CERN-TH.4358/86

SYMMETRIES OF THE INTERACTING GAUGE-COVARIANT BOSONIC STRING

A. Neveu

CERN -- Geneva

and

P. West

King's College, Strand,
London WC2R 2LS, United Kingdom

ABSTRACT

We derive in detail the local gauge transformations and interaction Lagrangian of the second quantized bosonic string. We verify at lowest non-trivial order in the coupling constant the gauge invariance of the interacting action, and the closure of the field transformations. This gives the structure constants of the corresponding local gauge group. This group has its generators labelled by all the string states of the theory. We show that the fermionic part of the interaction is just the overlap condition for the ghost co-ordinates of the three strings, multiplied by an appropriate extra ghost at the point common to the three strings. Our treatment is based on an extensive use of Goto-Naka-type overlap equations. It avoids the use of ill-defined infinite matrices or series, and can be straightforwardly extended to other string theories. We explicitly derive the conformal transformation relating the gauge-covariant three-point function to the old Caneschi-Schwimmer-Veneziano vertex. We also deal with the closed bosonic string.

CERN-TH.4358/86
January 1986

1. - INTRODUCTION

Recently, the interacting gauge-covariant open bosonic string was found. In particular, the cubic interaction for the open bosonic string was given in a gauge-covariant formulation¹⁾ as well as in the BRS formulation²⁾. The gauge-invariant formulation used was that of Ref. 3), while the BRS formulation was found in Ref. 4). An alternative approach to gauge-covariant string theory can be found in Ref. 5). In Refs. 1) and 2), the strings are assigned lengths, denoted by α_r , which are conserved in string interactions. Three strings interact by two strings joining and splitting at their end-points, in the same way as they do in the second quantized light cone string field theory⁶⁾⁻⁹⁾.

In this paper, we derive the cubic interaction for the open bosonic string given in Ref. 1) in more detail. We also give the transformation law of the string field and show that it leaves the action invariant to order g^0 and leads to a closed algebra at this order. The general forms of these results are in agreement with the generic forms given in Ref. 10), where it was also shown how a gauge-covariant formulation of the free bosonic string could be found by the systematic introduction of supplementary fields.

There are several gauge-covariant formulations of the free bosonic¹¹⁾⁻¹⁴⁾ and supersymmetric^{11),15)} string theories; however, the simplest and easiest to handle at the free and interacting level is given in Ref. 3). This formulation consists of a string field χ defined on the space $x^\mu(\sigma)$, $c(\sigma)$, $\bar{c}(\sigma)$, where $\partial_\sigma x^\mu$, $\partial_\sigma c$ and $\bar{c}$ vanish at the ends of the string, i.e., at $\sigma = 0$ and π . The free action is invariant under $\delta|\chi\rangle = Q|\Lambda\rangle$ and is given by $\langle\chi|Q|\chi\rangle$, where Q is related to the BRS charge for the first quantized theory investigated by Kato and Ogawa¹⁶⁾, and $|\chi\rangle$ is subject to a suitable ghost number constraint. Thus, it appears that the first quantized string theory provides the setting for the second quantized gauge invariant string theory. The reader's attention is drawn to Ref. 3) for more details, and it is this formulation which is used in this paper. The open supersymmetric string is dealt with in Ref. 17). In fact, the form of the action given above is quite general, and applies to all string theories for the appropriate Q .

In Section 2, the gauge identities for vertices are discussed in general. Under the assumption that the vertex contains in co-ordinate space an overlap δ function, gauge identities are derived. These are sufficient to determine the cubic vertex, and its form is given. In Section 3, we prove that the bosonic

part of the vertex in the gauge-covariant theory is related to the old Caneschi-Schwimmer-Veneziano three-reggeon vertex by a conformal transformation, so that it gives the same couplings between three arbitrary on-shell physical states. In particular, the string lengths turn out to be pure gauge artifacts. The fermionic generalization of the Caneschi-Schwimmer-Veneziano vertex is given. It is related to the fermionic part of the gauge-covariant vertex by the same conformal transformation. It is apparent from the derivation that the covariant vertex satisfies Goto-Naka-like conditions, which means that it is an overlap δ function.

In the fourth section, the transformation law of χ is given. It is demonstrated that the same vertex given in Section 2 occurs in the transformation and the action, leading to a gauge-invariant action to order g^0 and closure at this order of the transformation laws.

Section 5 contains a derivation of the vertex in momentum space, where it is shown to be the overlap δ function for the three strings in the space $x^\mu(\sigma)$, $c(\sigma)$, $\bar{c}(\sigma)$ multiplied by appropriate ghost co-ordinates $c(\sigma)$ evaluated at the point common to the three strings. In Section 6, we extend our work to the free closed bosonic string, and we begin to discuss its interaction.

2. - GAUGE IDENTITIES AND THE VERTEX

The cubic vertex concerns the interaction of three strings. As explained in Ref. 1), we assign lengths α_r , $r = 1, 2, 3$ to the strings, and so they are parametrized by ξ_r ($0 < \xi_r < \alpha_r$). It is convenient to introduce another parametrization, namely η_r , where $\eta_r = (\pi/|\alpha_r|)\xi_r$ ($0 < \eta_r < \pi$). This method of parametrization is illustrated in Fig. 1, where the arrows indicate the direction in which the parameters increase. It is easily seen that the interaction point is given by $\eta_3 = \beta_2\pi$, where $\beta_1 = \alpha_1/|\alpha_3|$ and $\beta_2 = \alpha_2/\alpha_3$, and that the relation between the η_r 's is of the form

$$\eta_3 = \begin{cases} -\beta_2\eta_2 + \beta_2\pi & \text{for } 0 \leq \eta_2 \leq \beta_2\pi \\ -\beta_1\eta_1 + \pi & \text{for } \beta_2\pi \leq \eta_2 \leq \pi \end{cases} \quad (2.1)$$

If the vertex contains overlap δ functions in co-ordinate space, then the string co-ordinates denoted generically by $Z^r(\eta_r)$ will obey the relations

$$Z^3(\gamma_3) = \begin{cases} Z^1(\gamma_1) & \text{for } 0 \leq \gamma_3 \leq \beta_2 \pi \\ Z^2(\gamma_2) & \text{for } \beta_2 \pi \leq \gamma_3 \leq \pi \end{cases} \quad (2.2)$$

It is important to stress that the vertex may contain more than the overlap δ functions but we assume, at this point, that it is generically of the form

$$V = N \delta(Z^3(\gamma_3) - \theta(-\gamma_3 + \beta_2 \pi) Z^2(\gamma_2) - \theta(\gamma_3 - \beta_2 \pi) Z^1(\gamma_1)) \quad (2.3)$$

where N is some function of Z 's. In fact, we will see that N contains ghost co-ordinates evaluated at the interaction point. In particular, for the open bosonic string we have

$$Z^\lambda(\gamma) = \left\{ x_r^\lambda(\gamma), c^\lambda(\gamma), \frac{\bar{c}^\lambda(\gamma)}{\alpha^2} \right\}$$

By applying this equation, we find that the generic form of the identity is given by

$$V \sum_r \int_0^\pi d\gamma_r f(\gamma_r) S = 0 \quad (2.4)$$

where V is the vertex and S is a function of Z^r and $\delta/\delta Z^r$ and derivatives of these objects with respect to η_r . Under the change of variable from η_3 to η_1 and η_2 in the appropriate region, we assume that Q is adjusted to not scale.

As explained in Ref. 1), Eq. (2.4) is in general not well defined, since we must take a function f which has a convergent expansion in all three variables. Such a function is known to be a polynomial in $\bar{Z}$, where $\bar{Z}$ parametrizes the upper half plane onto which the string may be mapped. This map, introduced by Mandelstam¹⁸⁾, is given by

$$z = \alpha_1 \ln(z-1) + \alpha_2 \ln z \quad (2.5)$$

where the end-points of the three strings correspond to the points $z_1 = 1$, $z_2 = 0$, $z_3 = \infty$ respectively. The inverse¹⁸⁾ of this map is given by

$$z = \begin{cases} -\frac{1}{z_3} e^{-y_3(z_3)} & \text{for string 3} \\ e^{-\frac{1}{z_2} y_1(z_1)} & \text{for string 1} \\ z_2 e^{-\frac{1}{z_3} y_2(z_2)} & \text{for string 2} \end{cases} \quad (2.6)$$

where y_r is defined by the equation

$$y_r(x) = \gamma_r \ln(1 + x e^{\gamma_r}) = \sum_{n=1}^{\infty} \gamma_r f_n(\gamma_r) x^n \quad (2.7)$$

while

$$z_r = e^{i\gamma_r}, \quad \gamma_r = -\frac{\alpha_{r+1}}{\alpha_r}, \quad (\alpha_{3+1} \equiv \alpha_1)$$

It can be shown¹⁸⁾ that

$$f_n(\gamma) = \frac{1}{n\gamma} \binom{n\gamma}{n} = \frac{(n\gamma-1) \dots (n\gamma-n+1)}{n!}$$

We suspect that this is the unique type of function that has convergent expansions in z_r for all three strings.

While one could take any polynomial in Z , it is most convenient to take one which has only a z^{-n} pole in z . Such functions are of the form

$$F_m(z) = \sum_{p=0}^{n-1} \binom{n\gamma_3}{p} (-z)^{n-p} \quad (2.8)$$

To derive the gauge identities, we need to know the expansion of $F_m(Z)$ in terms of z_r for each r ; this is given by the following theorem:

Theorem

$$F_m(z) = \begin{cases} \frac{1}{z_3^m} - m \gamma_3 f_m(z_3) - m \sum_{n=1}^{\infty} N_{mn}^{33} z_3^n & \text{for string 3} \\ \sum_{p=0}^{m-1} \binom{m \gamma_2}{p} (-1)^{m-p} - m \sum_{n=1}^{\infty} N_{mn}^{31} z_1^n & \text{for string 1} \\ -m \sum_{n=1}^{\infty} N_{mn}^{32} z_2^n & \text{for string 2} \end{cases} \quad (2.9)$$

where

$$N_{mn}^{rs} = - \frac{\alpha_1 \alpha_2 \alpha_3}{m \alpha_r + n \alpha_s} \times \frac{mn}{\alpha_r \alpha_s} f_m(\gamma_r) f_n(\gamma_s) \quad (2.10)$$

The expansion coefficients N_{mn}^{rs} are none other than the Fourier transform of the Neumann functions which play such a central role in string vertices.

