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GAUSS-BONNET FROM KALUZA-KLEIN

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ABSTRACT

Tseytlin has calculated the curvature-squared terms in the bosonic string Lagrangian. They do not form the Gauss-Bonnet combination. From this, we use Kaluza-Klein techniques to deduce the corresponding terms in the superstring Lagrangians. They do not form the Gauss-Bonnet combination for the Type II (A or B) strings, but for the $E_8 \times E_8$ heterotic string they do.

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The status of curvature-squared terms in low-energy string Lagrangians is quite confusing. Candelas et al.¹⁾ introduced a term $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ into the $d = 10$ superstring Lagrangian in order to find compactifying solutions yielding chiral fermions in four dimensions. Subsequent string calculations confirmed that such a term was indeed present with the right coefficient²⁾. Then Zwiebach³⁾ pointed out that since $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ alone would give rise to ghost poles in the propagators, more satisfactory curvature-squared terms would be provided by the Gauss-Bonnet combination^{*)}

$$R_{\mu\nu\lambda\sigma} R^{\lambda\sigma\mu\nu} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \quad . \quad (1)$$

Such an addition would not modify the propagator because terms bilinear in fluctuations combine to a total derivative and integrate to zero. The cubic and higher terms yield non-trivial interactions, however. Another way to understand this dates back to the calculation of the graviton self-energy $\Pi_{\mu\nu\rho\sigma}(p)$ using dimensional regularization. There are five tensors formed from the momentum p^μ and Minkowski metric $\eta^{\mu\nu}$ with the symmetries $\Pi_{\mu\nu\rho\sigma} = \Pi_{\lambda\mu\rho\sigma} = \Pi_{\rho\sigma\mu\nu}$ but these are reduced to two by the Ward identity $p^\mu \Pi_{\mu\nu\rho\sigma} = 0$ ⁴⁾. It follows that there must be one linear combination of the three invariants $R_{\mu\nu\rho\sigma} R^{\rho\sigma\mu\nu}$, $R_{\mu\nu} R^{\mu\nu}$ and R^2 which does not contribute. Note that the Gauss-Bonnet combination is the topological Euler number density only for $d = 4$ ^{**)} .

An elegant summary of higher-derivatives in d -dimensional gravity theories has been given by Zumino⁶⁾.

Zwiebach's criterion is, in fact, more restrictive than mere ghost-freedom: it means that there are no new poles at all even with positive norm. One could still add an arbitrary amount of R^2 to the Lagrangian (1) without introducing ghosts⁷⁾⁻⁹⁾. (As we shall see later, however, neither the bosonic string nor the superstrings take advantage of this intermediate possibility.)

*) The ordering of the indices in (1) becomes important when the connection has torsion, as it does when the antisymmetric tensor $B_{\mu\nu}$ is included. For the most part we shall neglect the dependence on $B_{\mu\nu}$ and the dilaton ϕ .

**) Even in $d = 4$ one can still calculate how its coefficient gets renormalized since it is given by the gravitational trace anomaly. Similar remarks apply to R in $d = 2$ ⁵⁾.

Subsequent work provided evidence both for and against Zwiebach's conjecture. Romans and Warner¹⁰⁾ claimed that the Gauss-Bonnet combination in the $d = 10$ Lagrangian was an inevitable consequence of supersymmetry. Han et al.¹¹⁾, on the other hand, pointed out that Romans and Warner had implicitly assumed a ghost-free fermion Lagrangian. When this assumption was relaxed, an arbitrary combination was allowed. Similar ambiguities exist in four-dimensional supergravity Lagrangians¹²⁾⁻¹⁴⁾. After all, supersymmetry alone does not forbid ghosts; it merely requires that they form supermultiplets.

