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TOPOLOGICAL CHARGES
IN THE FIELD THEORIES WITH BROKEN SYMMETRY

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A b s t r a c t

The effective method on construction of the Higgs field topological charges expressions is proposed , provided the symmetry group is compact and semisimple. The topological charges are expressed as integrals on differential forms , defined with help of characteristic classes of bundle over the vacuum manifold.

INTRODUCTION

The theories with broken symmetry, which take an important place in modern physics, are attractive because of some interesting topological properties. For investigation of the localized topological solutions in the theories of such kind the methods of topology and differential geometry are applied. The classification of the solutions in terms of homotopy groups leads to integer topological "integral of movement" - topological charges.

Let's consider the system of fields containing the Higgs field $\mathbf{F}=(F^1, \dots, F^n)$. The set V of vacuum solution for the Higgs subsystem consists of the constant in all space fields $\mathbf{F} \neq 0$, for which Higgs potential $U(\mathbf{F})=\text{abs. min.}$.

We are interested in the localized solutions, thus the field $\mathbf{F}(\vec{r})$ must possess the vacuum asymptotic at the infinity. In other words, at $r \rightarrow \infty$ for every direction $\vec{n} = \frac{\vec{r}}{r}$ of tending to the infinity, $\mathbf{F}(\vec{r})$ tends to some vacuum value $\mathbf{F}$, which can be different for the different directions $\vec{n}$.

Thus the next boundary condition arises

$$\mathbf{F}(\vec{r}) \rightarrow \mathbf{F}(\vec{n}) \in V, \text{ at } r \rightarrow \infty, \quad (1.1)$$

defining by introducing the function $\mathbf{F}(\vec{n})$ on sphere S^2 . Classes of homotopic maps

$$\mathbf{F}(\vec{n}) : S^2 \rightarrow V \quad (1.2)$$

form homotopic group $\pi_2(V)$, which defines the topological charges of the soliton solutions.

The case, when the Higgs potential possesses the continuous group of symmetry is more important. Hence, the absolute minimum

of $U(F)$ is degenerate. We suppose that all degeneration caused by the action of group G . The set V of the vacuum solutions forms the homogeneous space G/H , where H is stationary subgroup of group G , $hF_0 = F_0, h \in H$ for some fixed vacuum F_0 .

In the case when Higgs field interacts with gauge field, nontriviality of the group $\mathcal{H}_2(G/H)$ leads to the appearance of monopole-like solutions /1,2/. Topological charges of those are investigated in the papers /3 - 5/*). The papers /3,4/ contains the expressions through the gauge field strength and Higgs field, which defines the topological charges modulo the set of normalization coefficients. The last are founded in /5/, provided the Higgs field transforms under adjoint representation of compact and simply connected group G .

In the present paper the normalization coefficients are founded for all representations any compact and semisimple group of symmetry, and the explicit topological charges expressions through the Higgs field at infinity in terms of integrals over sphere from closed differential forms are constructed.

The important point of consideration is the explicit realization the basis of cohomology group $H^2(G/H)$ in terms of characteristic classes of bundle over the vacuum manifold G/H , which give a useful differential geometry interpretation of topological charges (section 3). Also the explicit formulae are founded for the basic elements of group $\mathcal{H}_2(G/H)$ (section 4). In the section 5 the example of representations of the group $SU(2)$ is considered in detail.

*) See also the references in the /5/.

In the section 2 some preliminary explanations are given. It's necessary to note, that the problem of the topological charges determination has some common features with formalism of the geometry quantization /6,7/.*)

2. Algebraic topology approach to the topological charges determination

We assume that group of symmetry G is compact connected and semisimple. This case comprises a big number of physically interesting examples. Besides let's assume that stationary subgroup H is connected. Later we get rid of this restriction. One can note, since H is connected, the vacuum manifold $V=G/H$ is oriented.

Let $\widetilde{G/H}$ is universal (i.e. $\pi_1(\widetilde{G/H})=0$) covering manifold to G/H . So, the next isomorphism is suitable /9/:

$$\pi_2(\widetilde{G/H})=H_2(\widetilde{G/H}, \mathbb{Z}) \quad (2.1)$$

Projection $\widetilde{G/H} \rightarrow G/H$ induces isomorphism of the homotopy group /9/:

$$\pi_2(\widetilde{G/H})=\pi_2(G/H) \quad (2.2)$$

It's known /10/ that:

$$H^{m-2}(\widetilde{G/H}, \mathbb{Z})=Z^1=Z \oplus \dots \oplus Z \quad (1\text{-times}), \quad (2.3)$$

($m=\dim G/H$), where 1 is dimension of the centre of the Lie algebra of the subgroup H .

*) After finishing the work under this paper, the paper /8/ was published, where the problem of the topological charges determination is considered in different manner. Also /8/ doesn't contain the explicit expressions for the basic elements of the group $\pi_2(G/H)$ and explicit expressions for the topological charges through the Higgs field.

Provided H is connected /11/:

$$H^2(G/H) = \mathbb{R}^1 \quad (2.4)$$

From the Poincare duality /12/:

$$H_2(\widetilde{G/H}, \mathbb{Z}) = H^{m-2}(\widetilde{G/H}, \mathbb{Z}) \quad (2.5)$$

From the (2.1) - (2.3), (2.5) we have

$$\pi_2(G/H) = \mathbb{Z}^1 \quad (2.6)$$

Let's define through $[F]$ homotopy class of the map (1.2). It decomposes into linear combination $[S_a]$, $a=1, \dots, l$, where $[S_a]$ are basic elements of the group $\pi_2(G/H)$:

$$[F] = \sum_{a=1}^l N_a [S_a] \quad (2.7)$$

Differential forms on G/H may be transformed into forms on S^2 with help of the map (1.2). This transformation we denote F^* . F^* commutes with exterior differential $F^*d = dF^*$, and hence

$$F^* : H^2(G/H) \rightarrow H^2(S^2) \quad (2.8)$$

is homomorphism. Two homotopic maps (1.2) induce equal homomorphism (2.8) /12/. In other words, for any closed 2-form K , defining some element $H^2(G/H)$, $F_1^*K - F_2^*K$ is the exact 2-form on S^2 , if F_1 and F_2 are from the same homotopy class. Thus the expression

$\int_{S^2} F^*K$ depends on homotopy class of the map $F(\bar{n})$. Provided (2.7)

$$\int_{S^2} F^*K = \sum_{a=1}^l N_a \int_{S^2} S_a^*K, \quad (2.9)$$

where S_a is any map from the homotopy class $[S_a]$.

