GEOMETRY AND ANOMALY STRUCTURE OF SUPERSYMMETRIC σ -MODELS

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ABSTRACT

Supersymmetric σ -models, which describe low-energy interactions of Goldstone fields arising from spontaneous symmetry breaking, correspond to non-compact, non-homogeneous Kähler manifolds. We discuss systematically general features of this class of σ -models in comparison with the more familiar σ -models based on compact, homogeneous (coset) Kähler manifolds G/H . A key property of supersymmetric σ -models is their hidden invariance under local $\tilde{H}$ transformations, where $\tilde{H}$ is a subgroup of the complexification $\bar{G}$ of G , which characterizes the specific model. Contrary to homogeneous ones, non-homogeneous σ -models may be supplemented with local counter terms which cancel isometry anomalies. We illustrate our general discussion by means of a σ -model based on the coset space $U(n+2)/U(n) \times SU(2)$.

1. INTRODUCTION

Supersymmetric σ -models^{*)} describe interactions of nonlinearly transforming fields. To these belong Goldstone fields which arise from spontaneous symmetry breaking in supersymmetric theories. Supersymmetry requires that the Goldstone bosons are contained in chiral Goldstone superfields, together with additional scalar and fermionic degrees of freedom. The geometrical structure of supersymmetric σ -models is that of Kähler manifolds [2].

Supersymmetric σ -models may play an important role as effective Lagrangians for quarks and leptons [3, 4]. The idea that quarks and leptons are quasi-Goldstone fermions, i.e., superpartners of Goldstone bosons, leads to strong constraints on their quantum numbers. Particularly interesting are the coset spaces of exceptional groups [3-5]. Furthermore, effective Lagrangians resulting from spontaneous compactification can have the form of a σ -model with some nonlinearly realized global symmetry [6].

In supersymmetric theories the complex extension $\bar{G}$ of a global or local unitary symmetry group G plays an important role, as was first noticed in Ref. [7]. The crucial quantity in the construction of a supersymmetric σ -model associated with the symmetry breaking from G to H is $\tilde{H}$ [8, 9], a complex subgroup of $\bar{G}$, which is fixed by the symmetry breaking vacuum expectation values. $\tilde{H}$ determines the dimension of the Kähler manifold, its complex structure, the form of the Kähler potential and, as we will discuss in detail in this paper, the local Wess-Zumino type counter term which has to be added in order to cancel isometry anomalies.

Important progress has been made in the classification of homogeneous Kähler manifolds [10], the general construction of Kähler potentials [11, 12], and also in the analysis of supersymmetric σ -model anomalies [13-16]. In this paper we attempt a comprehensive study of the simplest class of supersymmetric σ -models, which can result from spontaneous symmetry breaking, and which differ 'minimally' from the familiar homogeneous σ -models.

The paper is organized as follows. In Section 2 we review the connection between supersymmetric σ -models and complex coset spaces $\bar{G}/\tilde{H}$, with particular emphasis on the difference between homogeneous and non-homogeneous σ -models. Our general discussion is illustrated in Section 3 by means of a σ -model based on the coset space $U(n+2)/U(n) \times SU(2)$. Section 4 deals with the geometry of supersymmetric σ -models, i.e., the transition from 'curved' to 'flat' indices, the holonomy group, and the important concept of $\tilde{H}$ gauge theories. In Section 5 we then turn to a discussion of isometry anomalies and of conditions for the

*) We focus on $d = 4$, $N = 1$; for a review, see Ref. [1].

existence of supersymmetric Wess-Zumino type local counter terms, which cancel these anomalies. Stronger than usually considered, 't Hooft type matching conditions are conjectured and illustrated by means of the $U(n+2)/U(n) \times SU(2)$ example. Summary and conclusions are presented in Section 6.

2. SUPERSYMMETRIC σ -MODELS AND COMPLEX COSET SPACES

In any field theory the spontaneous breaking of a symmetry group G to a subgroup H gives rise to a set of massless Goldstone bosons which is determined by the set of broken generators of the Lie algebra $\mathfrak{g}(G)$. The corresponding scalar Goldstone fields can be thought of as coordinates of the coset manifold G/H [17] at each point of space-time. This leads naturally to a non-linear realization of the symmetry group G on the Goldstone fields and to a G -invariant σ -model, which is the effective Lagrangian of the low energy interactions of the Goldstone bosons [17, 18].

In a supersymmetric theory the scalar Goldstone fields have to be embedded into chiral superfields and the σ -model has to be part of a supersymmetric σ -model. This requires additional degrees of freedom, quasi-Goldstone bosons and quasi-Goldstone fermions, which are related to the Goldstone bosons through supersymmetry, but which are not themselves Goldstone fields satisfying low energy theorems.

If the symmetry breaking from G to H leaves supersymmetry unbroken, the non-linear realization of G on the Goldstone fields has to be part of a non-linear realization of G on a set of chiral Goldstone superfields ϕ^α ,

$$\begin{aligned}\delta_A \phi^\alpha &\equiv [T_A, \phi^\alpha] = k_A^\alpha(\phi), \\ \delta_A \bar{\phi}^{\bar{\alpha}} &\equiv [T_A, \bar{\phi}^{\bar{\alpha}}] = \bar{k}_A^{\bar{\alpha}}(\bar{\phi}),\end{aligned}\tag{2.1}$$

where

$$\bar{k}_A^{\bar{\beta}}(\bar{z}) = -k_A^{\beta*}(z) \quad ;$$

the superfields $\phi^\alpha = (\varphi^\alpha, \psi^\alpha)$ contain complex scalars φ^α and Weyl fermions ψ^α , T_A , $A = 1 \dots \dim \mathfrak{g}(G)$, represent the generators of G , and $k_A^\alpha(\varphi)$ are the Killing vectors of the complex manifold spanned by the scalar fields φ^α . As the realization (2.1) of G does not mix chiral and anti-chiral superfields, it is automatically a realization of the complex extension $\bar{G}$ of G .