We will now prove this for $m = 1$, the method of proof for higher m then becoming clear. Let us first consider therefore $F_1(Z) = -Z$ and for the third string $-Z = (1/z_3)e^{-y_3(z_3)}$. From the definition of y in Eq. (2.7), we can derive the differential equation

$$e^{-y_r} \frac{dy_r}{dz_r} = \gamma_r + (\gamma_r - 1) z_r \frac{dy_r}{dz_r} \quad (2.11)$$

from which it follows that

$$\frac{d}{dz_r} e^{-y_r} = -\gamma_r - (\gamma_r - 1) \gamma_r \sum_{n=1}^{\infty} n f_n(\gamma_r) z_r^n \quad (2.12)$$

for any r . In particular, integrating this equation for $r = 3$, we find that

$$e^{-y_3} = 1 - \gamma_3 z_3 - \gamma_3(\gamma_3 - 1) \sum_{n=1}^{\infty} \frac{n}{n+1} f_n(\gamma_3) z_3^{n+1} \quad (2.13)$$

which agrees with the expansion for string 3 in Eq. (2.9).

For string 1, we must expand $-Z = -e^{(1/\gamma_1)y_1}$. Let us define $z_1 = s^{(\gamma_1-1)}$; using Eq. (2.11), one may show that

$$\frac{d}{ds} [s(e^{y_1/\gamma_1} - 1)] = z_1 \frac{dy_1}{dz_1} = \gamma_1 \sum_{n=1}^{\infty} n f_n(\gamma_1) s^{n(\gamma_1-1)} \quad (2.14)$$

Consequently,

$$e^{\frac{y_1}{\gamma_1}} = 1 + \sum_{n=1}^{\infty} \frac{n \gamma_1 f_n(\gamma_1)}{n(\gamma_1-1) + 1} z_1^n \quad (2.15)$$

which leads to the desired result.

Finally, for string 2 we must expand $-Z = z_2 \exp\{(\gamma_2-1/\gamma_2)y_2\}$. The proof is similar to that for string 1, except now we consider $t = z_2^{\gamma_2}$, and we find that

$$t^2 \frac{d}{dt} \left(\frac{z_2}{t} e^{\frac{\gamma_2-1}{\gamma_2} y_2} \right) = z_2 \frac{\gamma_2-1}{\gamma_2} \frac{dy_2}{dz_2} = \frac{\gamma_2-1}{\gamma_2} \sum_{n=1}^{\infty} n f_n(\gamma_2) t^{n\gamma_2} \quad (2.16)$$

The derivation of the required result is apparent from this line. A general proof of these expansions will be given elsewhere.

As explained in Ref. 1), one may obtain cycled identities by making the replacements

$$Z \rightarrow \frac{1}{1-Z} \quad \text{and} \quad Z \rightarrow -\frac{1}{Z} + 1 \quad (2.17)$$

in $F_m(Z)$.

Finally, we mention a subtlety associated with the gauge identities which concerns the treatment of the interaction points. For reasons that will be explained for each case, it is sometimes necessary to consider functions $f(Z)$ which vanish at the interaction point, in which case we consider the polynomials

$$\bar{F}_m(Z) = F_m(Z) - F_m(Z_{int}) \quad (2.18)$$

We note that

$$Z_{int} = \frac{1}{\gamma_2} = 1 - \gamma_3 = \frac{\gamma_1}{\gamma_1 - 1} \quad (2.19)$$

The above discussion is quite general and may be applied to any string or superstring theory. We now illustrate it for the gauge-covariant formulation of the bosonic open string which has a cubic vertex denoted by V . This vertex can be written in the form

$$V = V^B V^F \quad (2.20)$$

where V^B involves only the bosonic co-ordinates $x^\mu(\eta)$, and V^F involves only the fermionic co-ordinates $c(\eta)$ or $\bar{c}(\eta)$. In the oscillator basis, they are functions of $\alpha_n^{\mu r}$ and β_n^r , $\bar{\beta}_n^r$ respectively.

Let us first take Q to be

$$P_\mu^2(\eta_2) = \frac{1}{\pi\sqrt{2\alpha'}} \left(-it \frac{\delta}{\delta x_n^r} + \frac{1}{2\pi\alpha'} \frac{dx_n^r}{d\eta_2} \right) \quad (2.21)$$

In terms of $\alpha_n^{\mu r}$ oscillators, we have

$$P_{\mu}^r(\gamma_r) = \sum_{n=-\infty}^{+\infty} \alpha_n^{\mu r} z_r^{-n} \quad (2.22)$$

We may verify that P_{μ}^r does scale appropriately, and since there is no need in this case to subtract at the interaction point, the corresponding identity when written in terms of z_r rather than η_r is of the form

$$V \sum_{r=1}^3 \oint \frac{dz_r}{z_r} F_m(z) P_{\mu}^r(z_r) = 0 \quad (2.23)$$

Using Eq. (2.9), one easily finds that the α 's obey the identity

$$V \left[\alpha_{-m}^{\mu r} - m \gamma_r f_m(\gamma_r) \alpha_0^{\mu r} - m f_m(\gamma_r) \alpha_0^{\mu r+1} - m \sum_{\substack{n=1 \\ s=1,2,3}}^{\infty} N_{mn}^{\mu s} \alpha_n^{\mu s} \right] = 0 \quad (2.24)$$

The vertex which satisfies this equation is easily seen to be

$$V^B = \langle 0_{123} | \exp \left[\frac{1}{2} \alpha_m^{\mu r} N_{mn}^{\mu s} \alpha_m^{\mu s} + N_{mn}^{\mu s} \alpha_m^{\mu r} \alpha_0^{\mu s} \right] \\ \times \exp \tau \sum_n \frac{L_n - 1}{\alpha_n} \quad (2.25)$$

where

$$N_m^{\mu s} = \frac{1}{\alpha_r} f_m(\gamma_r) [\alpha_r \delta_{r+1,s} - \alpha_{r+1} \delta_{r,s}] \quad (2.26)$$

$$\alpha_0^r = p^r \sqrt{2\alpha'} , \quad < \alpha_{1,23} > = < \alpha_1 > < \alpha_2 > < \alpha_3 > , \quad (2.27)$$

$$\tau = \alpha_1 L_n \left(-\frac{\alpha_1}{\alpha_3} \right) + \alpha_2 L_n \left(-\frac{\alpha_2}{\alpha_3} \right)$$

Although we find that the factor $\exp \tau \Sigma_t (L_0^t - 1/\alpha_t)$ is required to obtain correct four-point functions, we will take the effects of this factor in gauge identities and vertices to be understood in order not to overcomplicate expressions. We could also apply the above strategy to the light cone formulation; in this case we find the above vertex, the only difference being that $\mu = 0, 1, \dots, 23$.

In the later sections, we will need the action of

$$L_n = \frac{\alpha'}{2i} : \oint \frac{dz}{z} \tilde{z}^n P_\mu^2 : \quad (2.28)$$

on V . This identity may be deduced directly, or from Eq. (2.24). Care must be taken, as L is normal-ordered. In this case, one must use the functions $\bar{F}_m(Z)$, which vanish at the interaction point. This is because P^{μ^2} is badly defined at the interaction point due to the occurrence of the step functions and derivatives in P^μ . The L_n 's do not scale with unity but $(L_n^{(r)}/\alpha^r)$ does. Applying the above reasoning, we find the identity

$$V \left[\frac{L_{-m}^r}{\alpha_r} - m \gamma_r f_m(\gamma_r) \frac{L_0^r}{\alpha_r} - m f_m(\gamma_r) \frac{L_0^{r+1}}{\alpha_{r+1}} - m \sum_{\substack{n=1 \\ s=1,2,3}}^{\infty} N_{mn}^{rs} \frac{L_n^s}{\alpha_s} \right. \\ \left. - \sum_m^r \sum_{s=1}^3 \frac{L_0^s}{\alpha_s} - \frac{D}{\alpha_2} \sum_{n=1}^{m-1} \frac{m(n-m)}{2} N_{m,n-m}^{22} \right] = 0 \quad (2.29)$$

where

$$S_m^r = \sum_{p=0}^{m-1} \binom{m}{p} (\gamma_2)^{m-p} \quad (2.30)$$

The constant term comes about since L is normal-ordered and its value may easily be checked by taking $|0\rangle_{123}$ on the left-hand side of (2.29).

We now turn to the determination of the V^F part of the vertex by considering the $\beta, \bar{\beta}$ identities

$$\beta(\gamma) = \frac{1}{\pi\sqrt{2}} \sum_{n=-\infty}^{+\infty} \beta_n z^{-n} = \frac{\delta}{\delta c(\gamma)} - \frac{1}{2\pi} \bar{c}(\gamma) \quad (2.31)$$

$$\bar{\beta}(\gamma) = \frac{1}{\pi\sqrt{2}} \sum_{n=-\infty}^{+\infty} \bar{\beta}_n z^{-n} = -\frac{\delta}{\delta \bar{c}(\gamma)} + \frac{1}{2\pi} c(\gamma) \quad (2.32)$$

As explained in Refs. 1) and 2), the part of the vertex involving c and $\bar{c}$ must contain three c ghosts at the vertex in order to recover the Lorentz covariant generalization of the light cone result. Due to the $\delta/\delta c$ derivatives in $\bar{\beta}$, we must insist that it satisfy an interaction-point-subtracted identity. The β identity, on the other hand, is unsubtracted. It is easily seen that $\bar{\beta}^n$ and $\alpha^r \beta^r$ scale with unity, and hence we have the identities

$$V \left[\bar{\beta}_{-m}^2 - m \gamma_2 f_m(\gamma_2) \bar{\beta}_0^2 - m f_m(\gamma_2) \bar{\beta}_0^{2+1} - m \sum_{\substack{n=1 \\ \Delta=1,2,3}}^{\infty} N_{mn}^{\Delta} \bar{\beta}_n^{\Delta} - S_m^r \sum_{\Delta=1}^3 \bar{\beta}_0^{\Delta} \right] = 0 \quad (2.33)$$

$$V \left[\alpha_r \beta_{-m}^2 - m \gamma_2 f_m(\gamma_2) \alpha_2 \beta_0^2 - m f_m(\gamma_2) \alpha_{2+1} \beta_0^{2+1} - m \sum_{\substack{n=1 \\ \Delta=1,2,3}}^{\infty} N_{mn}^{\Delta} \alpha_r \beta_n^{\Delta} \right] = 0 \quad (2.34)$$

as well as

$$V \sum_{n=1}^3 \alpha_n \beta_0^2 = 0 \quad (2.35)$$

From Eq. (2.33) we can deduce the vertex; it is found to be

$$V^F = \langle 0_{123}^+ | \exp \left[- \beta_n^r n N_{nm}^{rs} \bar{\beta}_m^s + R_n^{rs} \beta_n^r \bar{\beta}_0^s \right] \quad (2.36)$$

where

$$R_m^{rs} = -m f_m(\gamma_r) (\gamma_r \delta_{r,s} + \delta_{r+1,s}) - S_r^m,$$

$$\langle 0_{123}^+ | = \langle 0_1^+ | \langle 0_2^+ | \langle 0_3^+ | \quad (2.37)$$

It is easily verified that this vertex does satisfy Eqs. (2.34) and (2.35).

