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WARD IDENTITIES AND CHIRAL ANOMALIES IN STOCHASTIC QUANTIZATION

M.B. Gavela and N. Parga^{*)}

CERN - Geneva

ABSTRACT

We derive the Ward identities (WI) for vector and axial currents in stochastic quantization at any given fictitious time t . This is achieved through a functional integral representation of the fermionic Langevin equations. The currents for this effective field theory differ in general from the naive ones; if stochastic regularization is used they are both conserved. We establish the connection between those WI and the field theory ones. The physical source of chiral anomalies is identified: these result from the quantum fluctuations in the fictitious time evolution of the system. In this context, both a traditional regularization method (Pauli-Villars) and stochastic regularization are considered.

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^{*)}On leave from Centro Atomico Bariloche, Argentina.

1. - INTRODUCTION

Calculations using the naïve expression of the chiral current, i.e., $j_\mu^5(x,t) = \bar{\psi}(x,t)\gamma_5\gamma_\mu\psi(x,t)$, (x denotes the space-time co-ordinates and t the fictitious time), have been carried out recently and it was found that the noise average of its divergence may give, as $t \rightarrow \infty$, the usual chiral anomaly¹⁾⁻³⁾ (at least when an infra-red regulator is conveniently used). However, a correct treatment of the problem requires the derivation and analysis of the corresponding Ward identities (WI) from which one should extract the true chiral current and the anomaly.

In stochastic quantization⁴⁾ (SQ), it is possible to write Langevin equations describing the field evolution in the fictitious time which are chiral invariant (for zero fermionic mass) and gauge covariant^{6),7),1)}. Starting from these equations, the WI should be derived.

Furthermore, the question of the regularization procedure is a delicate one: SQ provides a regularization procedure which respects any symmetry of the unregularized theory. This stochastic regularization⁵⁾ (SR) procedure has raised many hopes in so far as it preserves supersymmetry. However, it is clear that a regulator which preserves the chiral and gauge invariances of the original equations cannot give rise to anomalous chiral WI for those equations. From this perspective, it is then an open question whether the connection between the stochastically regularized Langevin equations and field theory allows the recovery of chiral anomalies for the latter.

In this paper we obtain and discuss the WI associated with charge conservation and with the chiral properties of fermions on a background field in the framework of SQ. The connection between those and the WI and currents of the field theory is established. This allows a careful analysis of the source of chiral anomalies in SQ. Furthermore, we compare in this context a traditional regularization method (Pauli-Villars) with SR.

In Section 2 we derive an effective action corresponding to the Langevin equations using path integral methods. The WI are then obtained for any fictitious time t , and the conserved currents of the effective theory are analyzed through SR. Section 3 establishes the relationship between the above mentioned unregularized effective action and WI with the corresponding ones in field theory (as $t \rightarrow \infty$): in order to achieve this step, we develop a

Lagrangian version of the Fokker-Planck Hamiltonian analysis. In Section 4, we specifically discuss Pauli-Villars and SR for the resulting field theory. In Section 5 we conclude.

2. - WARD IDENTITIES FOR THE LANGEVIN EQUATIONS

The Langevin equations for fermions in an external gauge field read^{6),7),1)}

$$\frac{\partial \Psi(x, \tau)}{\partial \tau} = -(\not{D} - im)(\not{D} + im)\Psi(x, \tau) + i(\not{D} - im)\eta_1(x, \tau) + \eta_2(x, \tau)$$

$$\frac{\partial \bar{\Psi}(x, \tau)}{\partial \tau} = -(\not{D}' - im)^T (\not{D}' + im)^T \bar{\Psi}(x, \tau) + \bar{\eta}_1(x, \tau) + i(\not{D}' - im)^T \eta_2(x, \tau) \quad (2.1)$$

where

$$\begin{aligned} \not{D} &= \gamma^\mu (\partial_\mu - ie A_\mu) \\ \not{D}' &= \gamma^\mu (-\partial_\mu - ie A_\mu) \end{aligned} \quad (2.2)$$

and we use a Euclidean metric $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$.

The noise sources are Grassmann variables following a Gaussian distribution:

$$\langle \eta_i^\alpha(x, \tau) \rangle_\eta = \langle \bar{\eta}_i^\alpha(x, \tau) \rangle_\eta = 0 \quad (2.3a)$$

$i, k = 1, 2$

$$\langle \eta_i^\alpha(x, \tau) \bar{\eta}_k^\beta(x', \tau') \rangle_\eta = \delta_{ik} \delta^\alpha_\beta \delta^4(x-x') \delta(\tau-\tau') \quad (2.3b)$$

for the unregularized theory. In Eqs. (3), α and β are Dirac indices. The SR of ultraviolet divergences consists in the replacement of $\delta(t-t')$ by a smearing function in the fictitious time $\alpha_\sigma(t-t')$

$$\langle \eta_i^\alpha(x, \tau) \bar{\eta}_k^\beta(x', \tau') \rangle_\eta = \delta_{ik} \delta^\alpha_\beta \delta^4(x-x') \alpha_\sigma(\tau-\tau') \quad (2.4)$$

so that when $\sigma = 0$, $\alpha_\sigma(t-t')$ is the Dirac delta function $\delta(t-t')$ and the unregularized theory is recovered.

