quad (5.13)$$

can be viewed as components of an E_7 covariant field strengths in eleven dimensions. This "E₇-covariance" is, of course, just a consequence of the structure of the generalized vielbein postulate (5.2) and (5.3). The above combinations will also play a role in section 7.

Similar unusual features arise when one works out the Riemann tensor

with $d=4$ indices. From the basic formula for the coefficients of anholonomy in section 2 one can work out the transformation of $\omega'_{\mu\alpha\beta}$ under a general coordinate transformation with parameter $\xi^m(x,y)$ by use of (4.2). One finds

$$\delta\omega'_{\mu\alpha\beta} = \xi^m \partial_m \omega'_{\mu\alpha\beta} - e'_\mu [\alpha e'_{\beta}]^v D_v \partial_m \xi^m \quad (5.14)$$

Clearly $\omega'^{\alpha\beta}_\mu$ does not transform covariantly, so that we are led to introduce a modified spin connection

$$\hat{\omega}_{\mu\alpha\beta} \equiv \omega'_{\mu\alpha\beta} + e'_\mu [\alpha e'_{\beta}]^v \partial_m B^m_v \quad (5.15)$$

which (in a trivial background) coincides with the connection used in section 3 (cf. (3.36)) where it appears in the transformation rule for ψ^A_μ . The new connection transforms covariantly with respect to the above transformations, i.e.

$$\delta\hat{\omega}_{\mu\alpha\beta} = \xi^m \partial_m \hat{\omega}_{\mu\alpha\beta} \quad (5.16)$$

Obviously the connection (5.15) is no longer torsion-free, as one sees by recalling (4.6)

$$D_{[\mu} e'^\alpha_{\nu]} = -\frac{1}{2} e'^\alpha_{[\mu} \partial_{\nu]} B^m \quad (5.17)$$

where D_μ is the covariant derivative containing $\hat{\omega}$. The commutator of two covariant derivatives D_μ and D_ν yields among other terms the Riemann curvature tensor associated with $\hat{\omega}$,

$$R_{\mu\nu\alpha\beta}(\hat{\omega}) \equiv D_\mu \hat{\omega}_{\nu\alpha\beta} - D_\nu \hat{\omega}_{\mu\alpha\beta} + \hat{\omega}_{\mu\alpha}{}^\gamma \hat{\omega}_{\nu\gamma\beta} - \hat{\omega}_{\nu\alpha}{}^\gamma \hat{\omega}_{\mu\gamma\beta} , \quad (5.18)$$

which transforms covariantly,

$$\delta R_{\mu\nu\alpha\beta}(\hat{\omega}) = \xi^m \partial_m R_{\mu\nu\alpha\beta}(\hat{\omega}) \quad (5.19)$$

Because the connection has torsion, the curvature (5.18) does not vanish under cyclic permutation, but we have

$$R_{[\mu\nu\rho]}^{\alpha}(\hat{\omega}) = e_{[\mu}^{\alpha} \partial_m (D_{\nu} B_{\rho]}^m) + 2 D_{[\mu} B_{\nu]}^m \partial_m e_{\rho]}^{\alpha} \quad (5.20)$$

This implies that the Ricci tensor is not symmetric, but instead

$$R_{[\mu}^{\alpha}(\hat{\omega}) e_{\nu]}^{\alpha} = -e^{\alpha} e_{\mu}^{\alpha} e_{\nu}^{\beta} \partial_m (e^{\rho} e_{\alpha}^{\rho} e_{\beta}^{\sigma} D_{[\rho} B_{\sigma]}^m) \quad (5.21)$$

where

$$R_{\mu}^{\alpha}(\hat{\omega}) \equiv R_{\mu\nu}^{\alpha\beta}(\hat{\omega}) e_{\beta}^{\nu} \quad (5.22)$$

6. - SU(8) INVARIANT FERMIONIC LAGRANGIAN

The results obtained so far are in principle sufficient to establish the SU(8) covariance of the equations of motion from the commutator of two local supersymmetry transformations. As this would, however, require the knowledge of the supersymmetry variations of B_{μ} , A_{μ} , B_m and A_m , we will now explicitly demonstrate the SU(8) covariance of the fermionic field equations by rewriting the fermionic part of the d=11 lagrangian in terms of the new SU(8) quantities. The analogous construction is not possible for the bosonic part because of the explicit appearance of the gauge field A_{MNP} in the d=11 lagrangian.

As before it is advantageous to use flat $SO(1,10)$ indices. We thus start from

$$L_F = -\frac{1}{2}E \bar{\Psi}_A \tilde{\Gamma}^{ABC} D_B \Psi_C - \frac{\sqrt{2}}{192} E F_{ABCD} (\bar{\Psi}_E \Gamma^{ABCDEFG} \Psi_F + 12 \bar{\Psi}_A \tilde{\Gamma}^{BCD} \Psi^D) \quad (6.1)$$

where we have ignored higher-order fermionic terms. We then write out the $d=4$ and $d=7$ indices in (6.1) explicitly. For the first term in (6.1), this straightforwardly leads to

$$\begin{aligned} & -\frac{1}{2}E \bar{\Psi}_A \tilde{\Gamma}^{ABC} D_B \Psi_C = \\ & = -\frac{1}{2}E \bar{\Psi}_\alpha \gamma^{\alpha\beta\gamma} D_\beta \Psi_\gamma + \frac{1}{2}E \bar{\Psi}_a \Gamma^{ab} \gamma^\alpha D_\alpha \Psi_b \\ & \quad - \frac{1}{2}E \bar{\Psi}_a \Gamma^a \gamma_5 \gamma^{\alpha\beta} D_\alpha \Psi_\beta - \frac{1}{2}E \bar{\Psi}_\alpha \gamma_5 \gamma^{\alpha\beta} \Gamma^a D_\beta \Psi_a \\ & \quad + \frac{1}{2}E \bar{\Psi}_\alpha \gamma_5 \gamma^{\alpha\beta} \Gamma^a D_a \Psi_\beta - \frac{1}{2}E \bar{\Psi}_a \Gamma^{ab} \gamma^\alpha D_b \Psi_\alpha \\ & \quad - \frac{1}{2}E \bar{\Psi}_\alpha \gamma^\alpha \Gamma^{ab} D_a \Psi_b - \frac{1}{2}E \bar{\Psi}_a \gamma_5 \Gamma^{abc} D_b \Psi_c \end{aligned} \quad (6.2)$$

Although one may freely integrate the left hand side of (6.2) by parts, we have not yet simplified the right hand side by combining certain terms through integration by parts. The reason is that one picks up additional terms with derivatives on E if not all terms are treated in the same way. All these extra terms are needed when one reassembles (6.1) into a manifestly $SU(8)$ invariant expression. It is also important that the covariantizations in (6.2) act not only on the spinor indices (which we have suppressed) but also on the vector indices unlike in the original expression (1.5). To illustrate this, we write out a few of these terms below. For instance,

$$\begin{aligned}
 D_{\alpha} \Psi_a &= [\partial_{\alpha} - \frac{1}{4} \omega_{\alpha\beta\gamma} \gamma^{\beta\gamma} - \frac{1}{2} \omega_{\alpha\beta c} \gamma^{\beta} \gamma_5 \Gamma^c - \frac{1}{4} \omega_{\alpha bc} \Gamma^{bc}] \Psi_a - \\
 &\quad - \omega_{\alpha a\beta} \Psi^{\beta} - \omega_{\alpha ab} \Psi^b, \\
 D_a \Psi_b &= [\partial_a - \frac{1}{4} \omega_{a\alpha\beta} \gamma^{\alpha\beta} - \frac{1}{2} \omega_{a\beta c} \gamma^{\beta} \gamma_5 \Gamma^c - \frac{1}{4} \omega_{acd} \Gamma^{cd}] \Psi_b - \\
 &\quad - \omega_{ab\alpha} \Psi^{\alpha} - \omega_{abc} \Psi^c,
 \end{aligned} \tag{6.3}$$

etc.