Supersymmetric σ -models can be classified by means of complex subgroups of $\bar{G}$. In general, the following complex subgroups have to be distinguished: $\bar{H}$, the complex extension of H , whose generators H_i act linearly in (2.1); $\tilde{H} \supset \bar{H}$, whose Lie algebra contains in addition the broken generators Y_p , which are non-linearly realized in (2.1) but which do not shift the Goldstone fields, i.e., $k_p^\alpha(\phi)|_{\phi=0} = 0$; finally, the complex coset space $\bar{G}/\tilde{H}$ may itself form a complex subgroup of $\bar{G}$, whose Lie algebra is given by the broken generators X_a , which do shift the Goldstone fields, i.e., $k_a^\alpha(\phi)|_{\phi=0} \neq 0$. The number of generators X_a is equal to the number N_π of chiral Goldstone superfields ϕ^α , which in turn is equal to the complex dimension of the manifold $\bar{G}/\tilde{H}$,

$$N_\pi = \dim_c (\bar{G}/\tilde{H}) \quad (2.2)$$

The non-compact complex extension $\bar{G}$ of G appears as the invariance group of the superpotential [7] (i.e., of the scalar potential minimum) in the underlying linear theory and also in the supersymmetric Goldstone theorem [8, 19]. Its field theoretic proof requires only the existence of a symmetry breaking vacuum expectation value $\langle \Phi \rangle$ of a chiral superfield operator [20]. $\tilde{H}$ is the stability group of the vacuum, whereas $\bar{G}/\tilde{H}$ is generated by the 'broken' generators,

$$H_i \langle \Phi \rangle = Y_p \langle \Phi \rangle = 0, \quad (2.3a)$$

$$X_a \langle \Phi \rangle \neq 0 \quad (2.3b)$$

The complex subgroup $\tilde{H}$ plays a crucial role in defining the supersymmetric σ -model. According to (2.2) it determines the number of Goldstone multiplets which, contrary to the non-supersymmetric case, is not fixed by the real dimension of G/H . Instead one has

$$\frac{1}{2} \dim (G/H) \leq N_\pi \leq \dim (G/H) \quad (2.4)$$

Examples with different numbers of Goldstone superfields for the same symmetry breaking from G to H have been discussed in the literature [3].

The maximal number of Goldstone superfields is reached for $\tilde{H} = \bar{H}$. In this case one has

$$N_\pi = \dim_c (\bar{G}/\bar{H}) = \dim (G/H), \quad (2.5)$$

i.e., each Goldstone superfield ϕ^α contains only one Goldstone boson, and the number of additional quasi-Goldstone bosons equals the number of Goldstone bosons, which is referred to as 'total doubling'. As the Goldstone bosons

always form a real representation of H , the quasi-Goldstone fermions will also be a real representation of H .

Particularly interesting is the case where $\tilde{H}$ is larger than $\bar{H}$. Then one has

$$N_{\tilde{H}} = \dim_c(\bar{G}/\tilde{H}) < \dim_c(\bar{G}/\bar{H}) = \dim(G/H) . \quad (2.6)$$

This implies that some of the Goldstone superfields have to contain two Goldstone bosons, and their fermionic superpartners may therefore be chiral with respect to H .^{*})

It is very important to realize that the case of 'non doubling', or 'total pairing', where all scalars are Goldstone bosons (which corresponds to a compact, homogeneous (coset) Kähler manifold) can never be obtained as a result of spontaneous symmetry breaking [22, 8]. This is most easily seen by dividing the generators of $\bar{G}$ into the non-hermitian step generators E_n , $E_{-n} = E_n^\dagger$ ($n = 1 \dots 1/2(\dim G - \text{rank } G)$) and the $U(1)$ generators H_r ($r = 1 \dots \text{rank } G$) of the Cartan subalgebra, which satisfy the commutation relations

$$[E_n, E_{-n}] = \rho^r(n) H_r , \quad (2.7)$$

where $\rho^r(n)$ are the root vectors of G . The condition for 'total pairing' reads

$$N_{\tilde{H}} = \dim_c(\bar{G}/\tilde{H}) = \frac{1}{2} \dim(G/H) . \quad (2.8)$$

This means that the broken generators can be divided into two sets $\{Y_a\}$ and $\{X_a\}$, $a = 1 \dots 1/2 \dim(G/H)$, which leave the vacuum expectation values of the chiral and the anti-chiral fields invariant,

$$\begin{aligned} Y_{\bar{a}} \langle \Phi \rangle &= 0 , & X_a \langle \Phi \rangle &\neq 0 , \\ Y_{\bar{a}} \langle \bar{\Phi} \rangle &\neq 0 , & X_a \langle \bar{\Phi} \rangle &= 0 . \end{aligned} \quad (2.9)$$

Eq. (2.9) implies $Y_{\bar{a}} = X_a^\dagger$ and can be satisfied by a set of step generators,

$$\{Y_{\bar{a}}\} = \{E_a\} , \quad \{X_a\} = \{E_{-a}\} . \quad (2.10)$$

^{*}) In the context of preon theories, this case has first been considered in Ref. [21].

Furthermore, (2.9) requires that all generators H_r of the Cartan subalgebra are unbroken. This, however, is incompatible with (2.10). Namely, from the algebra (2.7) we obtain

$$[E_a, E_{-a}] \langle \Phi \rangle = g^r(a) H_r \langle \Phi \rangle \neq 0, \quad (2.11)$$

which implies that at least one $U(1)$ generator H_r is broken. This leads to a 'partial doubling', where the corresponding Goldstone boson sits together with an additional quasi-Goldstone boson in a chiral multiplet, the 'novino' field [23], which is necessarily a singlet with respect to H . Clearly, (2.8) is violated.

The case with just one such 'novino' H -singlet superfield is the most interesting one in that it allows isotropy representations with maximal chirality. (Note that for such models $\bar{G}/\tilde{H}$ is always a group.) In the following we will focus mainly on this case.

We have seen that supersymmetric σ -models are associated with complex coset spaces $\tilde{M} = \bar{G}/\tilde{H}$. The special case with 'non doubling' corresponds to $\tilde{M} \simeq M = G/H$, i.e., to a compact homogeneous Kähler manifold, on which each point can be reached from the origin by a symmetry (isometry) transformation which leaves the Kähler metric invariant. We will refer to such models as *homogeneous σ -models*. As we have shown, they cannot result from a spontaneous symmetry breaking.

The models, which can be obtained from spontaneous symmetry breaking, have 'partial doubling', i.e., they contain additional scalar degrees of freedom. More generally, such models occur automatically in theories with *flat directions*. They correspond to non-compact manifolds $\tilde{M} = \bar{G}/\tilde{H}$ with $M = G/H$ embedded as maximal compact submanifold. These models, however, are not invariant under any non-compact isometry group, and all kinetic terms are positive definite. Rather, the number of isometries of the Kählerian metric imposed on $\tilde{M}$ is smaller than the (real) dimension of $\tilde{M}$. Its only isometries correspond to unitary G -transformations which determine the 'shape' of the manifold in the compact directions. Along the non-compact directions the (Kählerian) metric is not restricted by any symmetry. Physically, this arbitrariness of the effective Lagrangian is due to the unconstrained interactions of the quasi-Goldstone bosons, which depend on the specific underlying theory. $\tilde{M} = \bar{G}/\tilde{H}$ is a homogeneous space with respect to $\bar{G}$ -transformations. On the other hand, the action of the compact subgroup G does not carry all points into each other. Hence, with respect to symmetry transformations, $\tilde{M} \supset M$ is not a homogeneous space. Therefore we refer to this kind of models as *non-homogeneous σ -models*.