Let us first consider the unregularized theory (2.1), (2.2) and (2.3). The connection with standard formulations of field theory is achieved through the following relation between mean values

$$\langle O \rangle_s = \lim_{t \rightarrow \infty} \langle O(t) \rangle_s \quad (2.5)$$

where O is a product of field operators and its mean value is taken with the usual action S . $O(t)$ is the corresponding product of the solution of Eq. (2.1) with all fields considered at the same time t .

It is easy now to check that the theory is invariant under gauge transformations, as well as under the following chiral ones (for $m = 0$):

$$\begin{aligned} \psi &\rightarrow e^{i\alpha\gamma_5} \psi & \bar{\psi} &\rightarrow \bar{\psi} e^{i\alpha\gamma_5} \\ \eta_1 &\rightarrow e^{-i\alpha\gamma_5} \eta_1 & \bar{\eta}_2 &\rightarrow \bar{\eta}_2 e^{-i\alpha\gamma_5} \\ \eta_2 &\rightarrow e^{i\alpha\gamma_5} \eta_2 & \bar{\eta}_1 &\rightarrow \bar{\eta}_1 e^{i\alpha\gamma_5} \end{aligned} \quad (2.6)$$

In order to obtain the WI, we first transform the averages over the random sources into averages over the fermion field itself. According to (2.3), the probability distribution of the noises is

$$\mathcal{P}(\bar{\eta}_i, \eta_i) \propto \exp \left\{ - \int d^4x \int_0^t d\tau (\bar{\eta}_1 \eta_1 + \bar{\eta}_2 \eta_2) \right\} \quad (2.7)$$

which leads to a chiral invariant probability distribution for the fields

$$\begin{aligned} \mathcal{P}(\bar{\psi}, \psi) \prod_{x,\tau}^t d\psi d\bar{\psi} &= N \int \prod_{x,\tau}^t d\bar{\eta}_1(x,\tau) d\eta_1(x,\tau) d\bar{\eta}_2(x,\tau) d\eta_2(x,\tau) \cdot \\ &\cdot d\psi d\bar{\psi} \mathcal{P}(\bar{\eta}_i, \eta_i) P(\bar{\psi}(0)) P(\psi(0)) \delta(\psi - \psi_\eta) \delta(\bar{\psi} - \bar{\psi}_\eta^{(2.8)}) \end{aligned}$$

In the above expression $P(\psi(0))$, $P(\bar{\psi}(0))$ are the probability distributions of the field at $t = 0$; the functional integral is to be understood over all noise configurations up to time t ; ψ_η , $\bar{\psi}_\eta$ are the solutions of the Langevin

equations for given noises η_1, η_2 .

Introducing sources $\bar{J}$ and J for the fermionic fields and integrating over these, we obtain the generating functional of an effective four-dimensional theory with fermion fields parametrized by τ

$$Z(\bar{J}, J) = \int \prod_{x, \tau} d\bar{\Psi}(x, \tau) d\Psi(x, \tau) \mathcal{P}(\bar{\Psi}, \Psi) \cdot \exp \left\{ - \int d^4x \int_0^t d\tau (\bar{J}\Psi + \bar{\Psi}J) \right\} \quad (2.9)$$

Finally, performing the integrals over the noises contained in $(\bar{\Psi}, \Psi)$ the effective action we are looking for is obtained. Equation (2.9) is the Lagrangian version of the Fokker-Planck Hamiltonian formalism⁸⁾. Writing

$$\mathcal{P}(\bar{\Psi}, \Psi) \propto \exp \left\{ - \int d^4x \mathcal{L}_{\text{eff}}(\bar{\Psi}, \Psi) \right\}$$

and after including in the measure factors independent of the fermionic field, we obtain

$$\begin{aligned} \mathcal{L}_{\text{eff}}(\bar{\Psi}, \Psi) = & -\frac{i}{2} \left\{ \bar{\Psi}(x, t) (\not{D} + im) \Psi(x, t) + \right. \\ & + \int_0^t d\tau \dot{\bar{\Psi}}(x, \tau) (\not{D} - im)^{-1} \dot{\Psi}(x, \tau) + \\ & \left. + \int_0^t d\tau \bar{\Psi}(x, \tau) (\not{D} + im) (\not{D} - im) (\not{D} + im) \Psi(x, \tau) \right\} \quad (2.10) \end{aligned}$$

Notice that only the first derivative in t appears, a fact which shows the Markovian character of the process before regularization. The operator $(\not{D} - im)^{-1}$ is not well defined if the infra-red regulator m is set equal to zero; nevertheless, the total operator $\delta_t (\not{D} - im)^{-1} \delta_t^\dagger$ does give well defined quantities for all m when estimating mean values.