Consequently, the vertex¹⁾ for the open bosonic string is given by Eqs. (2.36) and (2.26). These have been derived under the assumption that the vertex contains in co-ordinate space an overlap δ function. Important in the derivation has been whether the identity for a given object is subtracted at the interaction point. It is not clear that this is equivalent to the vertex required in an invariant theory; however, in Section 4 we will show that this is indeed the case. We recall¹⁾ that the above vertex does indeed yield the correct interacting theory at lowest order.

3. - CONFORMAL TRANSFORMATIONS AND PHYSICAL COUPLINGS

In order to show that the vertex V^B of Eq. (2.25) is a reasonable starting point for an interacting gauge-invariant string field theory, one must show that it gives rise to the correct coupling between three arbitrary on-shell physical states. Physical states satisfy

$$(L_0 - 1)|\psi\rangle = L_n|\psi\rangle = 0, \quad n=1, 2, 3, \dots \quad (3.1)$$

with all supplementary fields vanishing. On the other hand, their couplings are known to be given by the Caneschi-Schwimmer-Veneziano vertex

$$V^{CSV} = \langle 0_{123} | \exp - \sum_{m=0}^{\infty} \sum_{n=1}^3 \frac{\alpha_m^{\mu} \alpha_n^{\mu} (-1)^n \Gamma(n)}{\Gamma(1+m) \Gamma(1+n-m)} \quad (3.2)$$

By a straightforward calculation, pushing the creation operators to the left, one finds

$$V^{CSV} \left[A \sum_{n=-\infty}^{+\infty} \alpha_n^{\mu} (1-z)^n + B \sum_{n=-\infty}^{+\infty} \alpha_n^{\mu} \left(\frac{z-1}{z}\right)^n + C \sum_{n=-\infty}^{+\infty} \alpha_n^{\mu} z^{-n} \right] = 0 \quad (3.3)$$

provided that

$$-\frac{A}{z} + B - \frac{C}{1-z} = 0 \quad (3.4)$$

and that the binomial series encountered in the course of the calculation converge. This convergence implies that one of the three coefficients A,B,C must vanish, otherwise there is at least one divergent series. Suppose that B = 0. We can then take

$$C = -\gamma_2 \frac{1-z}{1-\gamma_2 z}, \quad A = \gamma_2 \frac{z}{1-\gamma_2 z} \quad (3.5)$$

Using the variables z_3 and s of Eqs. (2.13) and (2.14), we remark that

$$\frac{1-z}{1-\gamma_2 z} = -\gamma_2 \frac{z_3}{z} \frac{dz}{dz_3}, \quad -\frac{z}{1-\gamma_2 z} = \gamma_2 s(1-z) \frac{d(\frac{1}{1-z})}{ds} \quad (3.6)$$

We then perform the conformal transformations on lines 1 and 3 from $\bar{z}$ to s and z_3 . From (3.6), we see that the functions $C(z)$ and $A(z)$ are precisely reabsorbed by the transformation of $\tilde{P}(\bar{z})$ and $\tilde{P}(1/(1-\bar{z}))$; for example

$$\tilde{P}(\bar{z}) \equiv \sum_m \alpha_m^{\mu} \bar{z}^{-m} = \frac{z}{z_3} \frac{dz_3}{dz} P^{\mu}(z_3) \quad (3.7)$$

and Eq. (3.3) becomes the overlap condition between strings 1 and 3 as derived in the previous section. Similarly, taking $A = 0$, one obtains the overlap condition between strings 2 and 3. This shows that indeed V^B and V^{CSV} differ only by conformal transformations on the three legs and give the same couplings between three arbitrary on-shell physical states. These conformal transformations are those of the previous section, mapping the interacting string configuration of Fig. 1 onto the upper half plane. To obtain the explicit form, in terms of L_n , $n > 0$, of the operator which maps V^{CSV} into V^B , we start from the trivial identity:

$$V^{CSV} \left[\alpha_{-1}^{\mu_1} - \sum_{n=1}^{\infty} \alpha_n^{\mu_2} + \alpha_0^{\mu_3} - \alpha_1^{\mu_3} \right] = 0 \quad (3.8)$$

and we look for an operator $\exp \sum_{\substack{n=1 \\ r=1,2,3}}^{\infty} a_n^r L_n^r$, such that

$$\begin{aligned} & \exp \left[- \sum_{\substack{n=1 \\ r=1,2,3}}^{\infty} a_n^r L_n^r \right] \left[\alpha_{-1}^{\mu_1} - \sum_{n=1}^{\infty} \alpha_n^{\mu_2} + \alpha_0^{\mu_3} - \alpha_1^{\mu_3} \right] \exp \sum_{\substack{n=1 \\ r=1,2,3}}^{\infty} a_n^r L_n^r \\ &= \alpha_{-1}^{\mu_1} + \gamma_1 \alpha_0^{\mu_1} - \alpha_0^{\mu_2} - \sum_{\substack{n=1 \\ r=1,2,3}}^{\infty} N_{1n}^{12} \alpha_n^{\mu_2} \end{aligned} \quad (3.9)$$

The solution for the first few a_n^r coefficients of this equation can be found in Eq. (6) of Ref. 1).

Having found the conformal correspondence of V^B and V^{CSV} for the orbital degrees of freedom, it is natural to ask whether the analogue of V^{CSV} exists for the ghosts β and $\bar{\beta}$. In the previous section, the ghost part of the vertex was found by imposing the continuity equations (2.33) and (2.34) for $\bar{\beta}(z)$ and $\beta(z)$. For $\bar{\beta}(z)$, Eq. (2.33) is essentially the same as Eq. (2.29) for $P^2(z)$. On the other hand, it is straightforward to derive the identities for passing the Virasoro generators through V^{CSV} . One finds

$$V^{CSV} \left\{ L_{-n}^1 - L_0^1 - (-1)^n \sum_{p=0}^{\infty} L_{n-p}^3 \frac{\Gamma(n+2)}{\Gamma(n+2-p)} \frac{(-1)^p}{p!} + L_0^3 - (-1)^n \sum_{p=0}^{\infty} L_{n+p}^2 \frac{\Gamma(n+1+p)}{\Gamma(n+1)} \frac{1}{p!} + L_0^2 - L_1^2 \right\} = 0 \quad (3.10)$$

Replacing L_n by $\bar{\beta}_n$, one thus finds the ghost analogue of the Caneschi-Schwimmer-Veneziano vertex:

$$V^{CSV}(\beta, \bar{\beta}) = \langle 0_{123} + \exp \left\{ \sum_{n=1}^{\infty} \bar{\beta}_n^1 \left[-\bar{\beta}_0^1 - (-1)^n \sum_{p=0}^{\infty} \bar{\beta}_{n-p}^3 \frac{(-1)^p \Gamma(n+2)}{\Gamma(n+2-p) \Gamma(p+1)} + \bar{\beta}_0^3 - (-1)^n \sum_{p=0}^{\infty} \bar{\beta}_{n+p}^2 \frac{\Gamma(n+1+p)}{\Gamma(n+1) \Gamma(p+1)} + \bar{\beta}_0^2 - \bar{\beta}_1^2 \right] \right\} \right\rangle + \text{cyclic permutations} \quad (3.11)$$

This vertex can then be checked to satisfy continuity conditions for $\beta(z)$ and $\bar{\beta}(z)$ quite analogous to those of the orbital $P(z)$, Eq. (3.3), namely

$$V^{CSV}(\beta, \bar{\beta}) \left[A' \sum_n \bar{\beta}_n^1 (1-z)^n + B' \sum_n \bar{\beta}_n^2 \left(\frac{z-1}{z} \right)^n + C' \sum_n \bar{\beta}_n^3 z^{-n} \right] = 0 \quad (3.12)$$

provided that

$$\frac{A'}{z^2} + B' + \frac{C'}{(1-z)^2} = 0 \quad (3.13)$$

and

$$V^{CSV}(\beta, \bar{\beta}) \left[A'' \sum_n \beta_n^1 (1-z)^n + B'' \sum_n \beta_n^2 \left(\frac{z-1}{z}\right)^n + C'' \sum_n \beta_n^3 z^{-n} \right] = 0 \quad (3.14)$$

provided that

$$-A''z + B'' - C''(1-z) = 0 \quad (3.15)$$

We note the analogy between Eqs. (3.4), (3.13) and (3.15). The difference between them is entirely due to the different conformal dimensions of $\tilde{P}(z)$, $\bar{\beta}(z)$ and $\beta(z)$. The ghost vertex V^F of Eq. (2.36) is related to $V^{CSV}(\beta, \bar{\beta})$ by the same conformal transformation acting on β and $\bar{\beta}$ as V and V^{CSV} . The Virasoro operators acting on β and $\bar{\beta}$ can be found, for example, in Ref. 19).

Actual calculations of string scattering in the tree or one-loop approximation are much more easily performed with the old framework of Feynman-like diagrams²⁰⁾ than with the light-cone Lagrangian of Refs. 6) and 7). The two are related by conformal transformations^{6), 7), 18)} on the integration variables of the propagators of the internal lines. Hence, we suggest that in practical calculations, one should use $V^{CSV}(\beta, \bar{\beta})$ to compute the ghost contribution to an arbitrary Feynman-like diagram, with an arbitrary number of loops; in the one-loop case, this reduces to the calculation of Ref. 21); for physical external states, the ghosts provide the modification of the integration measure needed to project out the unphysical states which are not properly dealt with in the internal lines when using the formalism of Ref. 20).

In this section, we have seen the direct connection between the covariant vertex and the Caneschi-Schwimmer-Veneziano vertex, and we have worked out the conformal map which links the two. We can then consider starting from another conformal map. The covariant vertex of the previous section has the disadvantage of requiring the external strings to carry an unphysical parameter, which is the width of the strips of Fig. 1. Instead, we could start from the vertex described by Fig. 2. The conformal map taking Fig. 2 to the upper half plane is given by

$$\frac{dz}{dz'} \propto \frac{[(z-z'_1)(z-z'_2)(z-z'_3)]^{1/3}}{(z-z_1)(z-z_2)(z-z_3)} \quad (3.16)$$

The points $z_1, z_2, z_3, z_1', z_2', z_3'$ are the real images of the points $Z_1, Z_2, Z_3, Z_1', Z_2', Z_3'$ in Fig. 2. In Eq. (3.16) we can, by Moebius invariance, choose $z_1 = 1, z_2 = 0, z_3 = \infty$ as in Section 2, and one then has

$$\frac{dz}{dz} = \frac{[(z+1)(z-2)(z-\frac{1}{2})]^{1/3}}{(z-1)z} \quad (3.17)$$

In the vicinity of each of the points z_1, z_2, z_3 this mapping and its inverse enjoy the same analyticity properties as that of Eq. (2.5), and one can repeat the analysis of this section to find the conformal factors by which such a vertex would differ from V^{CSV} , or the calculations of the previous section to find it directly in oscillator form. An obvious advantage of this vertex is that all three strings have the same width, π . The disadvantage is that the standard lore built around the light-cone vertex, on which we draw in this paper, is lost, and it is not clear what higher point functions would be needed in the action. Further investigation of this possible direction goes beyond the scope of this work, and is left for future study.