The second term in (6.1) is treated in a similar fashion. The next task is to rewrite the resulting expressions in terms of the redefined fields (2.26) and (2.27). This leads to a host of new terms. We will not write out all of them but rather discuss one specific example below. The rest of the calculation required to obtain our final result (6.12) is completely analogous although quite tedious. In determining the SU(8) invariant form of the fermionic lagrangian, it is advisable to do the calculation "backwards" as in [1] by first guessing the correct structure and then adjusting free coefficients by comparison with the result obtained directly from (5.1) after all redefinitions have been performed.

The example to be explained in detail is the following: we consider all terms bilinear in the fields χ^{ABC} involving one internal derivative ∂_m and the SU(8) fields B_m and A_m . The only such terms compatible with SU(8) are

$$\alpha e' \epsilon^{ABCDEFGH} e_{AB}^m \bar{\chi}_{CDE} \left(\partial_m + \frac{3}{2} B_m \right)_F^{F'} \chi_{F'GH}$$

$$+ \beta e' e^{mAB} A_m^{CDEF} \bar{\chi}_{ABC} \chi_{DEF} + \text{h.c.} \quad (6.4)$$

where $e' \equiv \det e'_\mu{}^\alpha$ (see (2.6)) and α, β are free coefficients. To verify this ansatz and to compute α and β , we next substitute (2.26), (2.27), (3.7), (3.33) and (3.34) into (6.4) (as (6.4) is SU(8) invariant we work in the gauge $\Phi = \mathbb{1}$). The term containing ∂_m is evaluated as follows

$$\begin{aligned} & \alpha e' \varepsilon^{ABCDEFGH} e_{AB}^m \bar{\chi}_{CDE} \partial_m \chi_{FGH} + \text{h.c.} = \\ & = -\frac{9}{2} \alpha e' \Delta^{-1} \varepsilon^{ABCDEFGH} \Gamma_{AB}^b \bar{\Psi}_{aC} \gamma_5 \Gamma_{DE}^a \Gamma_{FG}^c \partial_b \Psi_{cH} = \\ & = -108 \alpha e' \Delta^{-1} \Gamma_{[AB}^a \Gamma_{CD]}^c \bar{\Psi}_{aA} \gamma_5 \Gamma_{CD}^b \partial_b \Psi_{cB} = \\ & = 72 \alpha e' \Delta^{-1} \bar{\Psi}_a \gamma_5 (\delta^{ab} \Gamma^c + \delta^{bc} \Gamma^a + \delta^{ca} \Gamma^b) \partial_b \Psi_c, \end{aligned} \quad (6.5)$$

where in going from the second to the third line, we have used (A.2). We have also dropped an additional term proportional to $\partial_b \Delta^{-\frac{1}{4}}$ (from the Weyl rescaling) which vanishes by the Majorana property of Ψ_a . In comparing (6.5) with the corresponding contribution from (6.2), we must keep in mind that the rediagonalization in (2.20) has not been taken into account so far and thus similar contributions will arise from $\phi \partial_m \chi$ and $\phi \partial_m \psi$ terms. Instead of analyzing these separately, we simply replace Ψ_α by $\Psi_\alpha + \frac{1}{2} \gamma_5 \gamma_\alpha \Gamma^a \Psi_a$ in (6.2) and retain only the terms relevant for (6.5). This yields

$$\begin{aligned} & \frac{1}{2} E (-\frac{1}{2} \bar{\Psi}_a \Gamma^a \gamma_5 \gamma_\alpha) \gamma_5 \gamma^{\alpha\beta} \Gamma^b \partial_b (\frac{1}{2} \gamma_5 \gamma_\beta \Gamma^c \Psi_c) - \frac{1}{2} E \bar{\Psi}_a \Gamma^{ab} \gamma_\alpha \partial_b (\frac{1}{2} \gamma_5 \gamma_\alpha \Gamma^c \Psi_c) \\ & - \frac{1}{2} E (-\frac{1}{2} \bar{\Psi}_a \Gamma^a \gamma_5 \gamma_\alpha) \gamma_\alpha \Gamma^{bc} \partial_b \Psi_c - \frac{1}{2} E \bar{\Psi}_a \gamma_5 \Gamma^{abc} \partial_b \Psi_c = \end{aligned}$$

$$= E \bar{\Psi}_a \gamma_5 \left(-\frac{3}{2} \Gamma^a \Gamma^b \Gamma^c + \Gamma^{ab} \Gamma^c + \Gamma^a \Gamma^{bc} - \frac{1}{2} \Gamma^{abc} \right) \partial_b \Psi_c \quad (6.6)$$

which is easily seen to agree with (6.5) for

$$\alpha = -\frac{1}{144} \quad (6.7)$$

because $E = e' \Delta^{-1}$ from (2.2) and (2.6). To calculate β , we insert (6.7) into (6.4) and write out the other terms in (6.4), using (2.27),

$$\begin{aligned} & -\frac{1}{144} e' \varepsilon^{ABCDEFGH} e_{AB}^m \bar{\chi}_{CDE} \left(\frac{3}{2} B_{mF}^{F'} \chi_{F'GH} \right) + \\ & + \beta e' e^{mAB} A_m^{CDEF} \bar{\chi}_{ABC} \chi_{DEF} + \text{h.c.} = \\ & -\frac{3\sqrt{2}}{128} i e' \Delta^{-\frac{1}{2}} \Gamma_{[AB}^a \Gamma_{CD]}^e F_{abcd} \bar{\psi}_{eA} \Gamma_{AE}^{bcd} \Gamma_{[EC}^f \gamma_5 \psi_{fD]} - \\ & -\frac{3\sqrt{2}}{32} i \beta e' \Delta^{-\frac{1}{2}} \Gamma_{[AB}^b \Gamma_{CD]}^{cd} F_{abcd} (\bar{\psi}_e \Gamma_e^a \Gamma^e)_A \Gamma_{BC}^f \gamma_5 \psi_{fD} \\ & + \frac{3\sqrt{2}}{8} i \beta e' \Delta^{-\frac{1}{2}} \Gamma_{[AB}^b \Gamma_{CD]}^{cd} F_{abcd} \bar{\psi}_A^a \Gamma_{BC}^f \gamma_5 \psi_{fD} . \end{aligned} \quad (6.8)$$

After some Γ -matrix algebra, the Weyl-rescaling and the γ_5 -redefinition, this becomes

$$\begin{aligned} & e' \Delta^{-1} F_{abcd} \left\{ \left(\frac{\sqrt{2}}{96} + \frac{3\sqrt{2}}{16} \beta \right) \bar{\Psi}_e \Gamma^{ebcd} \Psi^a + \right. \\ & \left. + \frac{\sqrt{2}}{192} \bar{\Psi}_e \Gamma^{abcd} \Psi^e - \left(\frac{3\sqrt{2}}{32} + \frac{15\sqrt{2}}{16} \beta \right) \bar{\Psi}^a \Gamma^{bcd} \Psi^d \right\} . \end{aligned} \quad (6.9)$$