In the group $H^2(G/H)$ one may choose the basis K_a , $a=1, \dots, l$, which consist of the linear independent elements represented by closed 2-forms on G/H . Hence

$$\int_{S^2} F^*K_a = \sum_b N_b \int_{S^2} S_b^*K_a = \sum_b N_b M_{ab}$$

The matrix

$$M_{ab} = \int_{S^2} S_b^* K_a \quad (2.10)$$

is reversible and independent on $F(\vec{n})$.

Therefore

$$K_a = \sum_b (M^{-1})_{ab} \int_{S^2} F^* K_b \quad (2.11)$$

The problem now is to give an explicit realization of the basis K_a , $a=1, \dots, l$ in the group $H^2(G/H)$ (see section 3), basis $[S_a]$ in the group $\mathcal{T}_2(G/H)$ and determine the normalization matrix (2.10) (see section 4).

3. Realization of the basis of the group $H^2(G/H)$

Let's consider the general bundle /13/ $G(G/H, H)$ with base G/H , projection

$$p : G \rightarrow G/H \quad (3.1)$$

and structure group (fibre) H , which acts on G by the right shifts.

With the curvature $B = dA + A \wedge A$ of some connection A on the bundle one can connect characteristic classes /14/, defining elements of the group $H^2(G/H)$. It will be enough to find an explicit form of the basis of the group $H^2(G/H)$.

Let $\mathcal{G}$ denotes the Lie algebra of the group G , $\mathcal{H}$ - the Lie algebra of the stationary subgroup H . $\mathcal{H}$ is the direct sum of its centre $\mathcal{K}$ and some semisimple algebra $\mathcal{B}$ /15/:

$$\mathcal{H} = \mathcal{K} \oplus \mathcal{B} \quad (3.2)$$

Subspaces $\mathcal{K}$ and $\mathcal{B}$ are orthogonal relatively the positive definite scalar production on $\mathcal{G}$, which for any elements

$X = \sum x_m I_m$, $Y = \sum y_m I_m$ (I_m are generators of the group G) of $\mathcal{G}$ has the following form:

$$\langle X, Y \rangle = g_{mn} x_m y_n \quad (3.3)$$

where g_{mn} expresses through the structure constants:

$$g_{mn} = - \sum_{i,j} f_{mj}^i f_{ni}^j \quad (3.4)$$

In the case of simple group $\langle X, Y \rangle = c \cdot \text{Tr}(XY)$ (c is some real constant). The algebra $\mathcal{G}$ represents in the following manner:

$$\mathcal{G} = \mathcal{H} + \mathcal{H}^\perp \quad (3.5)$$

where $\mathcal{H}^\perp$ is the orthogonal complement to the $\mathcal{H}$.

It's obvious that

$$h \mathcal{H}^\perp h^{-1} \subset \mathcal{H}^\perp \quad (3.6)$$

for any $h \in H$, since $\langle h, h m h^{-1} \rangle = \langle h^{-1} h h, m \rangle = 0$ for any $h \in \mathcal{H}$ and $m \in \mathcal{H}^\perp$.

It's following from (3.2) that for $\mathcal{K}$ -components of the connection $A_{\mathcal{K}}$ and curvature $B_{\mathcal{K}}$ the relationship is satisfied:

$$dA_{\mathcal{K}} = B_{\mathcal{K}} \quad (3.7)$$

Here $A_{\mathcal{K}} = \left(\sum_a A^a r_a + \sum_i A^{1+i} b_i \right)_{\mathcal{K}} = \sum_{a=1}^1 A^a r_a$, where $\{r_a\}$, $\{b_i\}$

are the basic elements in the $\mathcal{K}$ and $\mathcal{B}$ correspondingly.

$B_{\mathcal{K}}$ is defined analogously.

From (3.5), (3.6) it follows /13/, that the connection on $G(G/H, H)$ may be defined as $\mathcal{H}$ -component of the form $g^{-1} dg$:

$$A = (g^{-1} dg)_{\mathcal{H}} \quad (3.8)$$

It's known /14/ that with help of the linear map $f: \mathcal{H} \rightarrow \mathbb{R}$, satisfying the invariance condition

$$f(h h h^{-1}) = f(h) \quad (3.9)$$

for any $h \in \mathcal{H}$ and $h \in H$ ($\mathbb{R}$ is set of real numbers), one may construct the form $f(B)$, which is projected uniquely on the closed form $F(B)$ on G/H .

The form $F(B)$ defines the characteristic class, i.e. the

element of the group $H^2(G/H)$ independent on choice of the connection on the bundle $G(G/H, H)$. $f(B)$ and $\overline{f(B)}$ are connected by the relationship:

$$p^* \overline{f(B)} = f(B), \quad (3.10)$$

where p is the projection (3.1).

For any $b \in \mathcal{B}$ $f(b) = 0$, so f may be represented as any linear map from $\mathcal{B}$ to $\mathbb{R}$ (provided H is connected), since the invariance condition (3.9) is obviously satisfied. Hence it is enough to consider the forms

$$f_a(B) = \langle r_a, B \rangle, \quad a=1, \dots, l \quad (3.11)$$

where r_a , $a=1, \dots, l$ is the basis in the centre $\mathcal{K}$.