4. - INVARIANCE AND CLOSURE

In Ref. 10), the generic form of the action and transformation laws in a gauge-covariant string theory was given. In terms of the formulation of Ref. 3), the action must be of the form

$$A = \frac{1}{2} \langle \chi | Q | \chi \rangle + \frac{1}{3} g V | \chi_1 \rangle | \chi_2 \rangle | \chi_3 \rangle \quad (4.1)$$

and the transformation law of the generic form

$$\delta \langle \chi_3 | = \frac{1}{g} \langle \chi_3 | Q^3 + V' | \chi_1 \rangle | \chi_2 \rangle - V'' | \chi_1 \rangle | \chi_2 \rangle \quad (4.2)$$

The question arises as to whether the vertex V' in the transformation law is the same as the vertex V in the action. In fact, the V', V'' and V can be deduced from a knowledge of the bosonic part V^B of V alone. Using this procedure, in Ref. 1) the expressions for $\delta \langle \chi |$ are given at lowest order. The reader may readily verify that these transformation laws are reproduced by the equation

$$\delta \langle \chi_3 | = \frac{1}{g} \langle \Lambda_3 | Q^3 + V \left[\frac{\alpha_2}{\alpha_3} |\chi_1\rangle |\Lambda_2\rangle - \frac{\alpha_1}{\alpha_3} |\Lambda_1\rangle |\chi_2\rangle \right] \quad (4.3)$$

This result requires us to assign the $|\rangle_-$ vacuum to be odd in the sense of anti-commuting. The $|\rangle_+$ vacuum must then be even. We adopt this to be the correct transformation rule and examine the criterion for invariance. The variation of the action of Eq. (4.1) at order g^0 under Eq. (4.3) is

$$\begin{aligned} \delta A = & \frac{1}{3} V \left[\frac{\alpha_2}{\alpha_3} |\chi_1\rangle |\Lambda_2\rangle Q^3 |\chi_3\rangle - \frac{\alpha_1}{\alpha_3} |\Lambda_1\rangle |\chi_2\rangle Q^3 |\chi_3\rangle + \text{cyclic} \right] \\ & + \frac{1}{3} V \left[Q^1 |\Lambda_1\rangle |\chi_2\rangle |\chi_3\rangle + |\chi_1\rangle Q^2 |\Lambda_2\rangle |\chi_3\rangle + |\chi_1\rangle |\chi_2\rangle Q^3 |\Lambda_3\rangle \right] \end{aligned} \quad (4.4)$$

As a result, the necessary and sufficient condition for invariance is

$$V \sum_{t=1}^3 \frac{Q^t}{\alpha_t} = 0 \quad (4.5)$$

We can also consider the criterion for closure. Carrying out the commutator of two transformations on $\langle \chi |$, we find at order g^{-1}

$$[\delta_{\Lambda'}, \delta_{\Lambda}] \langle \chi_3 | = \frac{1}{g} V \left[\frac{\alpha_2}{\alpha_3} Q^1 |\Lambda'_1\rangle |\Lambda_2\rangle + \frac{\alpha_1}{\alpha_3} Q^2 |\Lambda'_1\rangle |\Lambda_2\rangle \right] \quad (4.6)$$

From the lowest order results, one finds that the closure at this order is consistent with the result

$$[\delta_{\Lambda'}, \delta_{\Lambda}] \langle \chi_3 | = -\frac{\alpha_1 \alpha_2}{g \alpha_3^2} V (|\Lambda'_1\rangle |\Lambda_2\rangle - \Lambda \leftrightarrow \Lambda') Q^3 \quad (4.7)$$

Taking this to be the result to all levels, we find that the condition for closure at order g^{-1} is Eq. (4.5). Thus the conditions for invariance and closure are the same, and this confirms Eqs. (4.3) and (4.7) to all levels.

It now only remains to show that V does indeed satisfy this condition. We recall that

$$\begin{aligned} Q &= : \sum_{n=-\infty}^{+\infty} \beta_n L_{-n} - \sum_{n,m,p} \frac{1}{2} \bar{\beta}_p \delta_{n+m}^p \beta_{-n} \beta_{-m} - \beta_0 : \\ &= : \sum_{n=-\infty}^{+\infty} \beta_n \mathcal{L}_{-n} - \beta_0 : \end{aligned} \quad (4.8)$$

where

$$\mathcal{L}_n = L_n + \frac{1}{2} L_n^\beta$$

and

$$L_n^\beta = - \sum_m : (n+m) \beta_m \bar{\beta}_{n-m} :$$

We note that Q is a normal-ordered version of the BRS charge corresponding to the Virasoro algebra, and that it satisfies $Q^2 = 0$ in $D = 26^{16)}$. In functional language, Q is given by

$$Q = \frac{\pi}{2i\sqrt{2}} \oint \frac{dz}{z} \beta \left[P^2 + z \frac{d}{dz} (\beta \bar{\beta}) + \beta \frac{d}{dz} \beta \right] \bar{\beta} \quad (4.9)$$

Clearly this form does satisfy Eq. (4.5) in functional language, but this ignores all questions of normal ordering, as indeed does Eq. (4.9) itself.

We now consider the proof of Eq. (4.5) in the oscillator formalism. We must show that

$$V \sum_{\substack{n=-\infty \\ n=1,2,3}}^{+\infty} : \left[\beta_{-n}^2 \frac{\mathcal{L}_n^2}{\alpha_n} - \frac{\beta_0^2}{\alpha_n} \right] : = 0 \quad (4.10)$$

Now L_n^β obeys a similar relation to L_n , namely Eq. (2.29), except that the constant due to L_n^β being normal-ordered is different; this identity reads

$$\begin{aligned}
 V \left[\frac{L_{-n}^{\beta_2}}{\alpha_2} - m \gamma_2 f_n(\gamma_2) \frac{L_0^{\beta_2}}{\alpha_2} - m f_n(\gamma_2) \frac{L_0^{\beta_{2+1}}}{\alpha_{2+1}} - \frac{m}{\alpha_2} R_m^{\beta_2} \right. \\
 \left. + \frac{1}{\alpha_2} \sum_{n=1}^{n-1} (n+m)(n-m) N_{m, n-m}^{\beta_2} - m \sum_{n=1}^{\infty} N_{mn}^{\beta_2} \frac{L_n^{\beta_2}}{\alpha_2} - \sum_{n=1}^{\infty} \sum_{s=1}^3 \frac{L_n^{\beta_s}}{\alpha_s} \right] = 0
 \end{aligned}
 \quad (4.11)$$

In deriving this equation, we have normal-ordered expressions involving anticommuting variables. When performing such normal ordering, we must, when moving anticommuting creation operators to the left past another anticommuting operator, assign a minus sign; for example,

$$: \beta_m^- \beta_n^+ : = - \beta_n^+ \beta_m^- \quad \text{for } m, n > 0
 \quad (4.12)$$

Normal-ordering Q before we take it through V , Eq. (4.10) becomes

$$V \sum_{r=1,2,3} \left[\sum_{m=1}^{\infty} \frac{\beta_m^r \alpha_m^r}{\alpha_r} + \beta_0^r \frac{\alpha_0^r}{\alpha_r} + \frac{\alpha_{-m}^r}{\alpha_r} \beta_m^r - \frac{\beta_0^r}{\alpha_r} \right] = 0
 \quad (4.13)$$

Using Eqs. (2.29) and (4.11) to replace α_{-m}^r and dragging β_{-m}^r and β_0^r explicitly through V , we find the following four terms:

$$\begin{aligned}
 V \{ & [-\beta_n^{\beta_2} m N_{nm}^{\beta_2} \frac{\alpha_n^{\beta_2}}{\alpha_2}] + [R_n^{\beta_2} \beta_n^{\beta_2} \frac{\alpha_0^{\beta_2}}{\alpha_2}] \\
 & + [m \gamma_2 f_n(\gamma_2) \frac{\alpha_0^{\beta_2}}{\alpha_2} + m f_n(\gamma_2) \frac{\alpha_0^{\beta_{2+1}}}{\alpha_{2+1}} + m N_{mn}^{\beta_2} \frac{\alpha_n^{\beta_2}}{\alpha_2} \\
 & + \sum_n^m \sum_{s=1}^3 \frac{\alpha_0^{\beta_s}}{\alpha_s} + \frac{D}{2\alpha_2} \sum_{p=1}^{n-1} N_{m, p-m}^{\beta_2} p(m-p)
 \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{2} \sum_{p=1}^{m-1} N_{p, m-p}^{\alpha\alpha} (m-p)(m+p) + \frac{m}{2} R_m^{\alpha\alpha} \Big] \beta_m^{\alpha} \\
 & + \left[m f_m(\gamma_2) \left(\gamma_2 - \frac{1}{\gamma_2} \right) + \alpha_2 S_m^{\alpha} \sum_{s=1}^3 \frac{1}{\alpha_s} \right] \frac{\beta_m^{\alpha}}{\alpha_2} \Big\} = 0
 \end{aligned}$$

(4.14)

Examining this expression, in which summation signs are to be understood, we find that the terms in $\mathcal{L}$ would cancel if the β_n^s 's were always on the same side of $\mathcal{L}$. Using the relation

$$[\beta_n^{\alpha}, \alpha_m^{\alpha}] = \frac{1}{2} (2m+n) \beta_{n+m}^{\alpha} \delta_{\alpha\alpha} \quad (4.15)$$

we find that

$$\begin{aligned}
 & V \left\{ \frac{\beta_n^{\alpha}}{\alpha_2} \left[\sum_{m=1}^{n-1} \frac{D}{2} m(n-m) N_{m, n-m}^{\alpha\alpha} - \sum_{m=1}^{n-1} (n+m) N_{m, n-m}^{\alpha\alpha} \right. \right. \\
 & \quad - \gamma_2 n^2 f_n(\gamma_2) - n S_n^{\alpha} + n f_n(\gamma_2) \left(\gamma_2 - \frac{1}{\gamma_2} \right) \\
 & \quad \left. \left. + \alpha_2 S_n^{\alpha} \sum_{s=1}^3 \frac{1}{\alpha_s} \right] \right\} = 0
 \end{aligned} \quad (4.16)$$

Upon using the identity

$$S_n^2 = \sum_{m=1}^{n-1} m N_{m,n-m}^{\alpha\alpha} + (\gamma_2 - 1) n f_n(\gamma_2) \quad (4.17)$$

the above equation can be written in the form

$$\begin{aligned} V \left\{ \frac{\beta_n^2}{\alpha_n} \left[\sum_{m=1}^{n-1} N_{m,n-m}^{\alpha\alpha} (m^2 (1 - \frac{D}{2}) + n^2 (\frac{D}{4} - 2) \right. \right. \\ \left. \left. + n \alpha_2 \sum_{t=1}^3 \frac{1}{\alpha_t} + n(n-1) (1-2\gamma_2) f_n(\gamma_2) \right] \right\} = 0 \end{aligned} \quad (4.18)$$

Recalling that we may write

$$N_{n-m,m}^{\alpha\alpha} = \gamma_2(\gamma_2-1) \frac{m(n-m)}{n} f_{n-m}(\gamma_2) f_m(\gamma_2) \quad (4.19)$$

one may verify that Eq. (4.18) is satisfied provided $D = 26$. This completes the proof of the invariance of the action and the closure at this level.