This must now be compared again with the corresponding expression deduced from (6.1); as before, we must not forget to keep track of the additional terms generated by the replacement $\Psi_\alpha \rightarrow \Psi_\alpha + \frac{1}{2} \gamma_5 \gamma_\alpha \Gamma^a \Psi_a$. The relevant

terms are

$$\begin{aligned}
& - \frac{\sqrt{2}}{192} E F_{abcd} \{ \bar{\Psi}_e \Gamma^{abcdef} \Psi_f + 12 \bar{\Psi}^a \Gamma^{bc} \Psi^d \\
& \quad - \frac{1}{2} (\bar{\Psi}_e \Gamma^e \gamma_5 \gamma_\alpha) \gamma^\alpha \gamma_5 \Gamma^{abcdf} \Psi_f + \frac{1}{2} \bar{\Psi}_e \Gamma^{abcde} \gamma_5 \gamma^\alpha (\gamma_5 \gamma_\alpha \Gamma^f \Psi_f) - \\
& \quad - \frac{1}{4} (\bar{\Psi}_e \Gamma^e \gamma_5 \gamma_\alpha) \gamma^{\alpha\beta} \Gamma^{abcd} (\gamma_5 \gamma_\beta \Gamma^f \Psi_f) \} = \\
& = \frac{\sqrt{2}}{192} E F_{abcd} \{ \bar{\Psi}_e \Gamma^{abcd} \Psi^e + 8 \bar{\Psi}_e \Gamma^{ebcd} \Psi^a - 48 \bar{\Psi}^a \Gamma^{bc} \Psi^d \} \quad (6.10)
\end{aligned}$$

This agrees with (6.9), if we put

$$\beta = \frac{1}{6} \quad (6.11)$$

At this point, all coefficients are determined and the rest of the argument just checks the consistency of our ansatz (6.4). For the terms containing f (i.e. $F_{\alpha\beta\gamma\delta}$) the calculation is straightforward, whereas the part involving the spin connection is somewhat more complicated. In particular, one must keep all the terms displayed in (6.3) to obtain agreement. As the calculation does not require any new elements beyond those illustrated by our example, we refrain from giving further details.

A rather tedious calculation, in which the procedure explained above has to be repeated several times, also yields the complete $SU(8)$ invariant lagrangian. We now drop all primes and make use of the fully covariant derivatives introduced in section 3. The final result is

$$L_F =$$

$$\begin{aligned}
&= -\frac{1}{2} e \bar{\psi}_\mu^A \gamma^{\mu\nu\rho} (D_\nu(\hat{\omega}, B) - \frac{1}{4} \partial_m B_\nu^m) \psi_{\rho A} \\
&- \frac{1}{12} e \bar{\chi}^{ABC} \gamma^\mu D_\mu(\hat{\omega}, B) \chi_{ABC} - \frac{\sqrt{2}}{6} e \bar{\chi}_{ABC} \gamma^\nu \gamma^\mu \psi_{\nu D} A_\mu^{ABCD} + \\
&+ e G_{\alpha\beta AB}^+ \{ -\bar{\psi}_{[\mu}^A \gamma^\mu \gamma^{\alpha\beta} \gamma^\nu \psi_{\nu]}^B + \frac{1}{\sqrt{2}} \bar{\psi}_{\mu C} \gamma^{\alpha\beta} \gamma^\mu \chi^{ABC} \\
&\quad + \frac{1}{72} \epsilon^{ABCDEFGH} \bar{\chi}_{CDE} \gamma^{\alpha\beta} \chi_{FGH} \} + \\
&+ \frac{1}{2} e e_{AB}^m \bar{\psi}_\mu^A \gamma^{\mu\nu} D_m(B) \psi_\nu^B + \frac{\sqrt{2}}{4} e e_{AB}^m \bar{\chi}^{ABC} \gamma^\mu D_m(B) \psi_{\mu C} + \\
&- \frac{1}{144} e \epsilon^{ABCDEFGH} e_{AB}^m \bar{\chi}_{CDE} D_m(B) \chi_{FGH} + \frac{1}{6} e e^{mAB} A_m^{CDEF} \bar{\chi}_{ABC} \chi_{DEF} - \\
&- \frac{\sqrt{2}}{8} e e_{AB}^m A_m^{ABCD} \bar{\chi}_{CDE} \gamma^\mu \psi_\mu^E - \frac{\sqrt{2}}{6} e A_{mABCD} e^{mDE} \bar{\chi}^{ABC} \gamma^\mu \psi_{\mu E} \\
&- \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} e_\mu^\alpha \partial_m e_{\nu\alpha} e_{AB}^m \bar{\psi}_\rho^A \psi_\sigma^B + \text{h.c.} \tag{6.12}
\end{aligned}$$

For the torus reduction of [1], where the fields no longer depend on y , all terms containing ∂_m , B_m and A_m simply vanish. The remainder in (6.12), of course, agrees with the results of [1].

7. - FIELD EQUATIONS

From the results of the previous section it is straightforward to derive the fermionic field equations (up to higher-order fermionic terms). They are (up to (7.11) we consistently delete the primes on e_μ^α)

$$R_A^\mu = 0 \text{ and } R_{ABC} = 0, \quad (7.1)$$

where

$$\begin{aligned} R_A^\mu &\equiv e^{-1} \frac{\delta S}{\delta \psi_\mu^A} \\ &= -\gamma^{\mu\nu\rho} (D_\nu - \tfrac{1}{2} \partial_m B_\nu^m) \psi_{\rho A} + \frac{1}{6\sqrt{2}} \gamma^\nu \gamma^\mu \chi^{BCD} A_{\nu ABCD} \\ &\quad + [-8 G_{AB}^{\mu\nu} + e_{AB}^m \gamma^{\mu\nu} D_m + \tfrac{1}{2} e^{-1} \partial_m (e \gamma^{\mu\nu}) e_{AB}^m \\ &\quad - \tfrac{1}{2} A_{mABCD} e^{mCD} \gamma^{\mu\nu} - \tfrac{1}{2} e^{-1} e^{\mu\nu\rho\sigma} e_{AB}^m e_\rho^\alpha \partial_m e_{\sigma\alpha}] \psi_\nu^B \\ &\quad + \tfrac{1}{2\sqrt{2}} [G_{\alpha\beta}^{-BC} \gamma^{\alpha\beta} \gamma^\mu - \tfrac{1}{2} e^{mBC} \gamma^\mu D_m \\ &\quad + \tfrac{1}{2} e^{-1} \partial_m (e \gamma^\mu) e^{mBC} - \tfrac{1}{2} e_{DE}^m A_m^{BCDE} \gamma^\mu] \chi_{ABC} \\ &\quad - \frac{1}{6\sqrt{2}} A_m^{BCDE} e_{AE}^m \gamma^\mu \chi_{BCD}, \end{aligned} \quad (7.2)$$

and

$$\begin{aligned} R_{ABC} &= e^{-1} \frac{\delta S}{\delta \chi^{ABC}} \\ &= -\frac{1}{6} \gamma^\mu D_\mu \chi_{ABC} - \frac{\sqrt{2}}{8} A_{\mu ABCD} \gamma^\nu \gamma^\mu \psi_\nu^D - \frac{1}{6\sqrt{2}} e^{mDE} A_{mEABC} \gamma^\nu \psi_{\nu D} \\ &\quad + \tfrac{1}{2\sqrt{2}} [G_{\alpha\beta}^+ [AB \gamma^\mu \gamma^{\alpha\beta} + \tfrac{1}{2} \gamma^\mu e_{[AB}^m D_m - \tfrac{1}{2} \gamma^\mu e^{mDE} A_{mDE[AB}]] \psi_{\mu C}] \\ &\quad - \frac{1}{72} \epsilon_{ABCDEFGH} [e^{mGH} D_m - \tfrac{1}{2} e^{mGH} e^{-1} \partial_m e - 2 G_{\alpha\beta}^{-GH} \gamma^{\alpha\beta}] \chi^{DEF} \\ &\quad + \frac{1}{12} [-3 e_{F[C}^m A_{mAB]DE} + 3 e_{[AB}^m A_{mC]DEF} + e_{DE} A_{mFABC}] \chi^{DEF}. \end{aligned} \quad (7.3)$$