From this action (and its straightforward extension including gauge fields whenever necessary), it is now possible to derive the WI of the theory. One can, for instance, use the standard procedure of performing an adequate infinitesimal change of variables on the field, and equating to zero the first order correction in the action and the measure⁹⁾. We will now explicitly discuss the chiral WI. Let us consider the transformation

$$\begin{aligned}\delta \psi(x, \tau) &= i \alpha(x) \gamma_5 \psi(x, \tau) \\ \delta \bar{\psi}(x, \tau) &= i \alpha(x) \bar{\psi}(x, \tau) \gamma_5\end{aligned}\quad (2.11)$$

with $\alpha(x)$ independent of the fictitious time. It is possible to perform a more general transformation $\alpha(x, \tau)$, but this one will be enough for our purposes. The resulting WI is:

$$\begin{aligned}& \left\langle \partial_\mu J_\mu^5(x, t) \right\rangle_\eta + 2im \left\langle \bar{\psi}(x, t) \gamma_5 \psi(x, t) \right\rangle_\eta + \\& + \left\langle \int_0^t d\tau \bar{\dot{\psi}}(x, \tau) \left\{ \gamma_5, \frac{im}{\not{D}^2 + m^2} \right\} \dot{\psi}(x, \tau) \right\rangle_\eta + \\& + \left\langle \int_0^t d\tau \bar{\psi}(x, \tau) \left\{ \gamma_5, im(\not{D}^2 + m^2) \right\} \psi(x, \tau) \right\rangle_\eta = a(x, t)\end{aligned}\quad (2.12)$$

where $a(x, t)$ is the contribution from the integration measure. Notice that all the terms on the left-hand side which cannot be written as a divergence would have not appeared if we had taken $m = 0$. The first term is a total divergence given by

$$\begin{aligned}-\partial_\mu J_\mu^5(x, t) &= \bar{\psi}(x, t) \left\{ \gamma_5, \not{D} \right\} \psi(x, t) + \\& + \int_0^t d\tau \bar{\dot{\psi}}(x, \tau) \left\{ \gamma_5, \frac{\not{D}}{\not{D}^2 + m^2} \right\} \dot{\psi}(x, \tau) + \\& + \int_0^t d\tau \bar{\psi}(x, \tau) \left\{ \gamma_5, \not{D}(\not{D}^2 + m^2) \right\} \psi(x, \tau)\end{aligned}\quad (2.13)$$

Although we do not write the explicit expression for J_μ^5 (for space-saving reasons), a simple calculation shows that it is invariant under gauge transformations independent of t . Note that besides the naïve current $\bar{\psi} \gamma_\mu \gamma_5 \psi$ given by the first term of (2.13), there are other terms which cannot be neglected.

In order to determine the contribution of the measure to the WI, a regularization procedure must be chosen. Let us consider SR: Eq. (2.3b) is

replaced by Eq. (2.4). Although we have considered different choices for $\alpha_\sigma(t-t')$ yielding the same conclusions, we present here our results only for the case¹⁰⁾

$$\alpha_\sigma(\tau-\tau') = \frac{\sigma}{2} |\tau-\tau'|^{\sigma-1} \quad (2.14)$$

The measure contribution can now be obtained by a direct calculation of the averages on the left-hand side of (2.12). Any of its terms can be evaluated by projecting the fermion fields on eigenfunctions of the Dirac operator ($D\psi_E = E\psi_E$) and proceeding as in Ref. 1). The regulated propagator reads

$$\begin{aligned} R_{SR}^{ba}(y, x; t', t) &= i \int_0^t d\tau \int_0^t d\tau' \alpha_\sigma(\tau-\tau') \cdot \\ &\cdot \sum_E \psi_E^b(y) \psi_E^{a*}(x) \cdot e^{-(E^2+m^2)(t+t'-\tau-\tau')} \end{aligned} \quad (2.15)$$

where

$$R^{ba}(y, x; t', t) \equiv \langle \psi^b(y, t') \bar{\psi}^a(t, x) \rangle_h$$

and the sum over E is symbolic. With this tool, we obtain the mean value of the different terms in the WI for large t . Throughout all the paper the limit $t \rightarrow \infty$ will be taken keeping m finite. In this case, we obtain

$$\langle \bar{\psi}(x, t) \{ \gamma_5, \not{D} - im \} \psi(x, t) \rangle_h = \frac{i}{2} \sum_E \frac{\psi_E^+(x) \gamma_5 \psi_E(x)}{(E^2+m^2)^\sigma} \Gamma(\sigma+1) \quad (2.16a)$$