We now demonstrate that the cubic vertex given in Ref. 1) and re-examined in this paper is indeed the same as that stated in Ref. 2). Using Eq. (4.17), the fermionic part of our vertex V^F given in Eq. (2.36) can be rewritten in the form

$$\begin{aligned} V^F = & \langle 0+ | \exp \left\{ - \sum_{\substack{n,m \\ r,s}} \beta_n n N_{n,m}^{\alpha\alpha} \bar{\beta}_m^{\alpha} \right. \\ & + \sum_{r,s,n} \bar{\beta}_0^{\alpha} \beta_n^{\alpha} n f_n(\gamma_2) \left[\frac{\alpha_{r-1} - \alpha_{r+1}}{\alpha_r} \delta_{r,s} \right. \\ & \left. \left. + \frac{\alpha_r}{\alpha_{r+1}} \delta_{r+1,s} - \frac{\alpha_r}{\alpha_{r-1}} \delta_{r-1,s} \right] \right\} \end{aligned} \quad (4.20)$$

which can be seen to agree with that of Ref. 2).

The only point that remains to finish the construction of the open bosonic string is to complete the closure and invariance to all orders in g . As explained

in Ref. 1), we expect the occurrence of a four-point function but no higher-point function in the action. This can be seen either as a requirement of invariance of the action, or from demanding that the four-point function possess duality. The latter method would follow closely the light cone calculation of Refs. 6) and 7), which deduced the existence of the light cone four-point function for the open string. Many features of this calculation are the same, except that now we must also compute the contribution from the anticommuting oscillators. It is apparent that the integration regions will not match and a quartic interaction will be required. This calculation will be reported elsewhere.

For the algebra, we suspect that

$$\delta \langle \chi_3 | = \frac{1}{g} \langle \Lambda_3 | Q^3 + V(1,2,3) \left[\frac{\alpha_2}{\alpha_3} |\chi_1\rangle |\Lambda_2\rangle - \frac{\alpha_1}{\alpha_3} |\chi_2\rangle |\Lambda_1\rangle \right] \quad (4.21)$$

holds to all orders in g and that the commutator of two Λ transformations is given to all orders by

$$[\delta_{\Lambda'}, \delta_{\Lambda}] = \delta_{\bar{\Lambda}} \quad (4.22)$$

where

$$\langle \bar{\Lambda}_3 | = - \frac{\alpha_1 \alpha_2}{\alpha_3^2} V(1,2,3) [|\Lambda'_1\rangle |\Lambda_2\rangle - \Lambda \leftrightarrow \Lambda'] \quad (4.23)$$

This equation defines the symmetry group underlying the string. The lowest component of this equation is none other than the group law for Yang-Mills theory; however, $|\Lambda\rangle$ contains as many gauge parameters as there are fields in the theory. The higher-level group relations can be deduced from Eq. (4.23), and the structure functions are related to the N 's. Thus Eq. (4.23) represents an enormous generalization of Lie group theory which deserves further study. It is straightforward to evaluate the necessary condition that Eq. (4.23) hold; carrying out the commutator of two Λ transformations on $|\chi\rangle$ and neglecting the g^{-1} term which was discussed earlier, we find that

$$\begin{aligned}
 [\delta_{\Lambda'}, \delta_{\Lambda}] < \chi_3 | = V(1,2,3) \left\{ \frac{\alpha_2}{\alpha_3} \bar{V}(1',2',1) \left[\frac{\alpha_2'}{\alpha_1} |\chi_1\rangle |\Lambda_2'\rangle \right. \right. \\
 \left. \left. - \frac{\alpha_1'}{\alpha_1} |\Lambda_1'\rangle |\chi_2'\rangle \right] |\Lambda_2\rangle \right. \\
 \left. - \frac{\alpha_1}{\alpha_3} |\Lambda_1\rangle \bar{V}(1',2',2) \left[\frac{\alpha_2'}{\alpha_2} |\chi_1\rangle |\Lambda_2'\rangle - \frac{\alpha_1'}{\alpha_2} |\Lambda_1'\rangle |\chi_2'\rangle \right] \right\} \\
 - (\Lambda \leftrightarrow \Lambda')
 \end{aligned} \tag{4.24}$$

In order that this should agree with the g^{-1} term, it must be of the form

$$\begin{aligned}
 -V(1,2,3) \frac{\alpha_2}{\alpha_3} |\chi_1\rangle \bar{V}(1',2',2) \frac{\alpha_1 \alpha_2'}{\alpha_2^2} |\Lambda_1'\rangle |\Lambda_2'\rangle \\
 + \frac{\alpha_1}{\alpha_3} \bar{V}(1',2',1) \frac{\alpha_1 \alpha_2'}{\alpha_1^2} |\Lambda_1'\rangle |\Lambda_2'\rangle |\chi_2\rangle - (\Lambda \leftrightarrow \Lambda')
 \end{aligned} \tag{4.25}$$

In the above equations

$$V(1,2,3) |\chi_3\rangle \equiv < \chi_3 | \bar{V}(1,2,3) \tag{4.26}$$

Defining

$$V(1,2,3) \frac{1}{\alpha_1} \bar{V}(1',2',1) = V(2,3; 1',2') \tag{4.27}$$

we find that the necessary and sufficient condition for closure to all orders in g is

$$\begin{aligned}
 [V(2,3; 1',2') - V(3,2; 1',2')] - [1' \leftrightarrow 2'] \\
 - [V(1',3; 2',2) - V(3,1'; 2',2)] + [2 \leftrightarrow 2'] = 0
 \end{aligned} \tag{4.28}$$

This is a generalized Jacobi identity. As explained in Ref. 1), we must, in order to ensure the correct field content at every level, add the Chan-Paton reversed vertex. In the above closure discussion, it is to be understood that this has been done; in fact, it is required in order for Eq. (4.28) to hold. This calculation will be given elsewhere.

5. - THE VERTEX IN "MOMENTUM" SPACE

Having the vertex in oscillator language, it is useful for some purposes to have it in momentum space. This is achieved using the transition function $\langle Z, \bar{Z} | p, \bar{p} \rangle$. To find this object, we first discuss coherent states $|Z, \bar{Z}\rangle$ for the anticommuting variables required in this context; they are defined by

$$|Z_n, \bar{Z}_n\rangle = \exp(\beta_n^\dagger \bar{Z}_n + \bar{\beta}_n^\dagger Z_n) |0\rangle \quad (5.1)$$

and consequently satisfy

$$\beta_n |Z_n, \bar{Z}_n\rangle = Z_n |Z_n, \bar{Z}_n\rangle, \quad \frac{d}{d\bar{Z}_n} |Z_n, \bar{Z}_n\rangle = -\bar{\beta}_n^\dagger |Z_n, \bar{Z}_n\rangle \quad (5.2)$$

as well as similar equations for $\bar{\beta}$. The scalar product of the coherent states is given by

$$\langle Z, \bar{Z} | Z', \bar{Z}' \rangle = \exp(\bar{Z}^* Z' + Z^* \bar{Z}') \quad (5.3)$$

and as such they satisfy the relation

$$1 = \int dZ d\bar{Z}^* dZ' d\bar{Z}'^* |Z, \bar{Z}\rangle \langle Z', \bar{Z}'| \exp(-\bar{Z}^* Z' - Z^* \bar{Z}') \quad (5.4)$$

The momentum operators conjugate to c and $\bar{c}$ are given by

$$\begin{aligned} \bar{p}_n &= \frac{\partial}{\partial \bar{c}_n} = -\frac{i}{\sqrt{2}} (\beta_n - \beta_n^\dagger) \\ p_n &= \frac{\partial}{\partial c_n} = \frac{i}{\sqrt{2}} (\bar{\beta}_n + \bar{\beta}_n^\dagger) \end{aligned} \quad (5.5)$$

The transition function is found in the usual way; we consider the expressions

$$\begin{aligned} \langle p, \bar{p} | p_n | Z, \bar{Z} \rangle &= \langle p, \bar{p} | p_n | Z, \bar{Z} \rangle \\ &= \frac{1}{\sqrt{2}} \langle p, \bar{p} | \bar{\beta}_n - \frac{d}{d\bar{Z}_n} | Z, \bar{Z} \rangle \end{aligned} \quad (5.6)$$

and

$$\begin{aligned}
 \langle p, \bar{p} | c_n | z, \bar{z} \rangle &= \langle p, \bar{p} | \frac{d}{dp_n} | z, \bar{z} \rangle \\
 &= \frac{1}{\sqrt{2}} \langle p, \bar{p} | z_n - \frac{d}{d\bar{z}_n} | z, \bar{z} \rangle
 \end{aligned} \quad (5.7)$$

as well as the analogous equations for $\bar{p}$ and $\bar{c}$. We find that

$$\langle p, \bar{p} | z, \bar{z} \rangle = \exp \left[-\bar{z}_n z_n - i\sqrt{2} (\bar{p}_n \bar{z}_n + p_n z_n) + i \bar{p}_n p_n \right] \quad (5.8)$$

In the above the r, s indices in z and $\bar{z}$ have been suppressed.