where we made use of (3.32) and (3.35). The bosonic field equations cannot

be found in the same way, as the d=11 lagrangian depends explicitly on the antisymmetric tensor field A_{MNP} for which no expression exists in terms of the SU(8) covariant quantities used in this paper. Of course, the d=11 field equations do not depend explicitly on A_{MNP} , and they can in principle be transcribed into the new formulation. However, the large number of SU(8) covariant terms that may appear makes it practically impossible to cast the d=11 field equations directly in SU(8) covariant form. Therefore, we have followed a less direct approach and analyzed the supersymmetry variation of the fermionic field equations. This yields a number of SU(8) covariant bosonic equations, which must correspond to the original d=11 bosonic field equations (1.6-7), the Bianchi identity (1.8), and the generalized vielbein postulates (5.2-3).

To evaluate the variations of (7.2) and (7.3) completely is straightforward but extremely tedious. One important intermediate check on the correctness of our results is that the field equations must be supercovariant, i.e. all terms proportional to $\partial_\mu \epsilon$ or $\partial_m \epsilon$ must cancel. Another check is that the equations should be consistent with the integrability relations (5.4-6). Here we present only partial results. The spin-3/2 field equation transforms under supersymmetry into five different terms, i.e.

$$\begin{aligned} \delta R_A^\mu = & B^{(1)\mu\nu} \gamma_\nu \epsilon - \frac{1}{16} B^{(2)B}_A \gamma^\mu \epsilon_B - \frac{1}{4} e^{-1} \epsilon^{\mu\nu\rho\sigma} B^{(3)B}_{\rho\sigma A} \gamma_\nu \epsilon_B \\ & + 8 B^{(4)\mu}_{AB} \epsilon^B - \frac{1}{2} B^{(5)}_{\nu AB} \gamma^{\mu\nu} \epsilon^B, \end{aligned} \quad (7.4)$$

where $B(1) - B(5)$ are defined as follows

$$\begin{aligned}
B^{(1)\mu}_{\nu} &= \frac{1}{2} R_{\nu}^{\mu}(\omega) + \frac{1}{2} e^{-1} \partial_m (e e^{\rho}_{[\alpha} e^{\sigma}_{\beta]} D_{\rho} B_{\sigma}^m) e_{\nu}^{\alpha} e^{\mu\beta} \\
&- \frac{1}{4} \delta_{\nu}^{\mu} R(\omega) - [e_{AB}^m G^{+AB\rho}(\mu + e^{mAB} G_{AB}^{-\rho}(\mu)] e_{\alpha}^{\sigma} \partial_m e_{\rho}^{\alpha} g_{\sigma\nu} \\
&+ \frac{1}{12} A_{\mu}^{ABCD} A_{\nu ABCD} + 16 G_{\nu\rho}^{-BC} G_{BC}^{+\mu\rho} \\
&+ \frac{1}{16} [e^{nAB} e^{mCD} D_n A_{mABCD} - e^{nAB} e_{CD}^m A_m^{CDEF} A_{nABEF} \\
&- \frac{1}{12} e^{nAB} e_{AB}^m A_m^{CDEF} A_{nCDEF} - \frac{2}{3} A_{\rho}^{ABCD} A_{ABCD}^{\rho}] \delta_{\nu}^{\mu} \quad (7.5)
\end{aligned}$$

$$\begin{aligned}
B^{(2)B}_A &= e_{AC}^m e^{nCD} (F_{mn} + \frac{2}{3} (A_m \cdot A_n - A_n \cdot A_m))_D^B \\
&- (F_{mn} + \frac{2}{3} (A_m \cdot A_n - A_n \cdot A_m))_A^C e_{CD}^m e^{nDB} \\
&- 2e_{AC}^n (F_{mn} + \frac{2}{3} (A_m \cdot A_n - A_n \cdot A_m))_D^C e^{mDB} \quad (7.6)
\end{aligned}$$

$$\begin{aligned}
B^{(3)B}_{\mu\nu A} &= (F_{\mu\nu} + \frac{2}{3} (A_{\mu} \cdot A_{\nu} - A_{\nu} \cdot A_{\mu}))_A^B \\
&- 8(\partial_m G_{AC\mu\nu}^{+} e^{mCB} - \partial_m G_{\mu\nu}^{-CB} e_{AC}^m) \\
&+ 8(e_{AE}^m A_m^{BECD} G_{CD\mu\nu}^{+} - e^{mBE} A_{mAECD} G_{\mu\nu}^{-CD}) - \frac{1}{8} \delta_A^B [\text{trace}] \quad (7.7)
\end{aligned}$$

$$\begin{aligned}
B^{(4)\mu}_{AB} &= e_{\alpha}^{\mu} e_{\beta}^{\nu} (D_{\nu} - \frac{1}{2} \partial_m B_{\nu}^m) G_{AB}^{+\alpha\beta} \\
&- \frac{1}{16} e^{mCD} D_m A_{ABCD}^{\mu} - \frac{3}{32} \partial_m \partial_n B_{\mu}^n e_{AB}^m \\
&+ A_{\nu ABCD} G^{-CD\mu\nu} - \frac{1}{12} e_{C[A}^m A_{B]}^{\mu} A_m^{CDEF} \quad (7.8)
\end{aligned}$$

$$B_{\mu AB}^{(5)} = e_{AC}^m \left(F_{m\mu} + \frac{2}{3} (A_m \cdot A_\mu - A_\mu \cdot A_m) \right)_B^C - \frac{3}{2} \partial_m \partial_n B_\mu^n e_{AB}^m + e^{mCD} (D_m A_\mu - D_\mu A_m + \partial_m B_\mu^n A_n)_{ABCD} \quad (7.9)$$

Since these expressions have been deduced from the spin-3/2 field equation, they must vanish on shell. Hence, we have derived the following bosonic field equations

$$B_{\mu}^{(1)\nu} = B_A^{(2)B} = B_{\mu\nu A}^{(3)B} = B_{\mu AB}^{(4)} = B_{\mu AB}^{(5)} = 0. \quad (7.10)$$

We will discuss in a moment what these equations correspond to in the original formulation d=11 supergravity, but the reader should keep in mind that there are further equations that follow from the variation of the spin-1/2 equation. We will return to those equations at the end of this section.

The above expression for $B^{(1)} - B^{(5)}$ has been written in this form by using (5.4-6) and (3.43), and are complete modulo some terms proportional to $\partial_m e_\mu^\alpha A_m$, $\partial_m e_\mu^\alpha A_\nu$, $\partial_m e_\mu^\alpha \partial_n B_\nu^n$ and $\partial_m \partial_n e_\mu^\alpha$, which we have not fully evaluated. Note that $B_{\mu\nu}^{(1)}$ is symmetric by virtue of (5.21), and that $B_A^{(2)B}$ is traceless, since

$$B_A^{(2)A} = -2e_{AC}^m e^{nCD} \left(F_{mn} + \frac{2}{3} (A_m \cdot A_n - A_n \cdot A_m) \right)_D^A = 2e_{AC}^m (D_m A_n - D_n A_m)^{ACDE} e_{DE}^n = 0, \quad (7.11)$$

where in the second line we used (5.4). Both $B^{(2)}$ and $B^{(3)}$ are manifestly antihermitean, and $B^{(3)}$ is manifestly traceless; $B^{(4)}$ is manifestly antisymmetric in A and B.