Even if the number of Goldstone superfields, i.e., $\dim_{\mathbb{C}}(\bar{G}/\tilde{H})$, is fixed, there can still exist different choices of $\tilde{H}$ which correspond to inequivalent

complex structures [10, 12] on a real manifold of dimension $2N_\pi$, and thereby to different representations of the superfields Φ^α with respect to H . A simple example is the homogeneous Kähler manifold $SU(n+2)/SU(n) \times U(1)^2$, which allows for two inequivalent complex structures [24],

$$\Omega_I = \bar{n}^{-1, -n-1} + \bar{n}^{-n-1, -1} + \bar{1}^{n, -n}, \quad (2.12a)$$

and

$$\Omega_{II} = \bar{n}^{-1, -n-1} + \bar{n}^{n+1, 1} + \bar{1}^{n, -n}. \quad (2.12b)$$

They correspond to the complex subgroups shown in the figure. Note that Ω_{II} has no $SU(n)$ anomaly, which is not the case for Ω_I . This illustrates that different complex subgroups $\tilde{H}$ and, correspondingly, different complex structures can describe very different physical systems.

The basic variable of a supersymmetric σ -model is the representative of the complex coset space $\bar{G}/\tilde{H}$,

$$L(\phi) = e^{iX \cdot \phi}, \quad X \cdot \phi = X_a \phi^a \delta_a^\alpha, \quad (2.13)$$

which is the complex analogue of the CCWZ variable $\Xi(\pi)$ [17]. $L(\phi)$ transforms under a G -transformation as

$$L(\phi) \rightarrow L(\phi') = g L(\phi) \tilde{h}^{-1}(\phi, g), \quad (2.14)$$

$$g \in G, \quad \tilde{h} \in \tilde{H}.$$

From $L(\phi)$ one obtains the Maurer-Cartan 1-form of $\bar{G}/\tilde{H}$,

$$L^{-1}(\phi) d L(\phi) = E_a(\phi) d\phi^a, \quad (2.15)$$

which is an element of the Lie algebra $\mathfrak{k}(\bar{G})$. In the particularly interesting case where $\bar{G}/\tilde{H}$ differs from a homogeneous Kähler manifold just by one or several H -singlet 'novino' fields, $E_a(\phi)$ is a linear combination of the broken generators X_a only,

$$E_a(\phi) = E_a^\alpha(\phi) X_a, \quad (2.16)$$

and the coefficients E_α^a are the components of the vielbein on $\bar{G}/\tilde{H}$. This is different from conventional, compact cosets G/H where the Maurer-Cartan 1-form contains the $\mathfrak{g}(H)$ valued spin connection in addition to the vielbein.

From (2.14) one can easily find explicit expressions^{*}) for the Killing vectors and field dependent $\tilde{H}$ transformations in terms of $L(\phi)$. For infinitesimal G transformations,

$$g = 1 + \varepsilon^A T_A, \quad (2.17a)$$

$$\tilde{h} = 1 + \varepsilon^A \mathcal{S}_A(\phi), \quad (2.17b)$$

$$\phi^\alpha = \phi^\alpha + \varepsilon^A k_A^\alpha(\phi), \quad (2.17c)$$

and obtains

$$k_A^\alpha(\phi) = D_A^\alpha(L(\phi)) E^{-1\alpha}_a(\phi), \quad (2.18a)$$

$$\mathcal{S}_A^i(L(\phi)) = D_A^i(L(\phi)), \quad \mathcal{S}_A^P(L(\phi)) = D_A^P(L(\phi)), \quad (2.18b)$$

where $\mathcal{S}_A^i$ and $\mathcal{S}_A^P$ denote the coefficients of the generators of $\tilde{H}$ and $\tilde{H}/\bar{H}$ respectively. E^{-1} is the inverse vielbein,

$$E^{-1\alpha}_a E^\alpha_\beta = \delta^\alpha_\beta, \quad E^{-1\alpha}_a E^b_\alpha = \delta^b_a, \quad (2.19)$$

and D_A^B the adjoint representation of G ,

$$g^{-1} T_A g = D_A^B(g) T_B. \quad (2.20)$$

The Lagrangian of a supersymmetric σ -model is given by the Kähler potential of the corresponding Kähler manifold [2],

$$\mathcal{L} = \int d^4\theta K(\bar{\Phi}, \Phi). \quad (2.21)$$

All Kähler potentials for a given complex subgroup $\tilde{H}$ can be systematically constructed from $L(\phi)$ in the fundamental representation. In the case of homogeneous Kähler manifolds K is, up to 'decay constants', uniquely determined by the geometry of G/H and has the generic form [2, 11, 12],

^{*}) We follow the treatment of Ref. [25].

$$K(\bar{\Phi}, \Phi) = \sum_{i=1}^n c_i \ln \det_{\eta_i} (L^\dagger(\bar{\Phi}) L(\Phi)) , \quad (2.22)$$

where n is the number of $U(1)$ factors (central charges) of H , i.e., the dimension of the centre of H , and η_i are appropriate projection operators.

On the other hand, for non-homogeneous models the geometry of G/H does not specify the Kähler potential completely. If m of the n central charges are broken, which gives rise to m 'novino' type superfields, the most general Kähler potential involves an arbitrary function of m variables [26] of the form $\det_{\eta_i} (L^\dagger L)$. In the case of a single central charge one has, for instance,

$$K(\bar{\Phi}, \Phi) = F(\det_{\eta} (L^\dagger(\bar{\Phi}) L(\Phi)) , \quad (2.23)$$

where F is an arbitrary function. If $\tilde{G}/\tilde{H}$ differs from a homogeneous model by more than just one H -singlet 'novino' field, the arbitrariness is even larger. For a case with 'total doubling' the most general Kähler potential has been obtained in Ref. [27].

If the choice of $\tilde{H}$ is compatible with the conditions (2.7) and (2.9) for some vacuum expectation values $v_i = \langle \bar{\Phi}_i \rangle$, which is necessary if the σ -model is related to a spontaneous symmetry breaking, the variables

$$z_i = v_i^\dagger L^\dagger(\bar{\Phi}) L(\Phi) v_i \quad (2.24)$$

are obviously G -invariant quantities (in (2.24) $L(\Phi)$ and v_i are in the same G -representation). Consequently, arbitrary functions of the variables z_i are suitable Kähler potentials [8]. In the case of a single vacuum expectation value $v = \langle \bar{\Phi} \rangle$ (and thus, $m = 1$) one has, for instance,

$$K(\bar{\Phi}, \Phi) = \hat{F}(v^\dagger L^\dagger(\bar{\Phi}) L(\Phi) v) , \quad (2.25)$$

which is equivalent to (2.23).