Further confusion arose from explicit calculation of curvature-squared effects in the closed bosonic string^{3), 15)-26)}. If one considers quantization of the two-dimensional string theory with background gravitational field $g_{\mu\nu}$, antisymmetric tensor field $B_{\mu\nu}$ and dilaton field ϕ , then the vanishing of the β -functions $\beta_{\mu\nu}^g$ and $\beta_{\mu\nu}^B$ (at one-loop) and β^ϕ (at two-loops) corresponds to the field equations obtained by varying the low-energy string Lagrangian linear in curvature¹⁶⁾⁻¹⁸⁾. One might expect a similar relationship between $\beta_{\mu\nu}^g$ and $\beta_{\mu\nu}^B$ (at two-loops) and in β^ϕ (at three-loops) and the curvature-squared terms in the low-energy Lagrangians. Indeed a term $R_{\mu\alpha\beta\gamma} R^{\alpha\beta\lambda\gamma}$ does appear¹⁵⁾ in the two-loop $\beta_{\mu\nu}^g$ and a term $R_{\mu\alpha\beta\gamma} R^{\mu\alpha\beta\gamma}$ does appear¹⁶⁾⁻¹⁸⁾ in the three-loop β^ϕ with coefficients consistent with a $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ Lagrangian. In spite of all this, the status of the $R_{\mu\nu} R^{\mu\nu}$ and R^2 terms remained very obscure^{*)}. For example, as discussed in Refs. 27)-30), vanishing β^ϕ yields the critical dimension formula

$$d-26 = \frac{k C_A}{2 + C_A} \quad (2)$$

for the bosonic string compactified on a non-Abelian group manifold G where $k = \dim G$ and C_A is the second order Casimir in the adjoint representation³¹⁾⁻³⁵⁾. Now, on the group manifold

$$4\alpha' R_{mnpq} = (1-\lambda^2) C_{mn}{}^r C_{pq}{}^r, \quad (3)$$

where C_{mnp} are the structure constants and the Riemann tensor is that of a connection with torsion $\lambda/2 H_{mnp}$, where

*) N.B. Given a non-linear σ -model, there should correspond unique β -functions for a given choice of variables $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ . There is no question of changing the coefficients of $R_{\mu\nu} R^{\mu\nu}$ and R^2 by using the field equations. This only becomes a possibility when one asks what is their coefficient in the equation $\beta = 0$. Different combinations of the $\beta = 0$ equations could then yield different coefficients.

$$\sqrt{\alpha'} H_{mnp} = C_{mnp} \quad (4)$$

and λ is a constant. Hence

$$\begin{aligned} 16\alpha'^2 R_{mnpq} R^{mnpq} &= (1-\lambda^2) k C_A^2, \\ 16\alpha'^2 R_{mn} R^{mn} &= (1-\lambda^2) k C_A^2, \\ 16\alpha'^2 R^2 &= (1-\lambda^2) k^2 C_A^2. \end{aligned} \quad (5)$$

Thus, if one performs an (illegal!) expansion in powers of C_A in (2), one would expect terms $k C_A^n$ to correspond to terms $(\alpha' \text{ curvature})^n$ in the perturbation expansion of $\alpha' \beta^\phi$. The crucial point is that whereas $R_{mnpq} R^{mnpq}$ and $R_{mn} R^{mn}$ are satisfactorily linear in k , R^2 is quadratic and hence ruled out^{*)} as a contribution to β^ϕ !

Our interest in group-manifold compactification was motivated by our attempt^{27), 28)} to derive properties of the $d = 10$ superstrings, both heterotic³⁶⁾ and Type II³⁷⁾, via Kaluza-Klein²⁹⁾ starting from the $d = 506$ bosonic string compactified on the group manifolds $E_8 \times E_8$ or $\text{spin } 32/Z_2$ ^{**) .} [For an alternative $d = 26$ approach to this problem, see Refs. 38)-40).] Now the $R_{\mu\alpha\beta\gamma} R^{\alpha\beta\gamma}_\nu$ terms in the $\beta^\phi_{\mu\nu}$ function are known to appear in the ratio 2:1:0 for the bosonic, heterotic and Type II strings respectively, and this was satisfactorily explained from a Kaluza-Klein point of view in Ref. 28), but $R_{\mu\nu} R^{\mu\nu}$ and R^2 terms were not discussed.

Of course, it was never clear whether the ghosts arising from an $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ Lagrangian are genuine or merely artifacts of the perturbation

*) Except for the parallelizing choice $\lambda = \pm 1$, which is in fact the value appropriate to $\beta^\phi_{\mu\nu} = R_{\mu\nu}$ ($\lambda = \pm 1$) at one loop. However, at one loop $\beta^\phi = R(\lambda = \pm 1/\sqrt{3})$ which suggests that $\lambda = \pm 1$ is not appropriate in the present context. It would be interesting to pursue this line of argument to higher orders and thereby restrict the form of the $(\text{curvature})^n$ terms for $n > 2$.