To check the bosonic field equations (7.10) directly, one substitutes the explicit expressions given in section 3 into (7.5-9) and compares the result with the original field equations (1.6), (1.7) and Bianchi identities (1.8). For this comparison, one must adopt the gauge $\Phi = \mathbb{I}$ [see (3.6)]. The calculation is analogous to the one in the previous section and again tedious, especially for the terms with many uncontracted SU(8) indices. We will therefore not present all details but rather discuss several specific examples.

For the Einstein equation $B_{\mu\nu}^{(1)} = 0$, see (7.5), the verification is straightforward. For instance, from (3.27), we obtain

$$\begin{aligned}
 A_{\mu}^{ABCD} A_{\nu ABCD} &= \\
 &= \Delta^{-1} (-g'_{\mu\nu} F_{\alpha\beta\gamma}^2 + 3e'_{\mu}{}^{\alpha} e'_{\nu}{}^{\beta} F_{\alpha\gamma\delta} F_{\beta\gamma\delta}) \\
 &+ \Delta^{-1} e'_{\mu}{}^{\alpha} e'_{\nu}{}^{\beta} F_{\alpha abc} F_{\beta abc} \\
 &+ 3\Delta^{-1} e'_{\mu}{}^{\alpha} e'_{\nu}{}^{\beta} (2\Omega_{\alpha(ab)} \Omega_{\beta ab} + \Omega_{\alpha a}{}^a \Omega_{\beta b}{}^b) , \tag{7.12}
 \end{aligned}$$

where we have used the formulas (A.16) and reinstated the primes. The Weyl-rescaling factor thus disappears when one reverts to the original metric and vielbein. The combination which appears in (7.5) is

$$\begin{aligned}
 A_{\mu}^{ABCD} A_{\nu ABCD} - \frac{1}{2} g'_{\mu\nu} g'^{\rho\sigma} A_{\rho}^{ABCD} A_{\sigma ABCD} &= \\
 &= -\frac{1}{2} g'_{\mu\nu} F_{\alpha\beta\gamma}^2 + 3F_{\mu\alpha\beta a} F_{\nu\alpha\beta a} + F_{\mu abc} F_{\nu abc} - \frac{1}{2} g'_{\mu\nu} F_{\alpha abc}^2 + \\
 &+ 6\Omega_{\mu(ab)} \Omega_{\nu ab} + 3\Omega_{\mu a}{}^a \Omega_{\nu b}{}^b - 3g'_{\mu\nu} \Omega_{\alpha(ab)}^2 - \frac{3}{2} g'_{\mu\nu} \Omega_{\alpha a}^2 , \tag{7.13}
 \end{aligned}$$

where, of course, $g_{\mu\nu} = \Delta^{-1} g'_{\mu\nu}$. To see what this expression corresponds to, we go back to (1.6). The $\mu\nu$ -component of this equation is equivalent to

$$R_{\mu\nu}^{(11)} - \frac{1}{2} g_{\mu\nu} R^{(11)} = \frac{1}{48} g_{\mu\nu} F_{ABCD}^2 - \frac{1}{6} F_{\mu ABC} F_{\nu}^{ABC} \quad (7.14)$$

where

$$R_{\mu\nu}^{(11)} \equiv R_{\mu MN} g^{MN} . \quad (7.15)$$

The right hand side of (7.14) is equal to

$$\begin{aligned} & \frac{1}{48} g_{\mu\nu} \{ 24f^2 + 4F_{\alpha\beta\gamma a}^2 + 6F_{\alpha\beta ab}^2 + 4F_{aabc}^2 + F_{abcd}^2 \} - \\ & - \frac{1}{6} \{ F_{\mu abc} F_{\nu abc} + 3F_{\mu\alpha ab} F_{\nu\alpha ab} + 3F_{\mu\alpha\beta a} F_{\nu\alpha\beta a} \} , \end{aligned} \quad (7.16)$$

where we have made use of the definition (3.22). Comparison with (7.13) shows that the terms with $F_{\mu\alpha\beta a}$ and $F_{\mu abc}$ coincide after multiplication of (7.13) by $-1/6$. The remaining terms in (7.13) have to be related to the second term in (7.15). For these terms, it is actually somewhat more convenient to start from the $d=11$ Einstein action which can be written in the form [1]

$$-\frac{1}{2}ER(E) = -\frac{1}{8} E \{ \Omega_{ABC} \Omega^{ABC} - 2\Omega^{ABC} \Omega_{BCA} - 4\Omega_{AB}^{B A} \Omega_C^{C} \} \quad (7.17)$$

up to a total derivative by means of the coefficients of anholonomy introduced in section 2. Contracting over one space-time index and two internal indices, one would obtain the contribution

$$- \frac{1}{2} E \{ \Omega_{\alpha(ab)} \Omega_{\alpha(ab)} - \Omega_{\alpha a}^2 \} \quad (7.18)$$

from the above expression. However, the Weyl-rescaling gives another contribution containing $\Omega_{\alpha a}^2$ from $\Omega_{\alpha\beta\gamma}$ by (2.18), which changes (7.18) to (after the Weyl-rescaling)

$$- \frac{1}{2} e' \{ \Omega'_{\alpha(ab)} \Omega'_{\alpha ab} + \frac{1}{2} \Omega'^2_{\alpha a} \} \quad (7.19)$$

This is indeed proportional to the combination which appears in (7.13). Since our calculation provides numerous other and independent checks on the various coefficients, we leave the computation of the relevant coefficient as a little exercise to the reader.

Further terms involving F_{ABCD} in (7.14) have to come from the terms containing B_m and A_m in the Einstein equation (7.5). We first note that the terms proportional to $e_{AB}^m A_m^{ABCD}$ do not give any such contributions because

$$e_{CD}^m A_m^{ABCD} = \frac{1}{2} \Delta^{-\frac{1}{2}} e_a^{n\partial} e_{mnb} \{ \Gamma_{AB}^a e^{bm} + \Gamma_{AB}^b e^{am} + \Gamma_{AB}^m \delta^{ab} \} \quad (7.20)$$

The two other terms, however, give rise to

$$\begin{aligned} & - \frac{1}{8} e^{mAB} e^{nCD} D_m A_{nABCD} \rightarrow \\ & - \frac{1}{4} e^{m[AB} e^{nCD]} B_{mA}^E nEBCD \rightarrow \frac{1}{32} \left(\frac{1}{2} \Delta^{-1} F_{abcd}^2 + \frac{96}{7} \Delta^{-1} f^2 \right), \end{aligned} \quad (7.21)$$

and

$$\frac{1}{96} e^{mAB} e^{nCD} A_{mAB}^{CDEF} A_{nCDEF} \rightarrow \frac{1}{32} \left(\frac{1}{6} \Delta^{-1} F_{abcd}^2 + \frac{16}{7} \Delta^{-1} f^2 \right). \quad (7.22)$$

Adding these two expressions and multiplying by $g'_{\mu\nu}$, one obtains

$$g_{\mu\nu}'[(7.19) + (7.20)] = \frac{1}{48} g_{\mu\nu} (F_{abcd}^2 + 24f^2) \quad (7.23)$$

which agrees with the relevant terms in (7.16). The terms containing $F_{\alpha\beta ab}$ clearly originate from $G_{\mu\nu}^2$ in $B_{\mu}^{(1)\nu}$ and are easily checked. The final (and most tedious) check, which we do not perform here, consists in verifying that the terms originating from the curvature tensor, which are of the form $e \partial e$, also come out correctly.