An important feature of non-homogeneous σ -models is that they allow for totally G -invariant Kähler potentials. On the contrary, one necessarily has for homogeneous models [11] invariance only up to certain holomorphic functions $\gamma(\Phi, g)$, $g \in G$ (cf. (2.22)),

$$\ln \det_{\eta_i} (L^\dagger L) \xrightarrow{g} \ln \det_{\eta_i} (L^\dagger L) + i \operatorname{tr} [\eta_i (\bar{\gamma} - \gamma)^i Q_i] , \quad (2.26)$$

where Q_j denote the central charges of the homogeneous Kähler manifold. In the case of non-homogeneous models the trace in (2.26) is compensated by a term arising from the transformation of the 'novino' field [26]. The existence of a completely invariant Kähler potential is crucial for the coupling of a σ -model to supergravity.

The G-isometries of the σ -model are associated with currents,

$$J_A = \frac{\partial K}{\partial \Phi^a} \delta_A \Phi^a - \frac{\partial K}{\partial \bar{\Phi}^{\bar{a}}} \delta_A \bar{\Phi}^{\bar{a}}, \quad (2.27)$$

which are classically conserved,

$$\mathcal{D}^2 J_A = \bar{\mathcal{D}}^2 J_A = 0. \quad (2.28)$$

The non-conservation of J_A due to quantum corrections is the subject of Section 5. For the special Kähler potentials (2.25) one has

$$J_A = \hat{F}'(z) [v^\dagger L^\dagger L E_a v k_A^a - \bar{k}_A^{\bar{a}} v^\dagger E_{\bar{a}}^\dagger L^\dagger L v], \quad (2.29)$$

$$z = v^\dagger L^\dagger L v.$$

Finally, let us note that given the Kähler potential in terms of $L(\Phi)$, it is obvious how to gauge [4, 28] the G-isometries. One simply replaces $L^\dagger$ by $L^\dagger e^{2V}$, i.e., the Kähler potentials (2.25) becomes

$$K(\bar{\Phi}, \Phi; V) = \hat{F}(v^\dagger L^\dagger(\bar{\Phi}) e^{2V} L(\Phi) v). \quad (2.30)$$

Couplings of σ -models to additional matter fields have been discussed in Refs. [11], [29], [30].

3. AN EXAMPLE: THE COSET SPACE $U(n+2)/U(n) \times SU(2)$

In this section we want to discuss the simplest non-trivial example we know which illustrates the general features of non-homogeneous supersymmetric σ -models. It is a class of σ -models associated with the symmetry breaking $U(n+2) \rightarrow U(n) \times SU(2)$ [23] which deviate 'minimally', i.e., by a single 'novino' field, from a homogeneous σ -model. In this example the connection between the σ -model and an underlying linear theory, which realizes the symmetry breaking dynamically, can be studied explicitly.

The Goldstone bosons of the coset space $U(n+2)/U(n) \times SU(2)$ transform with respect to the unbroken group as $(n, 2) + (\bar{n}, 2) + (1, 1)$. They can be minimally embedded into $2n+1$ chiral superfields ($a = 1 \dots n$, $i = n+1, n+2$)

$$\phi_a^i \sim (\bar{n}, 2) \quad , \quad \phi \sim (1, 1) \quad , \quad (3.1)$$

where the 'novino' superfield ϕ contains one additional quasi-Goldstone boson.

On the Goldstone superfields (3.1) we have to realize the $U(n+2)$ Lie algebra ($\alpha = (a, i) = 1 \dots n+2$)

$$[T_\beta^a, T_\delta^b] = \delta_\delta^a T_\beta^b - \delta_\beta^b T_\delta^a \quad , \quad (3.2)$$

which contains the unbroken $U(n) \times SU(2)$ subalgebra

$$\begin{aligned} L_b^a &= T_b^a - \frac{1}{n} \delta_b^a T_c^c \quad , \quad Q = T_a^a \quad , \\ L_j^i &= T_j^i - \frac{1}{2} \delta_j^i T_k^k \quad , \end{aligned} \quad (3.3)$$

and the broken generators

$$X_i^a = T_i^a \quad , \quad Y_a^i = (X_i^a)^\dagger \quad , \quad Z = T_i^i \quad . \quad (3.4)$$

A non-linear realization of the $U(n+2)$ algebra is given by the following set of Killing vectors [23]:

$$[L_b^a, \phi_c^i] = \delta_c^a \phi_b^i - \frac{1}{n} \delta_b^a \phi_c^i \quad , \quad [Q, \phi_a^i] = \phi_a^i \quad , \quad (3.5a)$$

$$[L_j^i, \phi_a^k] = -\delta_j^k \phi_a^i + \frac{1}{2} \delta_j^i \phi_a^k \quad , \quad (3.5b)$$

$$[Y_a^i, \phi_b^j] = i \phi_b^i \phi_a^j \quad , \quad [Y_a^i, \phi] = \frac{1}{\sqrt{2}} \phi_a^i \quad , \quad (3.5c)$$

$$[X_i^a, \phi_b^j] = i \delta_b^a \delta_j^i \quad , \quad [X_i^a, \phi] = 0 \quad , \quad (3.5d)$$

$$[Z, \phi_a^i] = -\phi_a^i \quad , \quad [Z, \phi] = i\sqrt{2} \quad . \quad (3.5e)$$

One may check explicitly that the non-linear realization (3.5) corresponds to the choice of coset representative

$$L(\phi) = e^{iX_i^a \phi_a^i} e^{i\tau\phi} \quad (3.6)$$

rather than to the standard form

$$\hat{L}(\phi) = e^{i(X_i^a \phi_a^i + \tau\phi)} \quad (3.7)$$

The transition from $L(\phi)$ to $\hat{L}(\phi)$ corresponds to a redefinition of the fields ϕ_a^i in (3.5). Using $L(\phi)$ makes the connection between the non-homogeneous σ -models for $U(n+2)/U(n) \times SU(2)$ and the homogeneous model for the Grassmann manifold $U(n+2)/U(n) \times U(2)$ particularly transparent.