**) In fact this works only³⁰⁾ for $E_8 \times E_8$!

expansion. As discussed by Weinberg⁴¹⁾, the location of these poles corresponds to energies beyond which the perturbation ceases to be trustworthy. Nevertheless, one might be tempted to think that if apparent ghost-freedom is a good thing, it should be equally applicable to all string theories. The purpose of this paper is to point out that this is not the case. We find the curvature-squared Lagrangians

$$L_{\text{bosonic}} \sim 2 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2, \quad (6)$$

$$L_{\text{heterotic}}(E_B \times E_F) \sim R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2, \quad (7)$$

$$L_{\text{type II}} \sim -4 R_{\mu\nu} R^{\mu\nu} + R^2. \quad (8)$$

The Type I theory would have to be treated separately since it does not lend itself to a Kaluza-Klein derivation²⁶⁾ from the closed $d = 506$ bosonic string²⁷⁾. Curiously enough, this is also the case for the $\text{spin}(32)/\mathbb{Z}_2$ heterotic string³⁰⁾.

Our analysis is based on the results of Tseytlin⁴²⁾ for the closed bosonic string. He points out that, if we neglect the dependence on $B_{\mu\nu}$ and ϕ , the β^ϕ function in $d = 26$ is given by^{*)}

$$16 \pi^2 \beta^\phi = -R - \frac{\alpha'}{4} (2 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu}). \quad (9)$$

Note that, as anticipated, there is no dependence on R^2 . However, the low-energy Lagrangian is not given by β^ϕ as one might naively expect but by its exponential

$$\alpha' L = \exp(-8 \pi^2 \alpha' \beta^\phi) - 1, \quad (10)$$

*) Our normalization of α' is one-half that of Tseytlin's.

i.e.

$$\alpha' L = -R - \frac{\alpha'}{4} (2 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2) + \dots (11)$$

Thus L does have an R^2 dependence but the total curvature-squared combination is still not Gauss-Bonnet⁴²⁾!

Using the Kaluza-Klein techniques of Refs. 27) and 28), it is now a trivial matter to compute the corresponding Lagrangians for the $d = 10$ heterotic string and the $d = 10$ Type II strings. Of course, we must first rewrite L in $d = 506$ rather than $d = 26$ but L is dimension-independent save for the cosmological term $(d-26)/3\alpha'$, and we must replace R by R ($\lambda = \pm 1/\sqrt{3}$) (H corrections to the curvature squared terms can be neglected to this order.) The recipe is (i) to make the Kaluza-Klein ansatz of Ref. 28) for the gauge bosons (A_μ, B_μ) corresponding to $G_L \times G_R$ so that β^ϕ in $d = 10$ becomes

$$16\pi^2\beta^\phi = -R + \frac{\alpha'}{2} (F_{\mu\nu}^i F^{i\mu\nu} + G_{\mu\nu}^i G^{i\mu\nu}) - \frac{\alpha'}{4} (2 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu}), \quad (12)$$

where $F_{\mu\nu}^i$ and $G_{\mu\nu}^i$ are the Yang-Mills field strengths of A_μ and B_μ respectively, and then (ii) to truncate to an $SO(8)$ subgroup of G_R (in the heterotic case) or of both G_L and G_R (in the Type II case) and then identify the $SO(8)$ gauge potentials with the spin-connection of the transverse space-time $SO(8)$, i.e., $G_{\mu\nu}^i G^{\mu\nu i} = \frac{1}{2} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ for the heterotic string, and $G_{\mu\nu}^i G^{\mu\nu i} = F_{\mu\nu}^i F^{i\mu\nu} = \frac{1}{2} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ for the Type II. The net result is to put the coefficient of $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ in the ratio 2:1:0 for the bosonic, heterotic, and Type II strings respectively but to leave the coefficients of $R_{\mu\nu} R^{\mu\nu}$ and R^2 the same! Hence we obtain the Lagrangians (6), (7) and (8).

Heterotically speaking, therefore, Zwiebach is vindicated but it remains a mystery why perturbative arguments should be more reliable here than in the bosonic or type II cases.

In an addendum to his paper, Tseytlin clarifies the relation between β -functions and the effective action and reports on an unpublished calculation with R. Metsaev. They also find the Gauss-Bonnet combination for the heterotic string and agree with our results for Type II.

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