In appendix 1 it is shown that the forms $K_a = \overline{f(B)}$ defines the basic elements of the group $H^2(G/H)$.

Thus the topological charges may be expressed through the characteristic classes $\overline{f_a(B)}$, $a=1, \dots, l$ of the bundle $G(G/H, H)$ over the vacuum manifold G/H :

$$N_a = \sum_b (M^{-1})_{ab} \int_{S^2} F^*(\vec{n}) \overline{f(B)} \quad (3.12)$$

Since the group G acts on $V=G/H$ transitively, any value $F(\vec{n}) \in G/H$ may be represented by the action of some element $g(\vec{n}) \in G$ on to fixed vacuum F_0 :

$$F(\vec{n}) = g(\vec{n}) F_0 = p g(\vec{n}) \quad (3.13)$$

However the map $g(\vec{n}) : S^2 \rightarrow G$ isn't in generally continuous, since the bundle $G(G/H, H)$ is not trivial. Therefore to represent $F(\vec{n})$ as in (3.13) is suitable locally only - over the disks V and W (see Fig. 1):

$$\begin{aligned} F(\vec{n}) &= g_V(\vec{n}) F_0, & \vec{n} \in V \\ F(\vec{n}) &= g_W(\vec{n}) F_0, & \vec{n} \in W \end{aligned} \quad (3.14)$$

Moreover

$$g_W(\vec{n}) = g_V(\vec{n})h_{VW}(\vec{n}), \quad \vec{n} \in V \cap W, \quad (3.15)$$

where $h_{VW} \in H$. In other words g_V , g_W , h_{VW} are the sections and the binding function of the bundle induced from $G(G/H, H)$ by means of the map $F(\vec{n})$.

The base of this induced bundle is sphere S^2 , the fibre under any point $\vec{n} \in S^2$ is the fibre under the point $F(\vec{n}) \in G/H$ from the bundle $G(G/H, H)$.

Let $F(S^2, H)$ denotes the induced bundle.

It will be necessary to us later.

Fig. 1

By means of (3.10) and (3.14) the formula (3.12) may be rewriting in the following manner

$$N_a = \sum_b (M^{-1})_{ab} \left(\int_V \langle x_b, g_V^*(\vec{n})B \rangle + \int_W \langle x_b, g_W^*(\vec{n})B \rangle \right) \quad (3.16)$$

Thus for the case of the canonical connection (3.8):

$$N_a = \sum_b (M^{-1})_{ab} \left(\int_V \langle x_b, dg_V^{-1}(\vec{n}) \wedge dg_V(\vec{n}) \rangle + \int_W \langle x_b, dg_W^{-1}(\vec{n}) \wedge dg_W(\vec{n}) \rangle \right) \quad (3.17)$$

Let one of the disks, for example W , shrinks in to a point. Thus the ambiguity of the form $dg^{-1}(\vec{n}) \wedge dg(\vec{n})$ is concentrated at this point (the symbol V is omitted). This ambiguity is eliminated when the scalar production is used. So at last:

$$N_a = \sum_b (M^{-1})_{ab} \int_{S^2} \langle x_b, dg^{-1}(\vec{n}) \wedge dg(\vec{n}) \rangle \quad (3.18)$$

In the coordinates (θ, φ) on the sphere S^2 the form $dg^{-1}(\vec{n}) \wedge dg(\vec{n}) = (\partial_\theta g^{-1}(\vec{n}) \partial_\varphi g(\vec{n}) - \partial_\varphi g^{-1}(\vec{n}) \partial_\theta g(\vec{n})) d\theta d\varphi$.

4. Realization of the group $\pi_2(G/H)$ basis and the normalization matrix determination

Before the group $\pi_2(G/H)$ basis construction one has to remind some information on the semisimple Lie group theory /15, 16/. Let $\mathcal{T}$ is maximal commutative subalgebra in $\mathcal{G}$, i.e. Cartan subalgebra. Since G is compact, Cartan subalgebra may be chosen to contain the centre $\mathcal{K}$ and Cartan subalgebra in $\mathcal{B}$. The complexification of the real algebra $\mathcal{G}$

$$\mathcal{G}^{\mathbb{C}} = \mathcal{G} \otimes \mathbb{C} = \mathcal{T} \otimes \mathbb{C} + \sum_{\alpha \in \Delta} \mathbb{C} \alpha \quad (4.1)$$

where $\mathbb{C} \alpha$ are complex onedimensional subspaces with basic vectors e_{α} , which are the eigen vectors for all elements of Cartan subalgebra $\mathcal{T} \otimes \mathbb{C}$ in $\mathcal{G}^{\mathbb{C}}$ at the same time:

$$[X, e_{\alpha}] = i\alpha(X)e_{\alpha}, \quad (4.2)$$

where X is any element in $\mathcal{T} \otimes \mathbb{C}$. The linear functional $\alpha(X)$ on $\mathcal{T} \otimes \mathbb{C}$ called by root is defined by the element $\alpha \in \mathcal{T}$ (called by root also) which satisfies $\alpha(X) = \langle \alpha, X \rangle$. Δ denotes the set of roots. The value $\alpha(X)$ is real number, provided $X \in \mathcal{T}$.