$$\begin{aligned} \int_0^t d\tau \langle \dot{\bar{\psi}}(x, \tau) \{ \gamma_5, \frac{1}{\not{D} - im} \} \dot{\psi}(x, \tau) \rangle_h &= \\ &= \frac{i}{2} \sum_E \frac{\psi_E^+(x) \gamma_5 \psi_E(x)}{(E^2+m^2)^\sigma} \Gamma(\sigma+1) \{ (E^2+m^2)t - (\sigma+1/2) \} \end{aligned} \quad (2.16b)$$

$$\begin{aligned} \int_0^t d\tau \langle \bar{\psi}(x, \tau) \{ \gamma_5, (\not{D} + im)(\not{D} - im)(\not{D} + im) \} \psi(x, \tau) \rangle_h &= \\ &= \frac{i}{2} \sum_E \frac{\psi_E^+(x) \gamma_5 \psi_E(x)}{(E^2+m^2)^\sigma} \Gamma(\sigma+1) \{ -(E^2+m^2)t - (\sigma+1/2) \} \end{aligned} \quad (2.16c)$$

as $t \rightarrow \infty$. Here we gathered the terms in the WI (2.12) according to its origin from the effective Lagrangian (2.10). Notice the cancellation of the dangerous terms linear in t . In fact, the sum of all the terms is zero and consequently there is no contribution from the measure $a(x,t)$. A further remark is that the generalized mass terms, those which would disappear for zero fermionic mass, also cancel exactly between themselves. We are finally left with a Ward identity for zero mass fermions

$$\langle \partial^\mu J_\mu^5(x,t) \rangle_\eta = 0 \quad (2.17)$$

and this current has no anomaly.

We conclude then that the theory defined by the effective Lagrangian (2.10) and SR has a non-anomalous chiral WI given by (2.13) and (2.17) as expected. However, we want to stress that nothing can be concluded with respect to the field theory interpretation from the above discussion. This interpretation needs a WI where the fermionic fields depend only on the final time t . Although in the WI, Eqs. (2.12) and (2.13), it is possible to split the naïve current from the other terms, and associate the average of the last ones with the anomaly, this is an arbitrary procedure. In fact, as will be seen in the next section, those terms give both field operator contributions and terms of an anomalous type.

A final remark to be done here is that the charge $U(1)$ current can be studied with the same techniques used for the axial one. It is straightforward to derive that, besides the naïve vector current, it contains two extra pieces. The current is of course conserved and the WI is

$$\begin{aligned} & \langle \bar{\Psi}(x,t) (\vec{D} - \vec{D}') \Psi(x,t) \rangle_\eta + \left\langle \int_0^t d\tau \dot{\bar{\Psi}}(x,\tau) \left(\frac{1}{\vec{D} - im} - \frac{1}{\vec{D}' - im} \right) \dot{\Psi}(x,\tau) \right\rangle_\eta + \\ & + \left\langle \int_0^t d\tau \bar{\Psi}(x,\tau) \left\{ (\vec{D} + im)(\vec{D}^2 + m^2) - (\vec{D}' + im)(\vec{D}'^2 + m^2) \right\} \Psi(x,\tau) \right\rangle_\eta = 0 \end{aligned} \quad (3.18)$$

3. - CONNECTION WITH THE UNREGULARIZED FIELD THEORY ($t \rightarrow \infty$ limit)

As was shown in the last section, SR yields a regularized effective theory with a conserved gauge invariant chiral current (for zero fermionic

mass). This is expected for a theory in which the defining equations and regularization procedure respect both symmetries.

The above remarks do not imply however that the field theory defined by the equilibrium regime of the stochastic equations (2.1) and (2.3) will also have a similar WI. Any operator meaningful in field theory should be expressed in terms of the fields at the final fictitious time t , $\phi(x,t)$, and the $t \rightarrow \infty$ limit taken. Clearly, the unregularized effective Lagrangian Eq. (2.10) and the unregularized WI Eq. (2.12) contain operators described in terms of $\phi(x,t)$ as well as operators described in terms of $\phi(x,\tau)$ for all $\tau \leq t$. These last dependences have to be integrated out before the connection with field theory is clarified.

We will first study the effective action and then the WI.