As a second example consider (7.9). The part of this expression which is antisymmetric in AB is nothing but the integrability condition (5.5) (after contraction with δ_p^m). In order to analyze the part symmetric in AB, we again substitute the explicit expressions of section 3 in the gauge $\Phi = 1$ into the combination

$$F_{\mu m B}^A + \frac{2}{3} (A_{\mu} \cdot A_m - A_m \cdot A_{\mu})^A_B \quad (7.24)$$

which was introduced in section 5. Being traceless in the indices AB, this expression can be decomposed into contributions proportional to $\Gamma^{(1)}$, $\Gamma^{(2)}$ and $\Gamma^{(3)}$, respectively. The $\Gamma^{(2)}$ -contribution containing e_m^a and its derivatives can be shown to add up to zero after some algebra. The $\Gamma^{(3)}$ component becomes proportional to

$$\{ \partial_{\mu} F_{mnpq} - 4 \partial_m F_{\mu npq} - B_{\mu}^r \partial_r F_{mnpq} + 4 B_{\mu}^r \partial_m F_{rnpq} \} \Gamma_{AB}^{npq} \quad (7.25)$$

After multiplication with $e^{mBC} \epsilon_{BC}^m$ and symmetrization in the indices (AC), this also vanishes on account of the Bianchi identities

$$\partial_{[\mu} F_{mnpq]} = \partial_{[m} F_{npqr]} = 0 \quad (7.26)$$

which are just special components of (1.8). The remaining contribution is found to be proportional to

$$E^{-1} \partial_{\mu} (EF^{\nu\rho\sigma\mu}) + E^{-1} \partial_m (EF^{\nu\rho\sigma m}) + \frac{\sqrt{2}}{144} i \varepsilon^{\nu\rho\sigma\tau} \varepsilon^{mnpqrst} F_{\tau mnp} F_{qrst} \quad (7.27)$$

upon multiplication by e^{mBC} and symmetrization in (AC). Eq. (7.27) is a particular component of the Maxwell-equation (1.7) and thus also vanishes. The reader should realize that the calculations required to establish these results are invariably rather tedious, and there are several subtleties inherent in them that we have not exposed. For example, the derivative operators ∂_{μ} and ∂_m implicit in (7.24) originally act on components of the four index field strength some of whose indices are flat; the $e \partial e$ terms in B and A then must conspire in such a fashion that the flat indices are converted into curved ones as required in (7.26) and (7.27), respectively.

From this analysis, it is now obvious that the SU(8) covariant equations derived in the first part of this section contain both field equations and Bianchi identities when viewed in the original formulation. This illustrates why we cannot start from the original $d=11$ action of [2]: the fact that the Bianchi-identities (which are also valid off-shell) are rotated into the field equations by SU(8) enforces the field equations. Thus, the two versions of the theory can be related directly only for field configurations which satisfy the field equations.

To have also an example involving the coefficients of anholonomy, we consider the terms containing $\Omega_{\mu ab}$ in $B_{\mu\nu A}^{(3)B}$, see (7.7). As before, straightforward substitution of the results in section 3 and a little

calculation shows that

$$F_{\mu\nu A}^B + \frac{2}{3} (A_\mu \cdot A_\nu - A_\nu \cdot A_\mu)_A^B \rightarrow$$

$$\rightarrow \{ D_{[\mu} \Omega_{\nu]}^{ab} + \Omega_{[\mu a}^c \Omega_{\nu]}^{cb} + D_{[\mu} B_{\nu]}^n e_a^m \partial_n e_{mb} \} \Gamma_{AB}^{ab} \quad (7.28)$$

where the last term comes from the extra contribution $2 D_{[\mu} B_{\nu]}^m B_{mA}^B$ in the definition of $F_{\mu\nu A}^B$. We now use (2.12) to evaluate (7.28); the result is

$$(7.28) = \{ e_a^m \partial_m B_{[\mu}^n \partial_n B_{\nu]}^p e_{pb} - e_a^m [\partial_m B_{\nu]}^n e_{nb} \} \Gamma_{AB}^{ab} \quad (7.29)$$

On the other hand, some further calculation reveals that

$$- [8e_{AC}^m (\partial_m + B_m)^{[C} G_{\mu\nu}^{-B]D} - \text{h.c.}] \rightarrow$$

$$\rightarrow \Delta^{-\frac{1}{2}} \Gamma_{AB}^{ma} \partial_m (\Delta^{\frac{1}{2}} e_{na}^D [\partial_\mu B_\nu]^n) + \Gamma_{AB}^{ma} e_{[a}^n \partial_m e_{b]n} e_p^b D_{[\mu} B_{\nu]}^p \quad (7.30)$$

and

$$8e_{AE}^m A_m^{BECD} G_{\mu\nu CD}^+ - \text{h.c.} \rightarrow$$

$$\rightarrow - \Gamma_{AB}^{ma} e_{(a}^n \partial_m e_{b)n} e_p^b D_{[\mu} B_{\nu]}^p - \frac{1}{2} \Gamma_{AB}^{ma} e_b^n \partial_m e_n^b e_{pa} D_{[\mu} B_{\nu]}^p \quad (7.31)$$

These results add up to

$$(7.30) + (7.31) = e_a^m \partial_m ([\partial_\mu B_\nu]^n) e_{nb} \Gamma_{AB}^{ab} \quad (7.32)$$

which cancels (7.29) by virtue of (5.12).

Further checks of (7.5-9) along the lines described above are possible, but our examples should be sufficient to enable the reader to

reproduce some of them for himself (if he wishes to do so). Although we have performed many more cross-checks on our results, we thus refrain from giving further examples. However, the reader may find the table below useful which indicates the correspondence between eqs. (7.5-9) and the original ones (1.6), (1.7) and (1.8).

	(1.6) (Einstein)	(1.7) (Maxwell)	(1.8) (Bianchi)
(7.5)	$(\mu\nu)$	-	-
(7.6)	-	-	$[\text{mnpqr}]$
(7.9)	-	$[\mu\nu\rho]$	$[\mu\text{mnpq}]$
(7.7)	-	$[\mu\nu\text{m}]$	$[\mu\nu\text{mnp}]$
(7.8)	(μm)	$[\mu\text{mn}]$	$[\mu\nu\rho\text{mn}]$

Table 1: Correspondence between SU(8) covariant and original d=11 equations.

The components which are still missing in this table (i.e. the (mn) , $[\text{mnp}]$ and $[\mu\nu\rho\sigma\text{m}]$ components) must be associated with further SU(8) covariant equations that follow from the variation of the spin- $\frac{1}{2}$ equation (7.3); we also remind the reader that the $[\mu\nu\rho\sigma\tau]$ component of (1.8) is trivially satisfied.