In Cartan subalgebra $\mathcal{T}$ one may choose the subsystem π of simple roots α_i , $i=1, \dots, k$ ($k=\text{rank } G$), such that any root is linear combination with integer coefficients of the same sign of the elements from $\alpha = \sum n_i \alpha_i$. π is basis in $\mathcal{T}$. There is basis in the semisimple algebra $\mathcal{G}^{\mathbb{C}}$ which consists of π and e_{α} , $\alpha \in \Delta$ such that $[e_{\alpha}, e_{-\alpha}] = i\alpha$, and the set

$$\pi \cup \left\{ w_{\alpha} = \frac{e_{\alpha} + e_{-\alpha}}{(2\langle \alpha, \alpha \rangle)^{1/2}}, v_{\alpha} = \frac{i(e_{\alpha} - e_{-\alpha})}{(2\langle \alpha, \alpha \rangle)^{1/2}}, \alpha \in \Delta_+ \right\} \text{ is the basis}$$

in the Lie algebra $\mathcal{G}$. The element $\alpha^* = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$ is called dual to

the root α . $\mathcal{T}^*$ denotes the set of the roots α_i^* which are dual to α_i , $i=1, \dots, k$. Any root α^* dual to α represents the linear combination of the elements $\mathcal{T}^*$, $\alpha^* = \sum n_i \alpha_i^*$ with integer coefficients of the same sign. For every α^* $\exp(2\pi\alpha^*)=1$.

Let E_0 denotes the free Abelian group formed by the integer linear combinations of the elements $\mathcal{T}^*$. E_0 is subgroup in the free Abelian group $E = \{X \in \mathcal{T}, \exp(2\pi X)=1\}$.

The group $\mathcal{T}_1(G)$ is isomorphic to E/E_0 [17]:

$$\mathcal{T}_1(G) = E/E_0 \quad (4.4)$$

I.e. the group $\mathcal{T}_1(G)$ is formed by the homotopic classes of loops $\exp(\varphi X)$, $X \in E$ ($0 \leq \varphi \leq 2\pi$) which are shrinkable if $X \in E_0$.

At first, we suppose that Cartan subalgebra lies in the Lie algebra of the stationary subgroup H completely (adjoint representation belongs to this type). It is easy to construct the basic elements of $\mathcal{T}_2(G/H)$ explicitly. In the up half-sphere we put (see Fig.1)

$$S_{\alpha V} = \exp\left(\frac{\alpha^*}{2}\varphi\right) \exp(\nu_\alpha \theta) \exp\left(-\frac{\alpha^*}{2}\varphi\right) \quad (4.5)$$

which is continuous map from V to G . The meaning of the coordinates is obvious. In the down half-sphere we put

$$S_{\alpha W} = \exp\left(\frac{\alpha^*}{2}\varphi\right) \exp(\nu_\alpha \theta) \exp\left(\frac{\alpha^*}{2}\varphi\right) \quad (4.6)$$

The map $S_{\alpha W}$ is single-valued at the point $\theta = \pi$, since

$$\exp(\nu_\alpha \pi) \frac{\alpha^*}{2} \exp(-\nu_\alpha \pi) = -\frac{\alpha^*}{2} \text{ as it is easy to see. The binding}$$

function $h_{\alpha VW} = \exp(\alpha^* \varphi) \in H$ binds $S_{\alpha V}$ and $S_{\alpha W}$:

$$S_{\alpha W} = S_{\alpha V} h_{\alpha VW} \quad (4.7)$$

and its homotopic class $[h_{\alpha VW}] \in \mathcal{T}_1(H)$.

Therefore

$$S_\alpha = \exp\left(\frac{\alpha^*}{2}\varphi\right) \exp(v_\alpha \beta) F_0. \quad (4.8)$$

defines the continuous map $S^2 \rightarrow G/H$. When Higgs field is transformed by the adjoint representation of the group G it is easy to show directly

$$S_\alpha = \langle F_0, \alpha \rangle (w_\alpha \cos\varphi \sin\theta + v_\alpha \sin\varphi \sin\theta + \frac{\alpha^*}{2} \cos\theta) F_0 - \frac{\alpha^*}{2} \langle F_0, \alpha \rangle, \quad (4.9)$$

i.e. S_α is the sphere of radius $\langle F_0, \alpha \rangle$ in the space G/H .

Let's choose the system of the simple roots, among them $\alpha_{i_1}, \dots, \alpha_{i_l}$ are not orthogonal to the centre $\mathcal{C}$. Therefore the matrix

$$M_{ab} = \int_{S^2} S^*_{i_b} K_a = -2\pi \langle r_a, \alpha^*_{i_b} \rangle \quad (4.10)$$

is nondegenerate, so all $S_{\alpha_{i_1}}, \dots, S_{\alpha_{i_l}}$ are homotopically nontrivial and independent.

Now it is necessary to show that they define the basic elements in $\pi_2(G/H)$. We consider the exact homotopic sequence /9/ of the bundle $G(G/H, H)$.

$$\begin{array}{ccccccc} \dots \rightarrow & \pi_2(G) & \rightarrow & \pi_2(G/H) & \xrightarrow{d_*} & \pi_1(H) & \xrightarrow{i_*} \pi_1(G) \rightarrow \dots \\ & \parallel & & \parallel & & \parallel & \\ & 0 & & \mathbb{Z}^1 & & \mathbb{H} & \\ & & & & & E/E_0 & \end{array} \quad (4.11)$$

The homomorphism d_* relate to every $[S_\alpha]$ the homotopy class of binding function $[h_\alpha v_W] = [\exp(\alpha^* \varphi)] \in \pi_1(H)$. It follows from (4.11) that d_* is one-to-one correspondance between $\pi_2(G/H)$ and its image in $\pi_1(H)$, and that the kernel $\ker i_* = \text{im } d_* = \mathbb{Z}^1$ is formed by the elements $[\exp(\alpha^*_1 \varphi)]$. Hence, $[h_{\alpha_{i_a}} v_W]$, $a=1, \dots, l$ are the basic elements in $\ker i_*$, and therefore $[S_{\alpha_{i_a}}]$, $a=1, \dots, l$ is the basis in $\pi_2(G/H)$.

The formula (4.10) determines the necessary normalization matrix. For the case of classical Lie algebras the dual roots α_i^* are listed in the appendix 2.