As described in Section 2, $L(\phi)$ is a representative of the complex coset space $\bar{G}/\tilde{H}$. The subgroup $\tilde{H}$, generated by L_b^a , Q , L_j^i and Y_a^i of $\mathfrak{k}(\bar{G})$, is left invariant by a vacuum expectation value obtained from a superfield $\phi_{\alpha\beta}$ in the $(n+1) \cdot (n+2)/2$ -dimensional antisymmetric representation $\square$ of $U(n+2)$:

$$L_b^a \langle \bar{\Phi}_{ij} \rangle = Q \langle \bar{\Phi}_{ij} \rangle = L_j^k \langle \bar{\Phi}_{ij} \rangle = Y_a^k \langle \bar{\Phi}_{ij} \rangle = 0 \quad (3.8)$$

According to (2.13) the most general Kähler potential for the 'novino' model takes the form

$$K(\bar{\Phi}, \phi) = F(\det_\eta (L^\dagger(\bar{\Phi}) L(\phi))) \quad (3.9)$$

where $L(\phi)$ is the coset representative (3.6) in the fundamental representation and the projection operator η [11, 12] reads in $U(n) \times U(2)$ block-diagonal form

$$\eta = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} \quad (3.10)$$

After some algebra one finds (cf. [31])

$$K = F(e^{-i\sqrt{2}(\phi - \bar{\Phi})} [1 + \bar{\Phi}_i^a \phi_a^i + \det(\bar{\Phi}_i^a \phi_a^i)]) \quad (3.11)$$

where F is an arbitrary function up to the constraint

$$\left. \frac{\partial^2 F}{\partial \bar{\phi}^{\bar{\alpha}} \partial \phi^{\beta}} \right|_{\phi = \bar{\phi} = 0} = f_{\alpha}^{\beta} \delta_{\bar{\alpha} \bar{\beta}} \quad (3.12)$$

The Kähler potential for the homogeneous Grassmann manifold $U(n+2)/U(n) \times U(2)$ is obtained from (3.11) by setting $\phi = \bar{\phi} = 0$ and $F(z) = v_0^2 \ln z$. We note that the coupling of the 'novino' field in (3.11) is very reminiscent of the couplings of the dilation in supergravity theories.

The Kähler potential (3.11) can also be obtained starting from the vacuum expectation value $v = \langle \phi_{ij} \rangle = 1/\sqrt{2} \epsilon_{ij} v_0$ defined in (3.8). One easily verifies

$$v^\dagger L^\dagger(\bar{\phi}) L(\phi) v = v_0^2 e^{-i\sqrt{2}(\phi - \bar{\phi})} (1 + \bar{\phi}_i^{\bar{\alpha}} \phi_a^i + \det(\bar{\phi}_i^{\bar{\alpha}} \phi_a^i)) \quad (3.13)$$

where $L(\phi)$ is now taken in the antisymmetric representation $\square$ of $U(n+2)$.

In the non-linear realization (3.5) the generators of $\bar{H}$ (L_b^a , Q and L_j^i) act linearly, whereas those of $\tilde{H}/H$ (Y_a^i) and $\bar{G}/\tilde{H}$ (X_i^a , Z) act non-linearly. H is in fact the largest unitary subgroup of $\bar{G}$ which can be realized linearly on the Goldstone fields ϕ_a^i and ϕ . Furthermore there can be a linear representation of the complex subgroups $\tilde{H}$ and $\bar{G}/\tilde{H}$ if the action of the generators Y_a^i , X_i^a and Z is defined through

$$\tilde{H}/\bar{H}: [Y_a^i, \phi_b^j] = 0, \quad [Y_a^i, \phi] = \alpha \phi_a^i \quad (3.14a)$$

$$\begin{aligned} \bar{G}/\tilde{H}: [X_i^a, \phi_b^j] &= \beta \delta_b^a \delta_i^j \phi, \quad [X_i^a, \phi] = 0, \\ [Z, \phi_a^i] &= \gamma \phi_a^i, \quad [Z, \phi] = \delta \phi \end{aligned} \quad (3.14b)$$

With the action of $\bar{H}$ given by (3.5a, b), one easily verifies that the $\tilde{H}$ and $\bar{G}/\tilde{H}$ Lie algebras, defined as subalgebras of $\mathfrak{f}(\bar{G})$ are satisfied for arbitrary values of α , β , γ and $\delta = \gamma - 1$. Although (3.5a, b) and (3.14) define the action of every $\bar{G}$ generator on the fields ϕ_a^i and ϕ , this is not a realization of $\bar{G}$. Only the non-commuting complex subgroups $\tilde{H}$ and $\bar{G}/\tilde{H}$ can be realized linearly on the fields ϕ_a^i and ϕ .

In our discussion of isometry anomalies in Section 5 we will need the G-currents of the 'novino' model. They can easily be computed from the general

expression (2.26). The chiral superfields in (3.5) are dimensionless; they are related to physical fields with canonical kinetic terms through

$$\phi_a^i \rightarrow \frac{1}{f_1} \phi_a^i, \quad \phi \rightarrow \frac{1}{f_2} \phi, \quad (3.15a)$$

where

$$f_1^2 = F'(1), \quad f_2^2 = 2(F'(1) + F''(1)). \quad (3.15b)$$

In terms of the rescaled physical fields the currents read:

$$J_b^a = 2 \left(\bar{\Phi}_i^a \phi_b^i - \frac{1}{n} \delta_b^a \bar{\Phi}_i^c \phi_c^i + \dots \right), \quad (3.16a)$$

$$J_a = 2 \bar{\Phi}_i^a \phi_a^i + \dots \quad (3.16b)$$

$$J_j^i = 2 \left(-\bar{\Phi}_j^a \phi_a^i + \frac{1}{2} \delta_j^i \bar{\Phi}_k^b \phi_b^k + \dots \right), \quad (3.16c)$$

$$J_a^i = 2 \left(-i f_1 \bar{\Phi}_a^i + \frac{1}{\sqrt{2}} \frac{f_2}{f_1} \phi_a^i (\bar{\Phi} - \phi) + \dots \right), \quad (3.16d)$$

$$J_i^a = 2 \left(i f_1 \bar{\Phi}_i^a + \frac{1}{\sqrt{2}} \frac{f_2}{f_1} \bar{\Phi}_i^a (\phi - \bar{\Phi}) + \dots \right), \quad (3.16e)$$

$$J_z = 2 \left(2 f_1^2 + i \sqrt{2} f_2 (\bar{\Phi} - \phi) + \lambda \bar{\Phi}_i^a \phi_a^i + \frac{1}{2} K (\bar{\Phi} - \phi)^2 + \dots \right), \quad (3.16f)$$

$$\lambda = \left(\frac{f_2^2}{f_1^2} - 1 \right), \quad K = -(4F'''(1) + 6f_2^2 - 4f_1^2)/f_2^2.$$

As expected, the broken currents contain terms linear in the fields and depend on the 'decay constants' f_1 , f_2 , κ , etc., i.e., on the function $F(z)$.

The 'novino' model [23] discussed in this section was originally invented and further studied [31, 32] as a toy model for composite vector bosons. Its global σ -model anomaly structure has been investigated in Ref. [33]. The underlying linear model is a renormalizable theory which exhibits dynamical symmetry breaking corresponding to the coset space $U(n+2)/U(n) \times SU(2)$, as described above. The basic fields are a set of $n+2$ chiral multiplets $\hat{\Phi}_{-\alpha}^p$ ($p = 1, 2$; $\alpha = 1 \dots n+2$), which are doublets under a local $SU(2)$ symmetry, and a $SU(2)$ triplet of vector superfields.