The vertex in the coherent state basis is given by

$$V^F(z, \bar{z}) = V^F | z, \bar{z} \rangle = \exp \left[-z_n^r N_{nm}^{rs} \bar{z}_m^s + R_n^{rs} z_n^r \bar{\theta}_0^s \right] \quad (5.9)$$

In the rest of this section, the superscript F will often be dropped. The vertex in momentum space is then given by

$$\begin{aligned}
 V(p, \bar{p}) &\equiv V | p, \bar{p} \rangle = \int dz d\bar{z} dz^* d\bar{z}^* V(z, \bar{z}) \\
 &\times \langle z, \bar{z} | p, \bar{p} \rangle \exp(-\bar{z}^* z - z^* \bar{z})
 \end{aligned} \quad (5.10)$$

Performing the $z^*, \bar{z}^*$ integrations, we find that

$$V(p, \bar{p}) = \int dz d\bar{z} V(z, \bar{z}) \exp \left[-z \bar{z} + i\sqrt{2} (\bar{p} \bar{z} + z p) - i \bar{p} p \right] \quad (5.11)$$

In the Appendix, a short proof of the fact⁷⁾⁻⁹⁾ that V^B is in fact none other than the overlap δ function is given, and the symbols needed are defined below. One finds that

$$N_{nm}^{20} = (C^{-1})_{nm} \delta^{20} - 2 (\tilde{A}^2 \Gamma^{-1} A^0)_{nm} \quad (5.12)$$

where $C_{nm} = n\delta_{nm}$. In fact, one cannot simply carry out the three Gaussian integrals $Z^r, \bar{Z}^r$, since the matrix between Z^r and $\bar{Z}^r$ does not have an inverse. We therefore first carry out the $Z_1, Z_2, \bar{Z}_1$ and $\bar{Z}_2$ integrations to find the result

$$\begin{aligned} V(p, \bar{p}) = & \int dZ_3 d\bar{Z}_3 \exp \left[-2 Z_3 \bar{Z}_3 + 2 Z_3 C \Gamma^{-1} \bar{Z}_3 \right. \\ & + i\sqrt{2} (\bar{p}_3 \bar{Z}_3 + Z_3 p_3) - i \bar{p}_3 p_3 + Z_3 R^3 - i \bar{p}^2 p^2 \\ & - 2 (Z^1 - Z^3 C \Gamma^{-1} A^2 M - \frac{i}{2} \sqrt{2} \bar{p}^2 M) (M^{-1})^{20} \\ & \times (\bar{Z}^1 - \frac{1}{2} (MR)^0 - M C \tilde{A}^1 \Gamma^{-1} \bar{Z}^3 - \frac{M}{\sqrt{2}} p^0) \left. \right] \\ & \times \exp \sqrt{2} Z^3 C \Gamma^{-1} A^2 \bar{p}^2 M^{20} \left(\frac{1}{2} R^1 + \frac{1}{\sqrt{2}} p^0 + C \tilde{A}^0 \Gamma^{-1} \bar{Z}^3 \right) \end{aligned} \quad (5.13)$$

where

$$\begin{aligned} (M^{-1})_{mn}^{20} &= \delta_{mn} \delta^{20} - (C \tilde{A}^2 \Gamma^{-1} A^0)_{mn} \\ R_n^1 &= R_n^1 \beta_0^1 \end{aligned}$$

and in this equation (5.13) and in Eq. (5.14) the r, s sums are over 1, 2 only. Using the fact that

$$M_{mn}^{20} = \begin{pmatrix} 1 + C \tilde{A}^1 C^{-1} A^1 & C \tilde{A}^1 C^{-1} A^2 \\ C \tilde{A}^2 C^{-1} A^1 & 1 + C \tilde{A}^2 C^{-1} A^2 \end{pmatrix} \quad (5.14)$$

we can carry out the $Z_3, \bar{Z}_3$ integration, and after manipulating the resulting expression we find the result

$$V(p, \bar{p}) = \delta(\bar{p}^2 \tilde{A}^2) \delta(A^2 p^2 + \frac{A^2 R^2}{\sqrt{2}}) \exp \left(\frac{i}{\sqrt{2}} \bar{p}^2 R^2 \right) \quad (5.15)$$

where in this last equation

$$\begin{aligned} A^3 &= 1 \\ \tilde{A}^2 &= C \tilde{A}^2 C^{-1} \end{aligned} \quad (5.16)$$

and the sum is now over $r, s = 1, 2, 3$.

We observe that if

$$R^2 = \tilde{A}^2 R^2 \quad (5.17)$$

then $\exp(i/\sqrt{2} \tilde{p}^2 R^2) = 1$ using the first δ function. A preliminary examination indicates that this is the case, and we will now assume it. The vertex in momentum space then has a particularly simple form just consisting of two δ functions.

We may now transform to co-ordinate space by using the identity

$$\begin{aligned} V(c, \bar{c}) &= V |c, \bar{c}\rangle = \int d p d \bar{p} V(p, \bar{p}) \langle p, \bar{p} | c, \bar{c} \rangle \\ &= \int d p d \bar{p} \delta(\bar{p}^2 \tilde{A}^2) \delta(A^2 p^2 + \frac{A^2 R^2}{\sqrt{2}}) \\ &\quad \times \exp i (p^2 c^2 + p^2 \bar{c}^2) \end{aligned} \quad (5.18)$$

$$\begin{aligned} &= \delta(\bar{c}^1 - \bar{A}^1 c^3) \delta(\bar{c}^2 - \bar{A}^2 \bar{c}^3) \delta(c^1 - c^3 A^1) \\ &\quad \delta(c^2 - c^3 A^2) \delta(\frac{1}{\sqrt{2}} c^3 A^2 R^2 - c_0^2) \end{aligned} \quad (5.19)$$

It only remains to interpret this result. Using Eq. (5.17) and the result (see Appendix)

$$f_m(\gamma_3) = -\sqrt{2} \Gamma^{-1} B \alpha_3 \quad (5.20)$$

we find that the argument of the last δ function may be rewritten as

$$\begin{aligned} c^3 \left(\frac{1}{\sqrt{2}} A^2 R^2 - c_0^2 \right) &= c^3 B \alpha_3 (\delta^{34} \gamma_3 + \delta^{14}) - c_0^2 \\ &\quad - \frac{1}{\sqrt{2}} c^3 \Gamma C^{-1} S_3 \end{aligned} \quad (5.21)$$

We then observe that this equation implies that

$$\sum_n \alpha_n c_0^n = 0, \quad c_0^2 - c_0^1 = -c^3 B \alpha_3 \quad (5.22)$$

Consequently, we may rewrite the vertex in the form

$$\begin{aligned} V(c, \bar{c}) = & \delta(\bar{c}^1 - \bar{A}^1 c^3) \delta(\bar{c}^2 - \bar{A}^2 c^3) \delta(c^1 - c^3 A^1) \\ & \times \delta(c^2 - c^3 A^2) \delta\left(\sum_n \alpha_n c_0^n\right) \delta(c_0^1 - c_0^2 - c^3 B \alpha_3) \\ & \times \delta\left[c_0^3 + c^3 \left(\frac{1}{2} \Gamma C^{-1} S_3 + B \alpha_1\right)\right] \end{aligned} \quad (5.23)$$

Let us consider what part of this result is due to overlap δ functions. In particular, let us suppose that

$$c^3(\gamma_3) = \begin{cases} c^2(\gamma_2) & \text{for } 0 \leq \gamma_3 \leq \beta_2 \pi \\ c^1(\gamma_1) & \text{for } \beta_2 \pi \leq \gamma_3 \leq \pi \end{cases} \quad (5.24)$$

Take Fourier transforms in the regions $0 < \eta_1 < \pi$ and $0 < \eta_2 < \pi$ of the expressions

$$c^2(\gamma_2) = c_0^2 + 2 \sum_{n=1}^{\infty} c_n^2 \cos n \gamma_2 \quad (5.25)$$

we find, as for $x_\mu^r(\eta)$, that

$$c^1 = \tilde{A}^1 c_3, \quad c^2 = \tilde{A}^2 c_3, \quad \sum_n \alpha_n c_0^n = 0 \quad (5.26)$$

and $c_0^1 - c_0^2 = \alpha_3 c^3 B$ where

$$\begin{aligned} (\tilde{A}_1)_{nm} &= \frac{(-1)^n}{\pi} \times \frac{-2m\beta_1}{n^2 - \beta_1^2 m^2} (-1)^m \sin m\beta_1 \pi \\ (\tilde{A}_2)_{nm} &= \frac{2m\beta_2}{n^2 - m^2 \beta_2^2} \times \frac{\sin m\beta_2 \pi}{\pi} \\ \alpha_3 B_m &= -\frac{4}{m\beta_1 \beta_2} \sin m\beta_2 \pi \end{aligned} \quad (5.27)$$

For $\bar{c}$ we take the overlap condition

$$\frac{\bar{c}^3(\eta_3)}{\alpha_3} = \begin{cases} \frac{\bar{c}^2}{\alpha_2}(\eta_2) & \text{for } 0 \leq \eta_3 \leq \beta_2 \pi \\ \frac{\bar{c}^2}{\alpha_2}(\eta_1) & \text{for } \beta_2 \pi \leq \eta_3 \leq \pi \end{cases} \quad (5.28)$$

The fact that $\bar{c}$ vanishes at $\eta = 0$ and $\eta = \pi$ means we should take Fourier transforms of the equations

$$\bar{c}^2(\eta_2) = 2 \sum_{n=1}^{\infty} \bar{c}_n^2 \sin n \eta_2 \quad (5.29)$$

One finds that

$$\bar{c}^1 = \tilde{A}^1 \bar{c}^3, \quad \bar{c}^2 = \tilde{A}^2 \bar{c}^3 \quad (5.30)$$

Consequently, the vertex consists of the overlap δ functions for c and $\bar{c}$ as well as an additional factor of

$$c_0^3 + c_3 \left(\frac{1}{\sqrt{2}} \Gamma C^{-1} S_3 + B \alpha_1 \right) \quad (5.31)$$

6. - THE CLOSED BOSONIC STRING

In preparation for giving the interacting closed bosonic string, we now present the most useful free action for this theory. The method of construction is very similar to that of Ref. 3) and begins with the formulation of the closed bosonic string in terms of a finite number of the fields¹¹⁾ $\phi, \xi_n^m, \bar{\xi}_n^m, \bar{\chi}_n^m, \chi_n^m$ and $\phi^n, \bar{\phi}^n$. Let us identify these fields as

$$\begin{aligned} \psi_{00}^0 &= \psi, \quad \xi_n^m = -\psi_{10}^{m0}, \quad \psi_{01}^{00} = \bar{\chi}_n^m, \\ \psi_{10}^{01} &= -\chi_n^m, \quad \psi_{01}^{01} = \bar{\xi}_n^m, \quad \phi_{00}^{10} = \phi^n, \quad \phi_{00}^{01} = \bar{\phi}^n \end{aligned} \quad (6.1)$$

With this notation, the equations of motion of Ref. 11) become

$$\begin{aligned}
 (K + \tilde{K}) \psi_{00}^{00} + D \phi_{00}^{'0} + \tilde{D} \phi_{00}^{0'} &= 0 \\
 d \psi_{00}^{00} - D \psi_{01}^{'0} - \tilde{D} \psi_{01}^{0'} + \eta \phi_{00}^{'0} &= 0 \\
 \tilde{d} \psi_{00}^{00} + D \psi_{01}^{'0} - \tilde{D} \psi_{01}^{0'} + \tilde{\eta} \phi_{00}^{0'} &= 0 \\
 d \phi_{00}^{'0} + (K + \tilde{K}) \psi_{01}^{'0} &= 0 \\
 \tilde{d} \phi_{00}^{'0} - (K + \tilde{K}) \psi_{01}^{0'} &= 0 \\
 d \phi_{00}^{0'} + (K + \tilde{K}) \psi_{01}^{'0} &= 0 \\
 \tilde{d} \phi_{00}^{0'} + (K + \tilde{K}) \psi_{01}^{0'} &= 0
 \end{aligned} \tag{6.2}$$

and the transformation laws are

$$\begin{aligned}
 \delta \psi_{00}^{00} &= D \Lambda_{00}^{'0} + \tilde{D} \Lambda_{00}^{0'} \\
 \delta \phi_{00}^{'0} &= -(K + \tilde{K}) \Lambda_{00}^{'0}, \quad \delta \phi_{00}^{0'} = -(K + \tilde{K}) \Lambda_{00}^{0'} \\
 \delta \psi_{01}^{'0} &= d \Lambda_{00}^{'0}, \quad \delta \psi_{01}^{0'} = -\tilde{d} \Lambda_{00}^{'0} \\
 \delta \psi_{01}^{'0} &= d \Lambda_{00}^{0'}, \quad \delta \psi_{01}^{0'} = \tilde{d} \Lambda_{00}^{0'}
 \end{aligned} \tag{6.3}$$