Now we consider the example of adjoint representation Lie group $SU(3)$. The fixed vacuum F_0 may be chosen in the form of $F_0 = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$, $\lambda_1 + \lambda_2 + \lambda_3 = 0$. The scalar production is $\langle X, Y \rangle = -6\text{Tr}(XY)$. The roots of $su(3)$ algebra are $\alpha = 1/6\text{diag}(1, -1, 0)$, $\beta = 1/6\text{diag}(0, 1, -1)$, $\gamma = 1/6\text{diag}(1, 0, -1)$. α and β are simple, $\gamma = \alpha + \beta$.

Provided all eigen values of F_0 are different, $\mathcal{H} = \mathcal{R}$ coincides with Cartan subalgebra in $su(3)$. The sphere S_γ is deformable on to S_α , S_β which determines the basic elements in $\pi_2(G/H)$. Provided $F_0 = \text{diag}(\lambda, \lambda, -2\lambda)$, $\mathcal{H} = \mathcal{R} \oplus su(2)$, $\beta + \gamma$ is one-dimensional centre $\mathcal{R}$ generator. In this case the sphere S_α turns into a point, the sphere S_γ is deformable on to the sphere S_β determining the basic element of the group $\pi_2(G/H)$.

Now let Cartan subalgebra $\mathcal{T}$ containing $\mathcal{R}$ and Cartan subalgebra in $\mathcal{B}$ doesn't lie in $\mathcal{H}$ completely. Taking into account the consequences of the exact sequence (4.11) considered above one has to extract the basis in the free Abelian group $\ker i_* = \mathbb{Z}^1$, which consists of the elements like $[\exp(\varphi\mu^*)]$, where μ^* is the linear combination with integer coefficients of elements from the π^* such that $\mu^* \in \mathcal{H}$ and μ^* is not orthogonal to the centre $\mathcal{Z}$ (otherwise the loop $\exp(\varphi\mu^*)$ is shrinkable to a point). Let $\mu_1^*, \dots, \mu_l^*$ is the basis in the free Abelian group consisting of elements μ^* pointed out. Thus $[\exp(\varphi\mu_a^*)]$, $a=1, \dots, l$ is the basis in $\ker i_* = \mathbb{Z}^1$. The normalisation matrix M_{ab} may be calculated with help of the homomorphism d_* . Let S_{bV} and S_{bW} represent the

element of the homotopy group $\pi_2(G/H)$. Let $h_{bVW} = \exp(i\varphi \mu_b^*)$ is corresponding binding function. Thus by means of Stokes theorem (see Fig. 1):

$$\begin{aligned} M_{ab} &= \int_V \langle r_a, d(S_{bV}^{-1} dS_{bV}) \rangle + \int_W \langle r_a, d(S_{bW}^{-1} dS_{bW}) \rangle = \\ &= \int_{S^1} \langle r_a, S_{bV}^{-1} dS_{bV} \rangle - \int_{S^1} \langle r_a, h_{bVW}^{-1} S_{bV}^{-1} d(S_{bV} h_{bVW}) \rangle = \\ &= - \int_{S^1} \langle r_a, h_{bVW}^{-1} dh_{bVW} \rangle = -2\pi \langle r_a, \mu_b^* \rangle \end{aligned} \quad (4.14)$$

So, the normalization matrix is completely defined. One has to point out that the explicit formulae for S representing basic elements of $\pi_2(G/H)$ may be useful for the topologically nontrivial solutions construction.

Below we express the topological charges through the Higgs field directly. Then we generalize the results in the case when the stationary subgroup H is non-connected. Note that the function $r(F(\vec{n})) = g_V(\vec{n}) r g_V^{-1}(\vec{n}) = g_W(\vec{n}) r g_W^{-1}(\vec{n})$ ($r \in \mathcal{R}$) on the $W \cap V$ (see Fig.1), since $h_{VW} r h_{VW}^{-1} = r$. So the function $r(F(\vec{n}))$ representing the generator of the stationary subgroup $H_{F(\vec{n})}$ centre corresponding to the point $F(\vec{n}) \in G/H$ is continuous and single-valued.

Let $\mathcal{H}_{r(F(\vec{n}))}$ denotes the set of Lie algebra elements $X \in \mathcal{G}$ of the following type: $[X, r(F(\vec{n}))] = 0$. For any value $F(\vec{n})$ the Lie algebra $\mathcal{G}$ may be decomposed into the sum of subspaces

$$\mathcal{G} = \mathcal{H}_{r(F(\vec{n}))} + \mathcal{H}_{r(F(\vec{n}))}^\perp \quad (4.15)$$

where $\mathcal{H}_{r(F(\vec{n}))}^\perp$ is orthogonal complement relative the scalar production on $\mathcal{G}$.

Since the operator $\text{ad}_{r(F(\vec{n}))}(Y) = [r(F(\vec{n})), Y]$ is non-degenerate

on $Y \in \mathcal{H}_{r(F(\vec{n}))}^\perp$ and the form $dr(F(\vec{n}))$ takes values in $\mathcal{H}_{r(F(\vec{n}))}^\perp$, the form $\langle dr(F(\vec{n})) \cdot \text{ad}_{r(F(\vec{n}))}^{-1}(dr(F(\vec{n}))) \rangle$ is defined correctly.

It is easy to verify [7] that $-1/2 \langle dr(F(\vec{n})) \cdot \text{ad}_{r(F(\vec{n}))}^{-1}(dr(F(\vec{n}))) \rangle = \langle r, dg_V^{-1}(\vec{n}) \cdot dg_V(\vec{n}) \rangle$, at $\vec{n} \in V$, and

$= \langle r, dg_W^{-1}(\vec{n}) \cdot dg_W(\vec{n}) \rangle$, at $\vec{n} \in W$. Therefore by using the results of sections 3 and 4 we write

$$N_a = -1/2 \sum_b (M^{-1})_{ab} \int_{S^2} \langle dr_b(F(\vec{n})) \cdot \text{ad}_{r_b(F(\vec{n}))}^{-1}(dr_b(F(\vec{n}))) \rangle \quad (4.16)$$

The example of the explicit construction of the functions $r(F(\vec{n}))$ through the field $F(\vec{n})$ we consider in section 5 for the $SU(2)$ group representations.