Let us remember^{4),7)} that the field theory action is expected to be recovered for $t \rightarrow \infty$ because in this limit the probability distribution of

$$\Psi(x) \equiv \Psi(x,t) \quad (3.1)$$

is given by

$$P(\bar{\Psi}(x), \Psi(x)) = \lim_{t \rightarrow \infty} P(\bar{\Psi}(x,t), \Psi(x,t)) = e^{-S} \quad (3.2)$$

S being the unregularized field theory action

$$S \equiv \bar{\Psi}(x) (\not{D} + im) \Psi(x) \quad (3.3)$$

Once this probability is obtained, quantum fluctuations are contained as usual in the path integral over $\phi(x)$. The relation of these probability distributions with the one contained in (2.9) $\mathcal{P}(\bar{\phi}(x,\tau), \phi(x,\tau))$ is given by

$$P(\bar{\Psi}(x,t), \Psi(x,t)) = \int \prod_{x,\tau < t} D\bar{\Psi}(x,\tau) D\Psi(x,\tau) \mathcal{P}(\bar{\Psi}(x,\tau), \Psi(x,\tau)) \quad (3.4)$$

($\mathcal{P}$ is also a functional of the fields at time t , $\phi(x,t)$, $\bar{\phi}(x,t)$ which remain unintegrated) in such a way that

$$\begin{aligned} \langle \theta \rangle_S &= \lim_{t \rightarrow \infty} \int \mathcal{D}\bar{\psi}(x,t) \mathcal{D}\psi(x,t) P(\bar{\psi}(x,t), \psi(x,t)) \theta = \\ &= \lim_{t \rightarrow \infty} \int \mathcal{D}\bar{\psi}(x,t) \mathcal{D}\psi(x,t) \prod_{\tau < t} \mathcal{D}\bar{\psi}(x,\tau) \mathcal{D}\psi(x,\tau) P(\bar{\psi}, \psi) \cdot \theta^{(3.5)} \end{aligned}$$

where θ is a fermionic operator.

Let us proceed to the explicit evaluation of the functional integrals over $\psi(x,\tau)$ and $\bar{\psi}(x,\tau)$ relevant for the effective action. We separate the evolution in the fictitious time of the field $\psi(x,\tau)$ into classical plus fluctuation components

$$\begin{aligned} \psi(x,\tau) &= \psi_{cl}(x,\tau) + y(x,\tau) \\ \bar{\psi}(x,\tau) &= \bar{\psi}_{cl}(x,\tau) + \bar{y}(x,\tau) \end{aligned} \quad (3.6)$$

The fields are kept fixed at $\tau = 0$ and $\tau = t$. This means that

$$y(x,0) = y(x,t) = 0 \quad (3.7a)$$

$$\psi_{cl}(x,t) = \psi(x,t) \quad (3.7b)$$

The initial condition $\psi(x,0) = 0$ gives also

$$\psi_{cl}(x,0) = 0 \quad (3.8)$$

because of the conditions (3.7) and (3.8).

The part of the effective action (2.10) that has to be integrated reads

$$S \equiv \frac{i}{2} \int d^4x \int_0^t d\tau \left\{ \dot{\bar{\psi}}(x, \tau) (\not{D} - im)^{-1} \dot{\psi}(x, \tau) + \bar{\psi}(x, \tau) (\not{D} + im) (\not{D}^2 + m^2) \psi(x, \tau) \right\} \quad (3.9)$$

The calculation amounts to evaluate the propagator K of a field theory defined by the quadratic action (3.9) (for $t \rightarrow \infty$), i.e.,

$$K(\psi(x, t), 0; t, 0) = \prod_{x, \tau < t} \int \mathcal{D}\psi(x, \tau) \mathcal{D}\bar{\psi}(x, \tau) e^{-S} \quad (3.10)$$

where the bracket in K refers to the propagation from time 0 to time t in which the field value changes from 0 to $\psi(x)$. Though the decomposition (3.5), the classical field is precisely defined as the one minimizing s

$$\left. \frac{\delta S}{\delta \bar{\psi}(x, \tau)} \right|_{\psi = \psi_{cl}} = 0 \quad (3.11)$$

together with the conditions (3.7) and (3.8). $\bar{\psi}_{cl}(x, \tau)$ is defined analogously. On the other hand, $y(x, \tau)$ can take any value in the interval $(0, t)$ except for the extremum conditions (3.7a). Finally, (3.10) can be written

$$K(\psi(x, t), 0; t, 0) = e^{-S_{cl}} \prod_{x, \tau < t} \int \mathcal{D}y(x, \tau) \mathcal{D}\bar{y}(x, \tau) \cdot e^{-\frac{i}{2} \int d^4x \int_0^t d\tau \left\{ \dot{\bar{y}}(x, \tau) (\not{D} - im)^{-1} \dot{y}(x, \tau) + \bar{y}(x, \tau) (\not{D} + im) (\not{D}^2 + m^2) y(x, \tau) \right\}} \quad (3.12)$$

where s_{cl} is the action evaluated at the classical value of the fields. Crossed terms in ψ_{cl} and y do not appear because of the classical equations which read

$$\begin{cases} \ddot{\Psi}_Q(x, \tau) - (\not{D}^2 + m^2)^2 \Psi_Q(x, \tau) = 0 \\ \ddot{\bar{\Psi}}_Q(x, \tau) - \bar{\Psi}_Q(x, \tau) (\not{D}^2 + m^2) = 0 \end{cases} \quad (3.13)$$

with the solution

$$\begin{aligned} \Psi_Q(x, \tau) &= \text{sh}[(\not{D}^2 + m^2)\tau] \cdot \text{sh}^{-1}[(\not{D}^2 + m^2)t] \Psi(x, t) \\ \bar{\Psi}_Q(x, \tau) &= \bar{\Psi}(x, t) \text{sh}^{-1}[(\not{D}^2 + m^2)t] \text{sh}[(\not{D}^2 + m^2)\tau] \end{aligned} \quad (3.14)$$