We have only partially evaluated the supersymmetry variation of the spin- $\frac{1}{2}$ field equation, and established the existence of the following terms

$$\delta R_{ABC} = 2\sqrt{2} \gamma_{\mu}{}^B{}^{(4)\mu}{}_{[AB} \varepsilon_{C]}$$

$$\begin{aligned}
& + \frac{1}{6} \sqrt{2} \gamma^{\mu\nu} \{ D_{\mu} A_{\nu ABCD} + 6 e_{[AB}^m A_{mCD]EF} G_{\mu\nu}^{-EF} \\
& \quad + e_{EF}^m A_{mABCD} G_{\mu\nu}^{-EF} \} \varepsilon^D \\
& + \frac{1}{8} \sqrt{2} \{ e_{AB}^m (e_{CE}^n F_{mn}^E D - 2 D_{[m} A_{n]CDEF} e^{nEF}) \\
& \quad + e^{mEF} [e_{AB}^n D_{[m} A_{n]CDEF} + e_{DA}^n D_{[m} A_{n]BCEF} \\
& \quad + e_{ED}^n D_{[m} A_{n]ABCF} + e_{EA}^n D_{[m} A_{n]BCDF}] \} \varepsilon^D \\
& + \frac{1}{6} \sqrt{2} \{ D_{\mu} A^{\mu}_{ABCD} - 48 (G_{\mu\nu}^{+} [AB G_{CD}^{+\mu\nu}] + \frac{1}{24} \varepsilon_{ABCDEFGH} G_{\mu\nu}^{-EF} G^{-\mu\nu GH}) \\
& \quad + e^{mEF} [-\frac{3}{2} e_{AB}^n D_{(m} A_{n)CDEF} + \frac{1}{8} e_{EF}^n D_{(m} A_{n)ABCD} \\
& \quad + e_{E[A}^n D_{[m} A_{n]BCD} F] \} \varepsilon^D \\
& + \dots
\end{aligned} \tag{7.33}$$

where $B^{(4)}$ was defined in (7.8). The set of terms containing F_{mn} will vanish upon antisymmetrization of $[ABCD]$ as a result of the integrability relation (5.4) (we have not exhibited terms proportional to $A_m A_n$ in (7.33) which are required for a full cancellation). The last set of terms starting with $D_{\mu} A^{\mu}_{ABCD}$ is selfdual in $[ABCD]$ and is obviously a field equation for the spin-0 degrees of freedom. Among possible other terms it contains the d=11 Bianchi identity with components $[\mu\nu\rho\sigma]$.

8. - OUTLOOK

In this paper, we have shown that the existence of "hidden symmetries" in $d=4$ supergravity can be understood from the existence of related structures in higher dimensional supergravity theories. The local chiral $SU(8)$ of $N=8$ supergravity is just a consequence of a corresponding but much bigger local $SU(8)$ group in $d=11$ supergravity. We have explicitly demonstrated that the equations of motion and constraints of the original version of $d=11$ supergravity can be combined into new equations which are fully $SU(8)$ covariant. These results will be of great importance in [4] where we address and solve the long-standing problem of whether $d=11$ supergravity compactified on S^7 can be consistently truncated to a theory which coincides with gauged $N=8$ supergravity to all orders.

The construction described in this paper has taken us a long way from conventional supergravity and there remain many unanswered questions. We have no idea why this construction should have been possible at all. The "miracles" that have occurred on several occasions ought to be explained, and are indicative of the existence of yet a bigger theory, which contains $d=11$ supergravity and could provide a more natural framework for the new structure discovered here. The emergence of E_7 remains likewise mysterious in the present context.

Whereas the original version of $d=11$ supergravity emphasizes the invariance under $d=11$ general coordinate transformations, this is not so in the new formulation. The reader may have noticed that the general coordinate transformations with parameters $\xi^m(x,y)$ in the "internal" dimensions take a rather unusual form. In fact, using the explicit expressions (3.33) and (3.34) for B_m and A_m one may check that neither

quantity transforms as a vector. One reason is that the $SO(7)$ part of B_m does not coincide with the spin connection ω_m^{ab} ; the latter is split into two parts which have been distributed between B_m and A_m . However, there is a subtle interplay between the general coordinate transformations with parameters $\xi^m(x,y)$ and E_7 transformations. To see this we return to (4.2), which gives the transformation rule for e_{AB}^m , viz.

$$\delta_{gct} e_{AB}^m = \partial_n \xi^m e_{AB}^n - \xi^n \partial_n e_{AB}^m + \frac{1}{2} \partial_n \xi^n e_{AB}^m . \quad (8.1)$$

Introducing E_7 transformation parameters

$$\omega_A^B \equiv -\frac{1}{2} e_a^n \partial_n \xi^m e_{mb} \Gamma_{AB}^{ab} , \quad (8.2)$$

$$\omega_{ABCD} \equiv \frac{3}{4} e_a^n \partial_n \xi^m e_{mb} \Gamma_{[AB}^a \Gamma_{CD]}^b , \quad (8.3)$$

we can rewrite (8.1) in the form

$$\delta_{gct} e_{AB}^m = -\xi^n \partial_n e_{AB}^m + \delta_{E_7} e_{AB}^m , \quad (8.4)$$

with

$$\delta_{E_7} e_{AB}^m \equiv \omega_{[A}^C e_{B]C}^m + \omega_{ABCD} e^{mCD} . \quad (8.5)$$

Therefore, the part of a general coordinate transformation on e_{AB}^m associated with the $d=7$ world index m can be interpreted as a local E_7 transformation with parameters (8.2-3)*). This surprising result suggests that there may be a possibility of extending the local $SU(8)$, although so far we have not been able to recognize a similar pattern for

*) The reader will notice that the factor $\Delta^{-\frac{1}{2}}$ in the definition of e_{AB}^m once more plays an essential role.

the transformation rules of fields other than e_{AB}^m . It is evident that (8.5) will be useful in explaining the rigid E_7 invariance of (ungauged) $N=8$ supergravity.

It is an obvious question how local $SU(8)$ transformations act on the bosonic fields of the original $d=11$ supergravity formulation, E_M^A and F_{MNPQ} (or even A_{MNP}). Then metric g_{mn} can be directly determined from (3.10), which shows that g_{mn} is inert under $SU(8)$. As the other components of g_{MN} are also $SU(8)$ invariant, this result thus applies to the full $d=11$ metric. A similar result can be found for the components of the $d=11$ field strength F_{MNPQ} , which appear in various $SU(8)$ covariant tensors. These components can be extracted in an $SU(8)$ invariant way by contraction with an appropriate number of e_{AB}^m tensors. For instance, for $F_{\alpha\beta ab}$ one finds

$$F_{\alpha\beta ab} = - \frac{i\sqrt{2}}{5!} \sqrt{g} \varepsilon_{mnpqrst} e^{pBC} e^q_{CD} e^{rDE} e^s_{EF} e^{tFA} \Delta^2 e^m_a e^n_b G^+_{\alpha\beta AB} + \text{h.c.} \quad (8.6)$$

where $g = \det g_{mn}$. Hence we conclude that $SU(8)$ plays no important role in the original formulation of this theory.

Finally, we want to mention the possible implications of our work for other supergravity theories. First of all, one may contemplate the possibility of reducing $d=11$ supergravity to dimensions other than $d=4$. Since this gives rise to different hidden symmetries (especially the noncompact groups E_{11-d}) [6], repeating the procedure of this paper might lead to yet more versions of $d=11$ supergravity. In this case, $d=11$ supergravity would be a chameleon-like theory which completely changes its guise depending on how it is looked at. Since the elusive E_{10} -symmetry may

also be hiding in it, a further search in this direction may well be worth the effort. Secondly, it is well-known that other supergravity theories (such as $d=10$ supergravity) also exhibit "hidden symmetries" after dimensional reduction. Again, one may expect that they admit reformulations along the lines described here.