Now we discuss the case when the stationary subgroup H is non-connected. Let H_e denotes the connected component of $e \in H$. $T=H/H_e$ is finite group.*) Since the covering map $t: G/H_e \rightarrow G/H$ induces the isomorphism between $\mathcal{H}_2(G/H)$ and $\mathcal{H}_2(G/H_e)/9$, one has to go from the field $F(\vec{n})$ to the field $\hat{F}(\vec{n})$ for which the problem is solved; $t(\hat{F}(\vec{n}))=F(\vec{n})$, $\hat{F}(\vec{n}) \in G/H_e$. It is needed to define $\hat{F}(\vec{n})$.

The induced bundle $F(S^2, H)$ (see section 3) admits the reduction to the structure group H_e . In other words we always may choose the sections $g_V(\vec{n})$ and $g_W(\vec{n})$ so that the binding function $h_{VW} \in H_e$. Let $g_V(\vec{n})$ and $g_W(\vec{n})$ are the sections of this reduced bundle. Therefore the field $\hat{F}(\vec{n})$ may be written in the following manner

*) In general, for $z \in T$ $zrz^{-1} \neq r$ for $r \in \mathcal{K}$. So it is necessary to go from G/H to its covering space G/H_e .

$$\hat{F}(\vec{n}) = \begin{cases} \mathcal{E}_V(\vec{n})H_0 & , \vec{n} \in V \\ \mathcal{E}_W(\vec{n})H_0 & , \vec{n} \in W \end{cases} \quad (4.17)$$

Indeed, $t(\hat{F}(\vec{n})) = \mathcal{E}_V(\vec{n})H = \mathcal{E}_V(\vec{n})F_0 = F(\vec{n})$, $\vec{n} \in V$, and analogous for $\vec{n} \in W$. Therefore the formula (4.16) where the functions $r(F(\vec{n}))$ are defined by means of sections of the reduced to the structure group H_0 bundle $F(S^2, H)$, turns out to be valid in general case. In the next section we will treat the case of non-connected stationary subgroup on the example of representations of the $SU(2)$ group.

5. Topological charges for the $SU(2)$ group representation

Let's consider the Higgs field transformed by irreducible unitary representation D^j of $SU(2)$ group. The vacuum F_0^j may always be chosen as $F_{0m}^j = c \delta_{0m}$, $m=j, j-1, \dots, -j$ ($c \neq 0$ is real constant). There are three different cases:

- a) j - half-integer, $H=e$, hence $N=0$
- b) j - integer, odd. The stationary subgroup is $H=U(1)$ which is formed by elements $\exp(i\alpha I_3)^*$ (since $I_3 F_0^j = 0$)
- c) j - integer, even. The stationary subgroup is formed by all productions of elements from $U(1) = \{ \exp(i\alpha I_3) \}$ and $Z_2 = \{ 1, a \}$, where $a = \exp(i\pi I_2)$, $a_{mm'} = (-1)^{j-m} \delta_{m-m'}$, $a F_0^j = F_0^j$, $a^2 = 1$, $a I_3 a^{-1} = -I_3$.

In other words $H = U(1) \cdot Z_2$ is half-direct production of $U(1)$ and Z_2 (since Z_2 is not a normal subgroup of the group H).

*) I_1, I_2, I_3 is canonical basis in the Lie algebra of the representation D^j .

In the case b) $G/H=S^2$, in the case c) $G/H=RP^2$.

As it is above we write $F^j(\vec{n})$ as follows

$$F^j(\vec{n}) = \begin{cases} g_V^j(\vec{n})F_0^j & , \vec{n} \in V \\ g_W^j(\vec{n})F_0^j & , \vec{n} \in W \end{cases} \quad (5.1)$$

The binding function $h_{VW}^j(\vec{n}) \in H$. In case even j if it turns out that $h_{VW}^j \notin H_e$, one has to go to the sections $\hat{g}_V^j(\vec{n}) = g_V^j(\vec{n})a$, $\hat{g}_W^j(\vec{n}) = g_W^j(\vec{n})$ and to the new binding function $\hat{h}_{VW}^j = a^{-1}h_{VW}^j \in H$.

For the following treatment it is useful to shrink the disk W into a point $\vec{n}_0$ and represent $F^j(\vec{n})$ as $F^j(\vec{n}) = g^j(\vec{n})F_0^j$, where

$$g^j(\vec{n}) = \exp(-iI_3 \varphi(\vec{n})) \exp(-iI_2 \theta(\vec{n})) \exp(-iI_3 \psi(\vec{n})) \quad (5.2)$$

under the parametrization by Euler angles (θ, φ, ψ) . At the point $\vec{n}_0$ we have an ambiguity walking over the group H (H_e - when j is

even). This ambiguity is not essential as it is pointed in section 3. By means of formula (5.2) for the matrix of rotations $g^j(\vec{n})$

one may find that the generator of stationary subgroup of the point $F^j(\vec{n})$ is

$$\begin{aligned} r(F^j(\vec{n})) &= g^j(\vec{n}) iI_3 g^j(\vec{n})^{-1} = \cos \theta(\vec{n}) iI_3 + \cos \varphi(\vec{n}) \sin \theta(\vec{n}) iI_1 + \\ &+ \sin \varphi(\vec{n}) \sin \theta(\vec{n}) iI_2 = iI_3 g_{00}^1(\vec{n}) - iI_+ g_{10}^1(\vec{n}) + iI_- g_{-10}^1(\vec{n}) \quad (5.3) \\ (I_{\pm} &= (I_1 \pm iI_2)/(2)^{1/2}) \end{aligned}$$