Classically, its global symmetry group is

$$G = U(n+2) \times U(1)_R \quad (3.17)$$

In the quantum theory, due to instanton effects, only one $U(1)$ factor $U(1)_X$ is a true global symmetry,

$$X = T_\alpha^\alpha + \frac{n+2}{n-2} R, \quad (3.18)$$

where T_α^α and R denote the generators of the $U(1)$ factor in $U(n+2)$ and R symmetry. In the 'Higgs phase' G is broken to

$$H = U(n) \times SU(2) \times U(1)_R \quad (3.19)$$

through the vacuum expectation value

$$\langle \hat{\Phi}_i^P \rangle = v \delta_i^P \quad (3.20)$$

In the 'confining phase' the same symmetry breaking takes place through a vacuum expectation value of a gauge invariant operator,

$$\langle \hat{\Phi}_{ij} \rangle = \frac{1}{f} \langle \varepsilon_{pq} \hat{\Phi}_i^P \hat{\Phi}_j^q \rangle = \frac{1}{f} \varepsilon_{ij} v_0 \quad (3.21)$$

This is identical with (3.8) which we postulated above for purely group-theoretical reasons. There are strong dynamical arguments [32] that the 'confining phase' is just a gauge invariant formulation of the 'Higgs phase' and that the theory cannot be a 'preon theory' of strongly interacting bound states.

The coset space $U(n+2)/U(n) \times SU(2)$ corresponds to the classical symmetry breaking. The true symmetry breaking of the quantum theory involves in a non-trivial way the R symmetry, and therefore the chiral as well as the vector superfields of the underlying theory.

4. GEOMETRY OF SUPERSYMMETRIC σ -MODELS

The coupling of fermions to a σ -model involves geometrical quantities of the target manifold on which the scalars live, i.e., vielbein and spin-connection. In the usual, non-supersymmetric case, these can always be chosen according to the canonical CCWZ prescription [17]. In the supersymmetric case, however, the choice of the spin connection is restricted. As we will discuss

below, it has to be different from the canonical spin connection unless the Kähler manifold is homogeneous and symmetric. This is important in the context of anomalies.

It was shown in Section 2 that supersymmetric σ -models are naturally associated with complex coset spaces $\tilde{G}/\tilde{H}$. Correspondingly, there exists an alternative formulation of these theories in which this structure becomes manifest. As we will see in Section 5, this formulation is particularly suited to analyze the anomalies of supersymmetric σ -models.

4.1 Spin Connection and Holonomy Group

The Lagrangian of a supersymmetric σ -model (2.21),

$$\mathcal{L} = \int d^4\theta K(\bar{\Phi}, \Phi) ,$$

reads in terms of the scalar and fermionic component fields^{*)} ($\partial_\alpha \equiv \partial/\partial\phi^\alpha$, $\partial_{\bar{\alpha}} \equiv \partial/\partial\bar{\phi}^{\bar{\alpha}}$)

$$\begin{aligned} \mathcal{L} = & -g_{\alpha\bar{\beta}} \partial^\mu \phi^{\alpha\bar{\beta}} \partial_\mu \phi^\alpha - \frac{i}{2} g_{\alpha\bar{\beta}} \bar{\chi}^{\bar{\beta}} \bar{\sigma}^\mu \partial_\mu \chi^\alpha \\ & + \frac{1}{4} R_{\alpha\bar{\beta}\gamma\bar{\delta}} (\bar{\chi}^{\bar{\beta}} \bar{\chi}^{\bar{\delta}}) (\chi^\alpha \chi^\gamma) , \end{aligned} \quad (4.1)$$

where

$$\begin{aligned} g_{\alpha\bar{\beta}} &= \partial_\alpha \partial_{\bar{\beta}} K(\phi, \bar{\phi}) , \quad (\mathcal{D}_\mu)_\beta^\alpha = \delta_\beta^\alpha \partial_\mu + \Gamma_{\beta\gamma}^\alpha \partial_\mu \phi^\gamma , \\ \Gamma_{\beta\gamma}^\alpha &= g^{\alpha\bar{\delta}} \partial_\beta g_{\gamma\bar{\delta}} , \quad R_{\alpha\bar{\beta}\gamma\bar{\delta}} = g_{\alpha\bar{\epsilon}} \partial_{\bar{\delta}} \Gamma_{\beta\gamma}^\epsilon . \end{aligned}$$

$g_{\alpha\bar{\beta}}$, $\Gamma_{\beta\gamma}^\alpha$ and $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ are metric, affine connection and curvature tensor chosen on the Kähler manifold $\tilde{M}$. Obviously, $g_{\alpha\bar{\beta}}$ satisfies the Kähler property

$$\partial_{[\bar{\gamma}} g_{\bar{\delta}]\beta} = \partial_{[\delta} g_{\beta]\bar{\alpha}} = 0 . \quad (4.2)$$

The Lagrangian (4.1) is invariant under supersymmetry transformations

$$\begin{aligned} \delta_\zeta \phi^\alpha &= \sqrt{2} \zeta \chi^\alpha , \\ \delta_\zeta \chi^\alpha &= i\sqrt{2} \sigma^\mu \bar{\zeta} \partial_\mu \phi^\alpha - \sqrt{2} \Gamma_{\beta\gamma}^\alpha (\zeta \chi^\beta) \chi^\gamma , \end{aligned} \quad (4.3)$$

*) We use the conventions of Wess and Bagger [34].

as well as under holomorphic isometry transformations (cf. (2.1)),

$$\begin{aligned}\delta_A \varphi^\alpha &= k_A^\alpha(\varphi) \ , \\ \delta_A \chi^\alpha &= \partial_\beta k_A^\alpha(\varphi) \chi^\beta \ ,\end{aligned}\tag{4.4}$$

where the vector fields k_A^α satisfy the Killing equation

$$g_{\alpha\bar{\gamma}}(\partial_{\bar{\beta}} \bar{k}_A^{\bar{\gamma}} + \bar{\Gamma}_{\bar{\beta}\bar{\delta}}^{\bar{\gamma}} \bar{k}_A^{\bar{\delta}}) + g_{\gamma\bar{\beta}}(\partial_\alpha k_A^\gamma + \Gamma_{\alpha\delta}^\gamma k_A^\delta) = 0 \ .\tag{4.5}$$