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APPENDIX

We here list the $d=7$ Γ -matrix identities needed throughout this paper. The interested reader is advised to consult the appendices of [1] for further details. The defining relations of the $d=7$ Clifford algebra have already been given in (2.22). From these we obtain in a straightforward fashion the formulas

$$\begin{aligned}\Gamma^{abcdef} &= -i\epsilon^{abcdefg} \Gamma^g \\ \Gamma^{abcde} &= \frac{i}{2} \epsilon^{abcdefg} \Gamma^{fg} \\ \Gamma^{abcd} &= \frac{i}{6} \epsilon^{abcdefg} \Gamma^{efg} \\ \Gamma^{abc} &= -\frac{i}{24} \epsilon^{abcdefg} \Gamma^{defg}\end{aligned}\tag{A.1}$$

The following equations play an important role

$$\Gamma^a_{[AB} \Gamma^b_{CD]} = \frac{1}{24} \epsilon_{ABCDEFGH} \Gamma^a_{EF} \Gamma^b_{GH}\tag{A.2}$$

$$\Gamma^a_{[AB} \Gamma^{ab}_{CD]} = \frac{1}{24} \epsilon_{ABCDEFGH} \Gamma^a_{EF} \Gamma^b_{GH}\tag{A.3}$$

$$\Gamma^{[a}_{[AB} \Gamma^{bc]}_{CD]} = -\frac{1}{24} \epsilon_{ABCDEFGH} \Gamma^{[a}_{EF} \Gamma^{bc]}_{GH} = \frac{i}{24} \epsilon^{abcdefg} \Gamma^{de}_{[AB} \Gamma^{fg]}_{CD]}\tag{A.4}$$

A consequence of these is the (complex) selfduality of the tensors A_{μ}^{ABCD} and A_m^{ABCD} as well as the relation

$$e^m_{[AB} e^n_{CD]} = \frac{1}{24} \epsilon_{ABCDEFGH} e^{mEF} e^{nGH},\tag{A.5}$$

whose proof follows directly from (3.6), (A.2) and SU(8) covariance. From the triality properties of the d=7 Clifford algebra, one derives the following identities

$$\Gamma_{AB}^{ac} \Gamma_{CD}^{cb} - \Gamma_{AB}^{bc} \Gamma_{CD}^{ca} = - \Gamma_{AB}^a \Gamma_{CD}^b + \Gamma_{AB}^b \Gamma_{CD}^a - 2(\delta_{AC} \Gamma_{BD}^{ab} \pm 3 \text{ terms}) , \quad (\text{A.6})$$

$$\Gamma_{AB}^a \Gamma_{CD}^{ab} - \Gamma_{AB}^{ab} \Gamma_{CD}^a = - 2(\delta_{AC} \Gamma_{BD}^b \pm 3 \text{ terms}) , \quad (\text{A.7})$$

$$\Gamma_{[AB}^{ac} \Gamma_{CD]}^{cb} = 5 \Gamma_{[AB}^a \Gamma_{CD]}^b - \delta^{ab} \Gamma_{[AB}^c \Gamma_{CD]}^c , \quad (\text{A.8})$$

$$\Gamma_{[AB}^{ab} \Gamma_{CD]}^c = - \frac{1}{3} \delta^c [a \Gamma_{[AB}^b] d \Gamma_{CD]}^d + \Gamma_{[AB}^{[ab} \Gamma_{CD]}^c] , \quad (\text{A.9})$$

$$\begin{aligned} \Gamma_{[AB}^{ab} \Gamma_{CD]}^{cd} &= - \delta^{ac} \Gamma_{[AB}^b \Gamma_{CD]}^d + \delta^{bc} \Gamma_{[AB}^a \Gamma_{CD]}^d \\ &\quad + \delta^{ad} \Gamma_{[AB}^b \Gamma_{CD]}^c - \delta^{bd} \Gamma_{[AB}^a \Gamma_{CD]}^c \\ &\quad + \frac{2}{3} \delta_{cd}^{ab} \Gamma_{[AB}^e \Gamma_{CD]}^e + \Gamma_{[AB}^{[ab} \Gamma_{CD]}^{cd]} . \end{aligned} \quad (\text{A.10})$$

We next quote some formulas needed to convert d=7 vector into spinor indices. The inverse of

$$\lambda_{ABC} = i \Gamma_{[AB}^a \Psi_{aC]} \quad (\text{A.11})$$

is

$$\Psi_{aC} = \frac{3}{4} i \{ \Gamma_{AB}^a \lambda_{ABC} - \frac{1}{9} (\Gamma^a \Gamma^b)_{CA} \Gamma_{BD}^b \lambda_{ABD} \} . \quad (\text{A.12})$$

An immediate consequence of (A.11) is the relation

$$\bar{\lambda}_{ABC} \lambda_{ABC} = \frac{4}{3} \bar{\Psi}_a (\delta^{ab} + \frac{1}{2} \Gamma^a \Gamma^b) \Psi_b . \quad (\text{A.13})$$

In the course of the construction described in this paper, one encounters many further relations of this type. An especially useful one is

$$\begin{aligned} \Gamma_{[AB}^X \Gamma_{CD]}^Y = \\ = \Gamma_{[AB}^a \left\{ -\frac{1}{4} \Gamma_a^X \Gamma^Y - \frac{1}{4} \Gamma_a^Y \Gamma^X + \frac{1}{36} \Gamma_a^b \Gamma_b^X \Gamma^Y + \frac{1}{36} \Gamma_a^b \Gamma_b^Y \Gamma^X + \right. \\ \left. + \frac{1}{8} \text{tr}(\Gamma^X \Gamma^b) \left(\delta_{ab} - \frac{1}{9} \Gamma_a \Gamma_b \right) \Gamma^Y + \frac{1}{8} \text{tr}(\Gamma^Y \Gamma^b) \left(\delta_{ab} - \frac{1}{9} \Gamma_a \Gamma_b \right) \Gamma^X \right\}_{C]D} \end{aligned} \quad (\text{A.14})$$

where $X = a$ or $[ab]$ and $Y = c$ or $[cd]$. From this, one derives

$$\Gamma_{[AB}^{[ab} \Gamma_{C]D}^{c]} - \Gamma_{[AB}^{[a} \Gamma_{C]D}^{b]c]} + \frac{1}{3} \Gamma_{[AB}^d \Gamma_{C]D}^{dabc} = 0 \quad (\text{A.15})$$

$$\Gamma_{[AB}^{ab} \Gamma_{C]D}^b - \Gamma_{[AB}^b \Gamma_{C]D}^{ab} - 4 \Gamma_{[AB}^a \Gamma_{C]D}^{\delta} = 0 \quad (\text{A.16})$$

In section 7 we have used the following formulas

$$\begin{aligned} \Gamma_{[AB}^c \Gamma_{CD]}^{ca} \Gamma_{AB}^d \Gamma_{CD}^{db} &= -192 \delta^{ab} \\ \Gamma_{[AB}^{[a} \Gamma_{CD]}^{bc]} \Gamma_{AB}^{[d} \Gamma_{CD]}^{ef]} &= -32 \delta_{def}^{abc} \\ \Gamma_{[AB}^a \Gamma_{CD]}^b \Gamma_{AB}^c \Gamma_{CD}^d &= \frac{16}{3} (\delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc} + \delta^{ab} \delta^{cd}) \\ \Gamma_{[AB}^a \Gamma_{CD]}^b \Gamma_{AB}^c \Gamma_{CD}^{cd} &= 0, \text{ etc.} \end{aligned} \quad (\text{A.17})$$

Finally we recall the Γ -matrix completeness relation

$$2 \Gamma_{AB}^a \Gamma_{CD}^a - \Gamma_{AB}^{ab} \Gamma_{CD}^{ab} = -16 \delta_{CD}^{AB} \quad (\text{A.18})$$