In consequence, $s_{c\lambda}$ given by

$$\begin{aligned} s_Q &= \frac{i}{2} \int d^4x \bar{\Psi}(x, t) \frac{\text{ch}[(\not{D}^2 + m^2)t]}{\text{sh}[(\not{D}^2 + m^2)t]} (\not{D} + im) \Psi(x, t) \\ &\xrightarrow{t \rightarrow \infty} \frac{i}{2} \int d^4x \bar{\Psi}(x, t) (\not{D} + im) \Psi(x, t) \end{aligned} \quad (3.15)$$

and

$$\begin{aligned} S_{\text{eff}} &= \frac{i}{2} \left\{ \int d^4x \bar{\Psi}(x, t) (\not{D} + im) \Psi(x, t) + s_Q + s_y \right\} \\ &\xrightarrow{t \rightarrow \infty} i \left\{ \int d^4x \bar{\Psi}(x, t) (\not{D} + im) \Psi(x, t) + s_y \right\} \end{aligned} \quad (3.16)$$

where s_y is contained in Eq. (3.12)

$$J_y \equiv \int d^4x \int_0^t d\tau \left\{ \dot{\bar{y}}(x,\tau) (\not{D} - im)^{-1} \dot{y}(x,\tau) + \bar{y}(x,\tau) (\not{D} + im) (\not{D}^2 + m^2) y(x,\tau) \right\} \quad (3.17)$$

The important fact to note here is that, from the point of view of field theory, s_y is a constant, a quantity not depending on the fermionic fields.

We have then recovered the standard field theory action $S \equiv \bar{\psi}(x)(\not{D} + im)\psi(x)$, plus a term independent of the fields $\psi(x)$. The above is a proof that in this functional integral formalism the probability distribution of $\psi(x)$ converges to the right one, e^{-S} . It is in fact the Lagrangian version of the usual derivation using the Fokker-Planck Hamiltonian^{8),7)}.

Consequently, for any quantity depending only on $\psi(x,t)$, $\bar{\psi}(x,t)$, the bilinear dependence of S_{eff} on $y(x,\tau)$ is irrelevant: it will factorize when taking mean values; this is the case for field theory operators such as the naive terms in the WI, Eq. (2.12).

Conversely, the s_y part of the action cannot be overlooked when considering quantities which depend on the fictitious time history of the fields $\psi(x,\tau)$ such as the extra terms in (2.12), which are expressed as integrations over all $\tau \leq t$. For those, one should extract first the $\psi_{\text{cl}}(x,\tau)$ dependent part [which will be analyzed along the lines of the above paragraph after applying (3.14)]. The remaining $y(x,\tau)$, $\bar{y}(x,\tau)$ dependent part will be just a functional of the external gauge fields and its average must be taken only with s_y . Here the irrelevant part of the action is the bilinear dependence on $\psi(x)$.

Equation (2.13) can be rewritten in the following way:

$$i \bar{\psi}(x,t) \left\{ \not{x}_5, \not{D} \right\} \psi(x,t) + 2m i \bar{\psi}(x,t) \not{x}_5 \psi(x,t) + \\ + d(x,t) + \mu(x,t) = 0 \quad (3.18)$$

where $d(x,t)$ collects those terms which are a total divergence, excluding the naive ones

$$d(x,t) \equiv i \int_0^t d\tau \left\{ \dot{\bar{\psi}}(x,\tau) \left\{ \gamma_5, \frac{D}{D^2+m^2} \right\} \dot{\psi}(x,\tau) + \bar{\psi}(x,\tau) \left\{ \gamma_5, D(D^2+m^2) \right\} \psi(x,\tau) \right\} \quad (3.19)$$

and $\mu(x,t)$ collects the generalized mass terms

$$\mu(x,t) \equiv \int_0^t d\tau \left\{ \dot{\bar{\psi}}(x,\tau) \left\{ \gamma_5, \frac{i m}{D^2+m^2} \right\} \dot{\psi}(x,\tau) + \bar{\psi}(x,\tau) \left\{ \gamma_5, i m (D^2+m^2) \right\} \psi(x,\tau) \right\} \quad (3.20)$$

d and μ receive contributions both from the classical and fluctuation components of the field $\phi(x,\tau)$.