In the Lagrangian (4.1) the kinetic term for the fermions χ^α involves the metric $g_{\alpha\bar{\beta}}$. In order to get a canonical expression for the anomaly one usually switches from world to tangent space indices^{*}) which yields fermions with canonical kinetic terms,

$$\psi^a = e_a^a(\varphi^*, \varphi) \chi^\alpha \ ,\tag{4.6}$$

where

$$g_{\alpha\bar{\beta}} = e_a^a \eta_{ab} \bar{e}^{\bar{b}}_{\bar{\beta}} \ .\tag{4.7}$$

The vielbein e_a^a is defined only up to local frame rotations,

$$e'^a_a(\varphi^*, \varphi) = L^a_b(\varphi^*, \varphi) e^b_a(\varphi^*, \varphi) \ ,\tag{4.8}$$

which constitute the tangent frame group with flat metric η_{ab} . In terms of the fields ψ^a the fermionic part of the Lagrangian (4.1) reads [15]

$$\begin{aligned}L_f &= -i \bar{\psi}_b \bar{\nabla}^\mu D_\mu(\omega)_a^b \psi^a + \frac{1}{4} R_a^b{}_{c^d} (\bar{\psi}_b \bar{\psi}_d) (\psi^a \psi^c) \ , \\ D_\mu(\omega)_a^b &= (\delta_a^b \partial_\mu + \omega_{\mu a}^b) \ , \quad \omega_{\mu a}^b(\varphi^*, \varphi) = \omega_{\mu a}^b \partial_\mu \varphi^a + \bar{\omega}_{\mu a}^b \partial_\mu \varphi^{*\bar{a}} \ , \\ R_a^b{}_{c^d} &= (e^{-1})_a^\alpha (\bar{e}^{-1})^{\bar{b}\bar{\beta}} (e^{-1})_c^\gamma (\bar{e}^{-1})^{\bar{d}\bar{\delta}} R_{\alpha\bar{\beta}\gamma\bar{\delta}} \ ,\end{aligned}\tag{4.9}$$

^{*}) For a thorough discussion, see Ref. [15].

where the spin connection $\omega_{\alpha a}^b$ is given by vielbein and affine connection,

$$\omega_{\alpha a}^b = e_{\gamma}^b \partial_{\alpha} (e^{-1})_{\bar{a}}^{\gamma} + e_{\beta}^b \Gamma_{\alpha \gamma}^{\beta} (e^{-1})_{\bar{a}}^{\gamma} . \quad (4.10)$$

Because of the Kähler property (4.2), the affine connection is torsion free,

$$T_{\alpha \beta}^{\gamma} \equiv \Gamma_{[\alpha \beta]}^{\gamma} - g^{\gamma \bar{\delta}} \partial_{[\alpha} g_{\beta] \bar{\delta}} = 0 . \quad (4.11)$$

This implies in turn that the spin connection ω is also torsion free,

$$\partial_{[\beta} e_{\alpha]}^a + e_{[\beta}^b \omega_{\alpha]}^a{}_b = 0 , \quad (4.12)$$

which, in terms of 1-forms, $e^a = e_{\alpha}^a d\varphi^{\alpha}$, $\omega_b^a = \omega_{\alpha b}^a d\varphi^{\alpha}$, reads

$$\partial e^a + \omega_b^a \wedge e^b = 0 . \quad (4.13)$$

The fact that the spin connection ω is torsion free is a special property of supersymmetric σ -models; it is of importance in the context of (holonomy) anomalies.

The fermionic Lagrangian (4.9) is invariant under local frame rotations,

$$\begin{aligned} \psi^a &\rightarrow \psi'^a = L_b^a \psi^b , \\ e^a &\rightarrow e'^a = L_b^a e^b , \\ \omega_b^a &\rightarrow \omega'_b{}^a = L_c^a \omega_d^c (L^{-1})_b^d + L_c^a \partial (L^{-1})_b^c . \end{aligned} \quad (4.14)$$

Those frame rotations which, in the so-called active framework (cf., e.g., [35]), are induced by general (holomorphic) coordinate transformations constitute the holonomy group $\mathcal{H}$. It is associated with the Lie algebra in which

ω_b^a takes values. The Lagrangian (4.9) behaves like a $\mathcal{H}$ gauge theory with ω playing the role of a (composite) gauge field. Isometry transformations of the metric are special coordinate transformations. The associated induced local frame rotations belong to a subgroup of $\mathcal{H}$, which for coset spaces $M = G/H$ is identical with the isotropy group H .

Clearly, if $\mathcal{H}$ is larger than H , ω_b^a carries values not only in $\mathfrak{f}(H)$ but in a larger algebra. Physically this means that the fermions do not couple only to unbroken currents, but also to currents of broken symmetries (cf. 3.16). Correspondingly, the massless fermions do not only contribute to anomalies of the unbroken group H , but also to mixed anomalies involving broken generators, as we will see. We denote couplings of the fermions to broken currents generically by g_A , in analogy to the axial coupling constant in QCD. From the above it is clear that in order to determine whether such g_A do vanish or not, one has to find the holonomy group $\mathcal{H}$. To provide a quick algorithm to determine $\mathcal{H}$, we recall that, conventionally, the holonomy group is defined to be generated by the curvature,

$$R^a_b = d\omega_b^a + \omega_c^a \wedge \omega_b^c \quad . \quad (4.15)$$

This means, $\mathfrak{f}(\mathcal{H})$ is given by the set $(R^c_d)^a_b$ for all values of c and d . It can therefore be directly inferred from the four-fermion interaction $R^c_d{}^a{}_b \bar{\psi}_a \bar{\psi}_c \psi^b \psi^d$ [36]. For a choice of coordinates where the Kähler potential K contains no cubic terms, $R^c_d{}^a{}_b$ can be read off immediately from the quartic term of K . An explicit example will be discussed in the following subsection.

4.2 Geometry of homogeneous σ -models

Let us recall the case of compact homogeneous Kähler manifolds, i.e., $M = G/H$, $H = H' \times U(1)$, where H' is the maximal subgroup of G which commutes with the chosen $U(1)$ factor. The fermion content of the associated supersymmetric σ -model is fixed and necessarily complex. Geometrically, the fermions are sections of the complexified tangent bundle $TM(G/H)$.

The canonical CCWZ spin-connection $\omega_{(c)\alpha}^i$ and the vielbein e_α^a are defined by the expansion of the left-invariant Maurer-Cartan 1-form on G/H ,

$$\ell^{-1} d\ell = \omega_{(c)}^i H_i + e^a X_a \quad , \quad (4.16)$$

or, more explicitly,

$$\begin{aligned} l^{-1}(\varphi^*, \varphi)(\partial + \bar{\partial}) l(\varphi^*, \varphi) = & (\omega_{(c)\alpha}^i d\varphi^\alpha + \bar{\omega}_{(c)\bar{\alpha}}^i d\varphi^{*\bar{\alpha}}) H_i \\ & + e_a^\alpha d\varphi^\alpha X_a + \bar{e}_{\bar{a}}^{\bar{\alpha}} d\varphi^{*\bar{\alpha}} Y_{\bar{a}} \end{aligned} \quad (4.17)$$

H_i , X_a and $Y_{\bar{a}} = X_a^\dagger$ denote the unbroken and broken generators, for which we have chosen a step generator basis adapted to the specific complex structure on G/H ; $l(\varphi^*, \varphi)$ is the non-holomorphic but unitary CCWZ coset representative.