The symbols d , D , K and η differ slightly from those given in Ref. 3) and are given by

$$(dC)_{n_1 \dots n_{e+1}, q_1 \dots q_d}^{m_1 \dots m_a, p_1 \dots p_c} = L_{\sum n_i} C_{n_2 \dots n_{e+1}, q_1 \dots q_d}^{m_1 \dots m_a, p_1 \dots p_c} + a W_{p[n_1}^{[m_1} C_{n_2 \dots n_{e+1}, q_1 \dots q_d]}^{p \dots m_a] p_1 \dots p_c}$$

$$-\frac{1}{2} b V_{[n_1, n_2}^p C_{p n_3 \dots n_{e+1}, q_1 \dots q_d}^{m_1 \dots m_a, p_1 \dots p_c}$$

$$(DC)_{n_1 \dots n_e, q_1 \dots q_d}^{m_1 \dots m_{a-1}, p_1 \dots p_c} = a [L_p C_{n_1 \dots n_e, q_1 \dots q_d}^{p m_1 \dots m_{a-1}, p_1 \dots p_c}$$

$$+ b W_{np}^q C_{q n_2 \dots n_e, q_1 \dots q_d}^{p m_1 \dots m_{a-1}, p_1 \dots p_c} - \frac{1}{2} (a-1) V_{be}^{[m_1, be, m_2 \dots m_{a-1}]} C_{n_1 \dots n_e, q_1 \dots q_d}^{p_1 \dots p_c}$$

$$(K C)_{m_1 \dots m_a, p_1 \dots p_c}^{m_1 \dots m_a, p_1 \dots p_c} = (L_0 + m + n) C_{n_1 \dots n_b, q_1 \dots q_d}^{m_1 \dots m_a, p_1 \dots p_c},$$

$$m = \sum_i m_i, n = \sum_i n_i,$$

$$(\gamma C)_{n_1 \dots n_{b+1}, q_1 \dots q_d}^{m_1 \dots m_{a-1}, p_1 \dots p_c} = a \gamma_{[m, q]} C_{n_2 \dots n_{b+1}}^{q m_1 \dots m_{a-1}, p_1 \dots p_c} q_1 \dots q_d.$$

(6.4)

The symbols $\tilde{d}$, $\tilde{D}$, $\tilde{K}$ and $\tilde{\eta}$ are the same except that they operate on $p_1 \dots p_c$ and $q_1 \dots q_d$, leaving $m_1 \dots m_a$ and $n_1 \dots n_{b+1}$ alone. The objects obey the relations

$$[d, K] = [D, K] = d^2 = D^2 = [d, D] - 2\gamma K = \{\eta, d\} = \{\eta, D\} = 0 \quad (6.5)$$

as well as similar relations for the tilda's operators.

We now wish to generalize this system to one containing the fields

$$\psi_{k_3 k_4}^{k_1 k_2} \quad \text{with} \quad k_1 + k_2 - k_3 - k_4 = 0$$

and

$$\phi_{k_3 k_4}^{k_1 k_2} \quad \text{with} \quad k_1 + k_2 - k_3 - k_4 = 1 \quad (6.6)$$

All these fields are subject to

$$(K - \tilde{K}) \psi = (K - \tilde{K}) \phi = 0 \quad (6.7)$$

as were those of the finite system. One soon discovers this generalization of the equations of motion to be

$$\begin{aligned} (K + \tilde{K}) \psi_{k_3 k_4}^{k_1 k_2} + d \phi_{k_3-1 k_4}^{k_1 k_2} + \tilde{d} \phi_{k_3 k_4-1}^{k_1 k_2} (-1)^{k_1+k_3} \\ + \mathbb{D} \phi_{k_3 k_4}^{k_1+1 k_2} (-1)^{k_3} + \tilde{\mathbb{D}} \phi_{k_3 k_4}^{k_1 k_2+1} (-1)^{k_1+k_3+k_4} = 0 \end{aligned} \quad (6.8)$$

$$\begin{aligned} d \psi_{k_3 k_4}^{k_1 k_2} + \tilde{d} \psi_{k_3+1 k_4-1}^{k_1 k_2} (-1)^{k_1+k_3+1} \\ + \mathbb{D} \psi_{k_3+1 k_4}^{k_1+1 k_2} (-1)^{k_3+1} + (-1)^{k_1+k_3+k_4+1} \tilde{\mathbb{D}} \psi_{k_3+1 k_4}^{k_1 k_2+1} \\ + (-1)^{k_3} \gamma \phi_{k_3 k_4}^{k_1+1 k_2} + (-1)^{k_4-1} \tilde{\gamma} \phi_{k_3+1 k_4-1}^{k_1 k_2+1} = 0 \end{aligned} \quad (6.9)$$

and the invariance to be

$$\begin{aligned}
 \delta \psi_{k_3 k_4}^{k_1 k_2} &= (-1)^{k_3} \mathbb{D} \wedge_{k_3 k_4}^{k_1+1 k_2} + \tilde{\mathbb{D}} \wedge_{k_3 k_4}^{k_1 k_2+1} (-1)^{k_1+k_3+k_4} \\
 &\quad + d \wedge_{k_3-1 k_4}^{k_1 k_2} + \tilde{d} \wedge_{k_3 k_4-1}^{k_1 k_2} (-1)^{k_1+k_3} \\
 \delta \phi_{k_3 k_4}^{k_1 k_2} &= -(\kappa + \tilde{\kappa}) \wedge_{k_3 k_4}^{k_1 k_2}
 \end{aligned} \tag{6.10}$$

These transformations contain extra and hidden symmetries which, when taken into account, lead to the correct count of states. The pattern is so similar to that for the open string in Ref. 3) that we shall not go through it here.

We now wish to place this generalized system in a more convenient formulation. Let us take our base space to contain $x^\mu(\sigma)$, $c(\sigma)$ and $\bar{c}(\sigma)$, where c and $\bar{c}$ are anticommuting. All these co-ordinates obey the same periodic boundary conditions, which for $\bar{c}(\sigma)$ read

$$\bar{c}(-\pi) = \bar{c}(\pi) \tag{6.11}$$

They have a similar Fourier expansion and for c this is

$$c(\sigma) = \sum_{n=-\infty}^{+\infty} c_n e^{in\sigma} \tag{6.12}$$

We now define creation and annihilation operators by

$$\begin{aligned}
 \frac{\delta}{\delta c(\sigma)} - \frac{1}{2\pi} \tau(\sigma) &= \frac{1}{\pi} \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{+\infty} \bar{\beta}_n e^{in\sigma} \\
 -\frac{\delta}{\delta \bar{c}(\sigma)} + \frac{1}{2\pi} c(\sigma) &= \frac{1}{\pi} \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{+\infty} \beta_n e^{in\sigma} \\
 \frac{\delta}{\delta c(\sigma)} + \frac{1}{2\pi} \bar{c}(\sigma) &= \frac{1}{\pi} \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{+\infty} \tilde{\beta}_n e^{in\sigma} \\
 \frac{\delta}{\delta \bar{c}(\sigma)} - \frac{1}{2\pi} c(\sigma) &= \frac{1}{\pi} \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{+\infty} \tilde{\beta}_n e^{in\sigma}
 \end{aligned} \tag{6.13}$$

The first quantized BRS charge associated with the closed string Virasoro algebra is

$$\begin{aligned}
 Q &= : \left[\sum_n \beta_n L_{-n} + \frac{1}{2} \sum_{n,m,p} f_{nm}^p \bar{\beta}_p \beta_{-n} \beta_{-m} - \beta_0 \right. \\
 &\quad \left. + \sum_n \tilde{\beta}_n \tilde{L}_{-n} - \frac{1}{2} \sum_{n,m,p} f_{nm}^p \tilde{\beta}_p \tilde{\beta}_{-n} \tilde{\beta}_{-m} - \tilde{\beta}_0 \right] : \\
 &= : \beta_0 K - 2 \bar{\beta}_0 \eta + d + D + \tilde{\beta}_0 \tilde{K} - 2 \tilde{\beta}_0 \tilde{\eta} + \tilde{d} + \tilde{D} :
 \end{aligned}
 \tag{6.14}$$

where η , d , D , $\bar{\eta}$, etc., are as in Eq. (6.4) with an appropriate scattering of minus signs. This charge Q , however, is not quite the one we wish to consider. For the closed string, the measure $\int \mathcal{D}x^\mu \prod_{n \neq 0}^\infty dc_n d\bar{c}_n$ is even, and so the action cannot be of the form $\langle \chi | Q | \chi \rangle$. Let us consider, however, the object

$$\begin{aligned}
 \hat{Q} &= Q - \delta_0 (K - \tilde{K}) + 2 \bar{\delta}_0 (\eta - \tilde{\eta}) \\
 &= \gamma_0 (K + \tilde{K}) - 2 \bar{\gamma}_0 (\eta + \tilde{\eta}) + Q'
 \end{aligned}
 \tag{6.15}$$

where

$$\begin{aligned}
 \gamma_0 &= \frac{1}{2} (\beta_0 + \tilde{\beta}_0) \quad , \quad \bar{\gamma}_0 = \frac{1}{2} (\bar{\beta}_0 + \tilde{\bar{\beta}}_0) \\
 \delta_0 &= \frac{1}{2} (\beta_0 - \tilde{\beta}_0) \quad , \quad \bar{\delta}_0 = \frac{1}{2} (\bar{\beta}_0 - \tilde{\bar{\beta}}_0)
 \end{aligned}$$

and

$$Q' = d + D + \tilde{d} + \tilde{D}
 \tag{6.16}$$

We note that

$$\hat{Q}^2 = -(\eta - \tilde{\eta}) (K - \tilde{K})$$

and so vanishes from the fields of interest to us.