The τ integration gives

$$d_{cl}(x,t) \xrightarrow{t \rightarrow \infty} \bar{\psi}(x,t) \left\{ \gamma_5, D \right\} \psi(x,t) = -\partial_\mu j_\mu^5(x,t) \quad (3.21)$$

$$\mu_{cl}(x,t) \xrightarrow{t \rightarrow \infty} 2 i m \bar{\psi}(x,t) \gamma_5 \psi(x,t) \quad (3.22)$$

and since the relevant weight for those is given by e^{-S} , their contribution to the WI is, for $t \rightarrow \infty$

$$\langle \partial_\mu j_\mu^5(x,t) \rangle_S = 2 i m \langle \bar{\psi}(x,t) \gamma_5 \psi(x,t) \rangle_S \quad (3.23)$$

duplicating the result of the naive terms.

Let us now turn to the fluctuation contributions d_y and μ_y . The relevant quantity to be evaluated is

$$\int \prod_{x,\tau < t} D y(x,\tau) (d_y + \mu_y) e^{-\frac{i}{2} \int d^4x \int_0^t d\tau \left\{ \bar{y} (D - i m)^{-1} \dot{y} + \bar{y} (D + i m) (D^2 + m^2) y \right\}} \quad (3.24)$$

Since y is zero at the endpoints of the integration region and the quadratic forms contain D , it is convenient to expand

$$y(x,\tau) = \sum_{E,m} a_{E,m} \sin\left(\frac{\pi m \tau}{t}\right) \psi_E(x) \quad (3.25)$$

n is an integer and $\psi_E(x)$ are eigenfunctions of the Dirac operator $D \psi_E = E \psi_E$.

In this way the quadratic forms both in the exponential and the "operator" are diagonalized (actually, the diagonal elements of the two matrices are proportional). It is easy to obtain

$$\mu_y = 4im \operatorname{tr} \sum_{E,m} \frac{\psi_E^+(x) \gamma_5 \psi_E(x)}{E + im} \quad (3.26)$$

$$d_y = 4i \operatorname{tr} \sum_{E,m} \psi_E^+(x) \gamma_5 \psi_E(x) - 4im \operatorname{tr} \sum_{E,m} \frac{\psi_E^+(x) \gamma_5 \psi_E(x)}{E + im}$$

and only the terms proportional to m cancel in the sum. The WI is then, for $t \rightarrow \infty$

$$\langle \partial_\mu j_5^\mu(x) \rangle_S = 2mi \langle j_5(x) \rangle_S + 2i A(x) \quad (3.27a)$$

where

$$j_5(x) = \bar{\psi}(x) \gamma_5 \psi(x) \quad (3.27b)$$

and

$$A(x) = \left(\sum_m 1 \right) \operatorname{tr} \sum_E \psi_E^+(x) \gamma_5 \psi_E(x) \quad (3.27c)$$

The last term is an anomalous contribution to the WI, it is ill-defined for the unregularized theory. Note that apart from the usual $0 \times \infty$ indeterminacy, there is an extra $\sum_n 1$ coming from the contribution of all the fluctuation modes.

The conclusion is that the WI (2.12) for the unregularized effective theory with fields parametrized by τ expresses the conservation of a non-anomalous current $J_\mu^5(x, t)$. This current contains (in the $t \rightarrow \infty$ limit) the unregularized field theory classical one, plus a term due to the fluctuations of the fields in all fictitious times $\tau < t$; these fluctuations are at the origin of the chiral anomaly.

Of course, the final analysis of the anomalies needs the consideration of the regulated theory. This will be done in the next section.

We should finally remark that in the case of the vector WI (2.18), a similar analysis shows that there is no contribution from the fluctuations of the fields in the fictitious times. This is consistent with the absence of anomalies in that WI.

4. - THE REGULARIZED WARD IDENTITIES

We will treat here the regularized version of the Langevin equations (2.1) and (2.3) in two cases: Pauli-Villars regularization and stochastic regularization [consisting, as explained, in the replacement of (2.3b) by (2.4)].

We implement Pauli-Villars regularization in the present context by considering, besides (2.1) and (2.3), a parallel set of Langevin equations for a new field Ψ with mass M and noises N_1 and N_2 , etc. The functional form of the equations is otherwise unchanged. Bose statistics is assumed not only for Ψ and $\bar{\Psi}$, as usual, but also for the noises $N_1, N_2, \bar{N}_1, \bar{N}_2$. Their vector and chiral charges are the same as the corresponding ones for $\phi, \bar{\phi}, \eta_1, \eta_2, \bar{\eta}_1, \bar{\eta}_2$.