Taking into account the following formulae for the matrix of rotations /18/

$$g_{m_1 m_2 m_3}^1 = \sum_{m_1 m_2 m_3} \frac{C_{j-1m-m_3, j m_3}^{1m} C_{j m_1 j m_2}^{j-1m-m_3}}{C_{j-10j0}^{10} C_{j0j0}^{j-10}} g_{m_1 0}^j g_{m_2 0}^j g_{m_3 0}^j \quad (5.4)$$

(odd j)

($C_{j m_1 j m_2}^{j m_3}$ is Clebsh-Gordon coefficient),

$$s_{m_0}^2 = \sum_{m_1 m_2 m_3} \frac{C_{j-2m-m_3}^{2m} C_{j m_1 j m_2}^{j-2m-m_3}}{C_{j-20j0}^{20} C_{j0j0}^{j-20}} s_{m_1}^j s_{m_2}^j s_{m_3}^j \quad (5.5)$$

(even j),

one reaches the expression for $r(F(\vec{n}))$:

$$r(F^j) = \sum_{m_1 m_2 m_3} \frac{i F_{m_1}^j F_{m_2}^j F_{m_3}^j}{(F^j, F^j)^{3/2} C_{j-10j0}^{j-10} C_{j0j0}^{j-10}} (I_3 C_{j-1}^{10} - m_3 j m_3 C_{j m_1 j m_2}^{j-1-m_3} -$$

$$- I_+ C_{j-11-m_3 j m_3}^{j-11-m_3} C_{j m_1 j m_2}^{j-11-m_3} + I_- C_{j-1-1-m_3 j m_3}^{j-1-1-m_3} C_{j m_1 j m_2}^{j-1-1-m_3}) \quad (5.6)$$

(odd j), and

$$r(F^j) = i I_3 (z_0(F^j))^{1/2} + i I_+ (z_+(F^j))^{1/2} + i I_- (z_-(F^j))^{1/2} \quad (5.7)$$

(even j), (where $(\rho \exp(i\varphi))^{1/2} = \rho^{1/2} \exp(i/2 \varphi)$) and

$$z_{\pm}(F^j) = \sum_{m_1 m_2 m_3} (2/3)^{1/2} \frac{F_{m_1}^j F_{m_2}^j F_{m_3}^j}{(F^j, F^j)^{3/2}} \frac{C_{j-2+2-m_3 j m_3}^{2+2} C_{j m_1 j m_2}^{j-2+2-m_3}}{C_{j-20j0}^{20} C_{j0j0}^{j-20}} \quad (5.8)$$

$$z_0(F^j) = 1+2/3 \sum_{m_1 m_2 m_3} \frac{F_{m_1}^j F_{m_2}^j F_{m_3}^j}{(F^j, F^j)^{3/2}} \frac{C_{j-2}^{20} C_{j-2-m_3 j m_3}^{20} C_{j m_1 j m_2}^{j-2-m_3}}{C_{j-20j0}^{20} C_{j0j0}^{j-20}} \quad (5.9)$$

Since the form $\langle i I_3, dg^j(\vec{n})^{-1} dg^j(\vec{n}) \rangle$ up to a constant equals the

$$\text{form } \sum_{mn} r(F^j(\vec{n}))_{mn} \frac{dF_m^j(\vec{n}) \wedge dF_n^j(\vec{n})}{(F^j, F^j)} = j(j+1) \sin \theta(\vec{n}) d\theta(\vec{n}) \wedge d\varphi(\vec{n}),$$

the following formula for the topological charge is valid, in

case Higgs field transforming by D^j representation of $SU(2)$ group:

$$N = \frac{1}{4j(j+1)} \sum_{mn} \int_{S^2} r(F^j(\vec{n}))_{mn} \frac{\overline{dF_m^j(\vec{n})} \wedge dF_n^j(\vec{n})}{(F^j, F^j)} \quad (5.10)$$

which is degree of map $S^2 \rightarrow G/H_0$.

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A P P E N D I X 1

In this appendix we show that $K_a = \overline{F_a(B)}$, $a=1, \dots, l$ is the basis of group $H^2(G/H) = R^1$. It is sufficient to establish that $f(B) = -\sum d_a f_a(B)$ ($\sum d_a^2 \neq 0$) determines the nontrivial element $\overline{F(B)} \in H^2(G/H)$. Since $\overline{F(B)} \in H^2(G/H)$ is independent of connection on $G(G/H, H)$, one may consider the connection (3.8).

Let X^* denotes vector field on G with components $(\frac{d}{dt} g \exp(tX))_{t=0} = gX$, $X \in \mathcal{G}$, at the point $g \in G$. They are leftinvariant, i.e. $L_a X^* = X^*$ (where L_a is left shift on the element $a \in G$: $g \mapsto ag$), and form Lie algebra isomorphic to the Lie algebra $\mathcal{G}$. Any field $\mathcal{H}^*$ corresponding to $\mathcal{H} \in \mathcal{H}$ is tangent to fibres of the bundle $G(G/H, H)$. $A(\mathcal{H}^*) = \mathcal{H}$ according to definition of connection. In the case of the connection (3.8) vector fields $\mathcal{M}^*$, $\mathcal{M} \in \mathcal{H}^{\perp}$ are horizontal (i.e. $A(\mathcal{M}^*) = 0$) vector fields.

On G $f(B) = f(B_{\mathcal{X}}) = df(A_{\mathcal{X}}) = df(A)$. Let's assume $\overline{F(B)} = d\tilde{C}$ on G/H , where $\tilde{C}$ is 1-form on G/H .