The most straightforward procedure is to consider a space in which δ_0 and $\bar{\delta}_0$ do not occur. The vacuum in this space is annihilated by β_n , $\tilde{\beta}_n$, $\bar{\beta}_n$, $\tilde{\bar{\beta}}_n$ for $n > 1$ and the $|+\rangle$ vacuum satisfies

$$\gamma_0 |+\rangle = 0$$

while

$$|- \rangle = 2 \bar{\delta}_0 | + \rangle \quad (6.17)$$

The most general functional of $x^\mu(\sigma)$, $c(\sigma)$ and $\bar{c}(\sigma)$ is given by

$$\begin{aligned} |\chi \rangle = & \psi_{n_1 \dots n_a}^{m_1 \dots m_e q_1 \dots q_d} \beta_{n_1}^+ \dots \bar{\beta}_{n_1}^+ \dots \tilde{\beta}_{p_1}^+ \dots \tilde{\beta}_{q_1}^+ |- \rangle \\ & + \phi_{n_1 \dots n_a}^{m_1 \dots m_e q_1 \dots q_d} \beta_{n_1}^+ \dots \bar{\beta}_{n_1}^+ \dots \tilde{\beta}_{p_1}^+ \dots \tilde{\beta}_{q_1}^+ | + \rangle \end{aligned} \quad (6.18)$$

For the gauge-invariant theory, we take only the bosonic fields in $|\chi \rangle$, and so it is subject to the constraint

$$\left[\sum_{n \neq 0} (\beta_n^+ \bar{\beta}_n - \bar{\beta}_n^+ \beta_n) + (\beta \leftrightarrow \tilde{\beta}, \bar{\beta} \leftrightarrow \tilde{\bar{\beta}}) + \frac{1}{2} \delta_0 \bar{\delta}_0 \right] |\chi \rangle = 0 \quad (6.19)$$

We also enforce the condition

$$(K - \tilde{K}) |\chi \rangle = 0 \quad (6.20)$$

The action which leads to the equations of motion of Eq. (6.10) is given by

$$\langle \chi | \hat{Q} | \chi \rangle \quad (6.21)$$

and it is invariant under

$$\delta |\chi \rangle = \hat{Q} | \chi \rangle \quad (6.22)$$

provided one uses Eq. (6.20).

We may also formulate the free action in an alternative way by reintroducing δ_0 and $\bar{\delta}_0$. The most general functional is then of the form

$$|\chi \rangle = \psi | - + \rangle + \phi | ++ \rangle + \tilde{\psi} | + - \rangle + \tilde{\phi} | -- \rangle \quad (6.23)$$

We take a - vacuum to be odd while a + vacuum is even, and hence if $|\chi \rangle$ is odd, ϕ

and $\tilde{\phi}$ are even while ϕ and $\tilde{\psi}$ are odd. We note that $\delta_0|\chi\rangle = 0$ implies that $\tilde{\phi} = \tilde{\psi} = 0$ and we recover the previous field content. The above vacua are defined by the equations

$$\gamma_0 |+\pm\rangle = 0 = \delta_0 |\pm+\rangle$$

and

$$\begin{aligned} | - + \rangle &= 2 \bar{\gamma}_0 | + + \rangle, \quad | + - \rangle = 2 \bar{\delta}_0 | + + \rangle, \\ | - - \rangle &= 4 \bar{\gamma}_0 \bar{\delta}_0 | + + \rangle \end{aligned} \quad (6.24)$$

We note that

$$\begin{aligned} \langle - - | + + \rangle &= \langle + + | - - \rangle^+ = 4 \langle + + | \bar{\delta}_0 \bar{\gamma}_0 | + + \rangle \\ &= - \langle + + | - - \rangle \end{aligned} \quad (6.25)$$

and as a result we may choose

$$\langle - - | + + \rangle = i \quad (6.26)$$

By further manipulations one finds that

$$\langle + - | - + \rangle = i = - \langle + + | + - \rangle \quad (6.27)$$

As a consequence of these scalar products, one finds that $\langle \chi | \chi \rangle = 0$, for example.

We impose the following algebraic constraints on $|\chi\rangle$:

$$(K - \tilde{K})|\chi\rangle = 0 \quad (6.28)$$

and

$$\begin{aligned} &\left[\sum_{n>0} (\beta_n^+ \bar{\beta}_n - \bar{\beta}_n^+ \beta_n + \tilde{\beta}_n^+ \tilde{\beta}_n - \tilde{\beta}_n^+ \beta_n) \right. \\ &\quad \left. + 2 \gamma_0 \bar{\gamma}_0 - 2 \bar{\delta}_0 \delta_0 \right] |\chi\rangle = 0 \end{aligned} \quad (6.29)$$

This last equation ensures that we have the field content of Eq. (6.6) as well as $\tilde{\phi}^n$ and $\tilde{\phi}$, which have the reversed index structure to ϕ and ϕ respectively.

As noted above, the gauge-covariant free action cannot be $\langle \chi | Q | \chi \rangle$, and the correct action is

$$i \langle \chi | \bar{\delta}_0 Q | \chi \rangle \quad (6.30)$$

We note that we can replace Q by $\hat{Q}$ since $\bar{\delta}_0(Q - \hat{Q})$ vanishes on $|\chi\rangle$. This action is Hermitian as a result of the fact that $i \langle \chi | \{ \bar{\delta}_0, Q \} | \chi \rangle$ vanishes due to Eq. (6.24). Evaluating this action, we find we can recover an action which leads to the correct field equations of Eqs. (6.8) and (6.9). In fact, if we remove the constraint $(K - \tilde{K})|\chi\rangle = 0$, we find that it is imposed on ψ and ϕ by the still-Hermitian action of Eq. (6.30), since $\tilde{\psi}$ and $\tilde{\phi}$ act as Lagrange multipliers. The action of Eq. (6.30) is invariant under

$$\delta |\chi\rangle = Q |\chi\rangle \quad (6.31)$$

and this reproduces the transformation laws of Eq. (6.10).

We now turn to the construction of the interacting closed bosonic string. This can be found by a straightforward application of the methods discussed in the context of the open string in this paper. We will only briefly give the results. The homogeneous term in the transformation law of $|\chi\rangle$ can be seen to require an additional anticommuting term. Since the closed string is somewhat like the product of two open strings, we expect the vertex to be

$$\begin{aligned} V^{\text{closed}} &= \langle 0_{123} | V(\beta, \bar{\beta}, \beta_0) V(\tilde{\beta}, \tilde{\bar{\beta}}, \tilde{\beta}_0) \\ &\quad \times V^c(\alpha) V^c(\tilde{\alpha}) \end{aligned} \quad (6.32)$$

We take the vacuum to be

$$\langle 0_{123} | = \langle +_{-1} | \langle +_{-2} | \langle +_{-3} | \quad (6.33)$$

We therefore postulate the transformation law to be

$$\delta \langle \chi_3 | = \frac{1}{g} \langle \chi_3 | Q^3 + 2 V^{\text{closed}} \delta_0^3 \left(\frac{\alpha_2}{\alpha_3} | \chi_1 \rangle | \chi_2 \rangle - \frac{\alpha_1}{\alpha_3} | \chi_2 \rangle | \chi_1 \rangle \right) \quad (6.34)$$

while the interaction term is

$$\frac{2ig}{3} V^{\text{closed}} | \chi_1 \rangle | \chi_2 \rangle | \chi_3 \rangle \quad (6.35)$$

We find the condition for closure and invariance of the action is

$$V^{\text{closed}} \sum_{n=1}^3 \frac{Q^2}{\alpha_n} = 0 \quad (6.36)$$

The proof of this equation is very similar to that for the open bosonic string and is straightforward provided one utilizes the constraint of Eq. (6.28), and realizes that the $\bar{\delta}_0$ terms in V^{closed} die due to the choice of the vacuum in V .

For the same reasons as discussed above as to why the open string requires a four-point function, we do not expect any more terms for the closed string, and we expect the above action and transformation laws to be complete.

APPENDIX

In this Appendix, we define the symbols used in Section 5, and take the opportunity to give a short derivation of the fact that V^B is, in co-ordinate or momentum space, none other than the overlap δ function⁷⁾⁻⁹⁾. Let us begin in co-ordinate space by insisting that $x^3(\eta_3)$ overlaps with $x^2(\eta_2)$ and $x^1(\eta_1)$ according to Eq. (2.2). Then the vertex is

$$V(x) = V|x\rangle = \delta(\alpha^2 x_0^2) \delta(x^2 - \tilde{A}^2 x^3) \\ \times \delta(x^1 - \tilde{A}^1 x^3) \delta(x_0^2 - x_0^1 - \alpha_3 B x^3) \quad (A1)$$

where

$$\tilde{A}_{nm}^1 = \frac{2}{m\beta_1} \frac{\sin m\pi\beta_1}{\pi} (-1)^m, \quad \tilde{A}_{nm}^2 = \frac{2}{m\beta_2} \frac{\sin m\pi\beta_2}{\pi} \\ B_m = -\frac{4}{m\beta_1\beta_2} \sin m\pi\beta_2 \quad (A2)$$

Transferring to momentum space using $\langle x|p\rangle = \exp(ipx)$, we find that

$$V(p) = \delta(p^3 + A^2 p^2 + A^1 p^1 + B\mathbb{P}) \delta(p_0^1 + p_0^2 + p_0^3) \quad (A3)$$

where $\mathbb{P} = (\alpha_1 p_0^2 - \alpha_2 p_0^1)$. The vertex in the coherent state basis is given by

$$V(Z) = \int dp \, V(p) \langle p|Z\rangle \quad (A4)$$

$$= \int dp \, du \exp iu(A^2 p^2 + B\mathbb{P}) \exp(-\frac{1}{2} \frac{p_n^2}{n} - \frac{3n}{2n} + \sqrt{2} \frac{p_n z_n}{n}) \quad (A5)$$

The sum over n being understood and $A_{nm}^3 = \delta_{nm}$. Doing the three p_r Gaussian integrals, we find that

$$V(Z) = \int du \exp\left(\frac{1}{2} Z C^{-1} Z + \frac{1}{2} u \Gamma u + i u W\right) \delta\left(\sum_n p_0^n\right) \quad (A6)$$

where

$$\begin{aligned} C_{nm} &= u \delta_{nm} \\ \Gamma &= \sum_{n=1}^3 A^n C \tilde{A}^n \\ W &= \sqrt{2} A Z^n + B P \end{aligned} \quad (A7)$$

Performing the final u integration, we find that

$$\begin{aligned} V(Z) &= \exp\left[\frac{1}{2} Z C^{-1} Z - \frac{1}{2} \tilde{W} \Gamma^{-1} W\right] \delta\left(\sum_n p_0^n\right) \\ &= \exp\left[\frac{1}{2} Z_n^2 N_{nm}^{rs} Z_m^s + N_m^r Z_m^2 P\right] \delta\left(\sum_n p_0^n\right) \end{aligned} \quad (A8)$$

and hence

$$\begin{aligned} N_{nm}^{rs} &= \delta^{rs} C^{-1} - 2 (\tilde{A}^r \Gamma^{-1} A^s)_{nm} \\ N_m^r &= -\frac{1}{\sqrt{2}} (\tilde{A}^r \Gamma^{-1} B) P = N_m^{rs} \alpha_0^s \\ K &= -\frac{1}{2} \tilde{B} \Gamma^{-1} B \end{aligned} \quad (A9)$$

It is shown in Ref. 9) that these expressions are indeed equal to those given for N_{nm}^{rs} and N_m^r in Eq. (2.27).

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FIGURE CAPTIONS

Fig. 1 : The parameters of the three interacting strings.

Fig. 2 : Another possible interacting string configuration.

Fig. 1

Fig. 2