It is now easy to see that the unregularized effective Lagrangian (2.10) is replaced by

$$\begin{aligned}
 -\mathcal{L}_{\text{eff}}^{\text{P-V}}(\psi, \bar{\psi}, \Psi, \bar{\Psi}) = & -\frac{i}{2} \left\{ \bar{\psi}(x, t) (\not{D} + im) \psi(x, t) + \right. \\
 & + \int_0^t d\tau \dot{\bar{\psi}}(x, \tau) (\not{D} - im)^{-1} \dot{\psi}(x, \tau) + \int_0^t d\tau \bar{\psi}(x, \tau) (\not{D} + im) (\not{D}^2 + m^2) \psi(x, \tau) + \\
 & + \bar{\Psi}(x, t) (\not{D} + iM) \Psi(x, t) + \int_0^t d\tau \bar{\Psi}(x, \tau) (\not{D} - iM)^{-1} \dot{\Psi}(x, \tau) + \\
 & \left. + \int_0^t d\tau \bar{\Psi}(x, \tau) (\not{D} + iM) (\not{D}^2 + M^2) \Psi(x, \tau) \right\} \quad (4.1)
 \end{aligned}$$

In order to make contact with field theory, the treatment of the classical and fluctuating components is analogous to the ϕ and $\bar{\phi}$ decomposition [Eqs. (3.6), (3.7) and (3.8)]. Denoting by $Y(x, \tau)$ the fluctuating component of $\Psi(x, \tau)$, its Fourier decomposition [see (3.25)] reads

$$Y(x, \tau) = \sum_{E, m} \alpha_{E, m} \exp\left(\frac{\pi m}{t} \tau\right) \quad (4.2)$$

The crucial point is that $\alpha_{E, m}$ are now ordinary numbers due to the Bose statistics, not Grassmann variables. This implies that the fluctuation contributions to the regularized WI from $(\phi, \bar{\phi})$ and $(\Psi, \bar{\Psi})$ cancel each other. The WI is now given by

$$\begin{aligned} \partial_\mu \langle \bar{\Psi}(x, t) \gamma_5 \gamma_\mu \Psi(x, t) \rangle_S &= 2m i \langle \bar{\Psi}(x, t) \gamma_5 \Psi(x, t) \rangle - \\ &- \partial_\mu \langle \bar{\Psi}(x, t) \gamma_5 \gamma_\mu \Psi(x, t) \rangle + 2M i \langle \bar{\Psi}(x, t) \gamma_5 \Psi(x, t) \rangle \end{aligned} \quad (4.3)$$

When the limit $M \rightarrow \infty$ is taken, the last two terms in Eq. (4.3) yield the usual anomalous WI

$$\begin{aligned} \partial_\mu \langle \bar{\Psi}(x, t) \gamma_5 \gamma_\mu \Psi(x, t) \rangle_{reg} &= \\ &= 2m i \langle \bar{\Psi}(x, t) \gamma_5 \Psi(x, t) \rangle_{reg} + \frac{i}{8\pi^2} \text{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \end{aligned} \quad (4.4)$$

Note that the cancellation of the τ -fluctuation contributions from ϕ and Ψ is analogous to the measure cancellations in Fujikawa's treatment of Pauli-Villars regularization ⁹⁾.

SR is also straightforward. In this case the WI are still given by Eq. (3.27a) but averages now are taken with the SR prescription (2.4) and (2.14).

The averages of the two terms containing the fermionic fields are computed as in Ref. 1), yielding the correct regularized chiral anomaly

$$A(x) \propto \sum_E \frac{\psi_E^+(x) \gamma_5 \psi_E(x)}{(E^2 + m^2)^\sigma} \quad (4.5)$$

5. - CONCLUSIONS

In this work we showed how to construct WI in SQ for any fictitious time t . The procedure has two steps. First the WI are derived for an effective field theory containing the fermion field at any intermediate time τ , in this way a SR conserved chiral current is obtained. Secondly, this WI cannot be seen as a classical Noether current from the point of view of the equilibrium field theory. It contains integrals over all fictitious times which include quantum fluctuations giving rise to a chiral anomalous term.

The Pauli-Villars regularization of the chiral WI is straightforward. SR presents an apparent contradiction. The non-anomalous conserved chiral current obtained at the first step is expected from the gauge and chiral invariances of the defining equations (2.1), (2.7) and the regularization procedure. At the second step the existence of a chiral anomaly seems to be in disagreement with that fact. However, it can be easily understood: although Eq. (2.8) has a measure (including the two Dirac deltas) which is gauge and chiral invariant, in stochastic quantization the anomaly does not come from the variation of the final measure; as we said before it is constructed during the evolution in the stochastic time τ .

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While this paper was still in the typing pool, a later CERN preprint, which confirms part of our results, was written and submitted for publication ("Conserved Noether currents in stochastic quantization" by R. Kirschner et al., CERN-TH.4406/86).

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