2-Form $f(B)$ is leftinvariant ($L_a^* f(B) = f(B)$, $a \in G$), since the connection (3.8) is leftinvariant, and L_a^* commutes with external derivative $L_a^* d = dL_a^*$. Since left shift L_a is commuting with the projection p (3.1), the form $\overline{F(B)}$ is leftinvariant also. (if it is not so, the form $\tilde{C} = \int_G dm(a) L_a^* \tilde{C}$ ($\int_G dm(a) = 1$, $dm(a)$ is invariant measure on compact Lie group G)) is obvious leftinvariant.

Let $C = p^* \tilde{C}$, therefore $f(B) = dC$. By means of Cartan formula /13/: $1/2(XC(Y) - YC(X) - C([X, Y])) = f(B_{\mathcal{X}}(X, Y))$, $B_{\mathcal{X}}(X, Y) = (dA_{\mathcal{X}})(X, Y) = 1/2(XA_{\mathcal{X}}(Y) - YA_{\mathcal{X}}(X) - A_{\mathcal{X}}([X, Y]))$, where X, Y are vector fields on G ; If they are leftinvariant, $C([X, Y]) =$

$=f(A_{\chi}([X,Y]))$, because C and A being leftinvariant $A(X)$ and $C(X)$ are constant functions on G . Take $r \in \mathcal{R}$, $r \neq 0$ so that $f(r) \neq 0$. Since $\mathcal{G}$ is semisimple, $[\mathcal{G}, \mathcal{G}] = \mathcal{G}$, hence there are X_1, X_2 so that $r = [X_1, X_2]$. Thus $f(r) = f(A_{\mathcal{R}}(r^*)) = f(A_{\mathcal{R}}([X_1^*, X_2^*])) = C(r^*)$. But $C(r^*) = (p^*C)(r^*) = C(p_*r^*) = 0$, because of r^* is tangent to fibre and therefore $p_*r^* = 0$. This contradict to $f(r) \neq 0$. So, $\overline{f(B)}$ is a nontrivial element of cohomology group $H^2(G/H)$.

A P P E N D I X 2

In this appendix we list dual roots for simple classical algebras $su(k+1, k \geq 1)$, $so(2k+1, k \geq 1)$, $sp(k, k \geq 2)$, $so(2k, k \geq 3)$ of rank k .

1. $su(k+1, k \geq 1)$. Cartan subalgebra is formed by diagonal matrices.

$\alpha_j^* = \text{diag}(0, \dots, 0, 1, -1, 0, \dots, 0)$, $j=1, \dots, k$, (1 is on j 'th place).

2. $so(2k+1, k \geq 1)$. Cartan subalgebra consist of matrices

$\begin{pmatrix} 0 & \theta & \theta \\ \theta^T & 0 & \lambda \\ \theta^T - \lambda & 0 & 0 \end{pmatrix}$, where θ is the row of k zeroes, λ are diagonal matrices of order k .

$\alpha_j^* = \begin{pmatrix} 0 & \theta & \theta \\ \theta^T & 0 & \lambda_j \\ \theta^T - \lambda_j & 0 & 0 \end{pmatrix}$, $j=1, \dots, k$

$\lambda_j = \text{diag}(0, \dots, 0, 1, -1, 0, \dots, 0)$, $j < k$ (1 on j 'th place),

$\lambda_k = \text{diag}(0, \dots, 0, 2)$.

3. $sp(k, k \geq 2)$. Cartan subalgebra is made of matrices

$\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$, where λ are diagonal matrices of order k .

$\alpha_j^* = \begin{pmatrix} \lambda_j & 0 \\ 0 & -\lambda_j \end{pmatrix}$, $j=1, \dots, k$

$\lambda_j = \text{diag}(0, \dots, 0, 1, -1, 0, \dots, 0)$, $j < k$ (1 is on j 'th place),

$\lambda_k = \text{diag}(0, \dots, 0, 1)$.

4. $so(2k, k \geq 3)$. Cartan subalgebra is formed by matrices

$\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$, where λ are diagonal matrices of order k .

$$d_j^* = \begin{pmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{pmatrix}, \quad j=1, \dots, k$$

$\lambda_j = \text{diag}(0, \dots, 0, 1, -1, 0, \dots, 0)$, $j < k$ (1 is on j 'th place)

$\lambda_k = \text{diag}(0, \dots, 0, 1, 1)$.

REFERENCES

1. Monastyrskii M.I., Perelomov A.M. JETP Lett., 1975, 21, 43.
2. Tyupkin Y.S., Fateev V.A., Schwarz A.S. JETP Lett., 1975, 21, 42.
3. Schwarz A.S. Nucl. Phys., 1976, 112B, 358.
4. Taubes C.H. Commun. Math. Phys., 1981, 81, 299.
5. Horvathy P.A., Rawnsley J.H. Commun. Math. Phys., 1984, 96, 497.
6. Kirillov A.A. The elements of representation theory, Moscow, 1978.
7. Karasev M.V., Maslov V.P. Usp. Mat. Nauk, 1984, 39, 114.
8. Horvathy P.A., Rawnsley J.H. Commun. Math. Phys., 1985, 99, 517.
9. Hu S.T. Homotopy theory. New York, 1959.
10. Bott R. in Representation Theory of Lie Groups. London Math. Soc. Lecture Notes, ser 34, Cambridge U.P., 1979.
11. Spivac M. A Comprehensive Introduction to Differential Geometry, vol. 5, Berkeley, 1975.
12. Massi W. Homology and cohomology theory, Moscow, 1981.
13. Kobayashi S., Nomizu K. Foundations of differential geometry, vol. 1, New York, 1963.
14. Kobayashi S., Nomizu K. Foundations of differential geometry, vol. 2, New York, 1969.
15. Pontrygin A.S. Continuous groups, Moscow, 1984.
16. Goto M., Grosshans F.D. Semisimple Lie algebras, New York, 1978.
17. Loos O. Symmetric Spaces, vol. 2, New York, 1969.
18. El-Baz E., Castel B. Graphical methods of spin algebras, New York, 1972.

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