In the non-supersymmetric case it is always possible to couple fermions to a σ -model by means of the canonical spin connection $\omega_{(c)}$. In the supersymmetric case, however, the spin connection has to be torsion free as a consequence of the Kähler property. The relation between the canonical and the torsion free connection on G/H can easily be determined by means of the Maurer-Cartan equation $d(l^{-1}dl) = -(l^{-1}dl)^2$. For complex manifolds it implies differential equations for $(2, 0)$, $(1, 1)$ and $(0, 2)$ forms. For the $(2, 0)$ forms one easily obtains

$$\partial \omega_{(c)}^i + \frac{1}{2} f_{jk}^i \omega_{(c)}^j \omega_{(c)}^k = 0, \quad (4.18a)$$

$$\partial e^a + f_{ib}^a \omega_{(c)}^i e^b + \frac{1}{2} f_{bc}^a e^b e^c = 0, \quad (4.18b)$$

where the structure constants involve only the generators corresponding to $\bar{H}$ and $\bar{G}/\bar{H}$,

$$[H_i, H_j] = f_{ij}^k H_k, \quad (4.19a)$$

$$[H_i, X_a] = f_{ia}^b X_b, \quad (4.19b)$$

$$[X_a, X_b] = f_{ab}^c X_c. \quad (4.19c)$$

The fact that eqs. (4.18) do not depend on structure constants involving the generators $Y_{\bar{a}}$, and the absence of a vielbein term in eq. (4.18a) are a consequence of the opposite sign of the central charges of the generators X_a and $Y_{\bar{a}}$.

The relation between the torsion-free connection w and the canonical connection $w_{(c)}$ can be read off from (4.18b). In analogy to manifolds without complex structure (cf., e.g., Ref. [25])

$$w_b^a = f_{ib}^a w_{(c)}^i + \frac{1}{2} f_{cb}^a e^c \quad (4.20)$$

satisfies the torsion-free condition (4.13). Therefore the two connections w and $w_{(c)}$ coincide only for symmetric spaces where $f_{ab}^c = 0$. This implies that only for symmetric spaces holonomy and isotropy groups coincide. In all other cases the holonomy group is necessarily larger than the isotropy group.

As discussed in the previous section the holonomy group is determined by the curvature tensor $R_{d\ b}^c\ a$ which, at $\phi = \bar{\phi} = 0$, is identical with the quartic term of the Kähler potential (if its cubic term vanishes). As an example, let us consider the Grassmann manifold $U(n+2)/U(n) \times U(2)$. From (3.11) one obtains $F(z) = v_0^2 \ln z$, $\phi = 0$,

$$K^{(4)} = \frac{1}{2} v_0^2 \left\{ (\bar{\phi}_i^a \phi_b^i)(\bar{\phi}_j^b \phi_a^j) + (\bar{\phi}_i^a \phi_a^j)(\bar{\phi}_j^b \phi_b^i) \right\} \quad (4.21)$$

Obviously, $(R_{d\ b}^c\ a)$ takes values in $U(n) \times U(2)$, i.e., holonomy and isotropy group coincide as expected for a symmetric space.

A simple example of a homogeneous non-symmetric Kähler manifold is $M = SU(n+2)/SU(n) \times U(1)^2$. For the complex structure Ω_I (cf. (2.12a)),

$$\Omega_I = \bar{h}^{-1, -n-1}(\varphi_i) + \bar{h}^{-n-1, -1}(\chi_i) + \bar{h}^{n, -n}(\varphi')$$

one obtains from Ref. [26] after some algebra ($K^{(3)} = 0$):

$$K^{(4)} = v^2 \left\{ -(\bar{\varphi}^i \varphi_j)(\bar{\varphi}^j \varphi_i) - \frac{1}{2}(\bar{\chi}^i \chi_j)(\bar{\chi}^j \chi_i) - (\bar{\varphi}^i \varphi_j)(\bar{\chi}^j \chi_i) \right. \\ \left. - \frac{1}{2}(\bar{\varphi}^i \varphi')^2 - (\bar{\varphi}^i \varphi')(\bar{\varphi}^j \varphi_j) + \frac{1}{2}(\bar{\chi}^i \varphi')(\bar{\varphi}^j \chi_j) \right\} \quad (4.22)$$

Clearly, $(R_{d\ b}^c\ a)$ generates $\mathcal{H} = U(2n+1)$ which is maximal and larger than $G = SU(n+2)$. The same holonomy group is obtained for the second complex structure Ω_{II} (cf. (2.12b)).

4.3 Geometry of non-homogeneous σ -models

In this chapter we will discuss mainly the geometry of non-homogeneous σ -models whose scalar field content differs from a homogeneous model by just one quasi-Goldstone boson. The fermionic representation content under H thus takes the form $\varrho_H = R + 1$, where R corresponds to a homogeneous model; we assume R to be complex. As we have seen in Section 2, these are the models with maximal chirality of the fermion representation, which can be obtained from spontaneous symmetry breaking.

In order to discuss the anomalies of these models one may switch from world to tangent space indices as described in Chapter 3.1, where the fermions are redefined by means of a vielbein depending on both holomorphic and antiholomorphic coordinates. Since G/H is necessarily not symmetric, it follows again from the torsion-free condition for the connection that there exist non-zero couplings g_A of the fermions to broken generators. In other words, the holonomy group $\mathcal{H}$ is always larger than H for non-homogeneous models. An interesting example is the 'novino' model. Whereas the holonomy group of the homogeneous Grassmann manifold $U(n+2)/U(n) \times U(2)$ is $H = U(n) \times U(2)$, as described in the previous section, one obtains for the non-homogeneous case $U(n+2)/U(n) \times SU(2)$ $\mathcal{H} = U(2n+1)$.

Supersymmetric σ -models are related to complex coset spaces $\tilde{M} = \tilde{G}/\tilde{H}$. This suggests to consider also an alternative formulation which emphasizes this structure. That is, we choose a 'tangent space' basis, where the structure group is $\tilde{H}$ instead of H , by means of a holomorphic vielbein associated with $\tilde{M}$. This means that we perform a CCWZ like construction adapted to $\tilde{M}$. As we will see, this is indeed the natural redefinition for non-homogeneous σ -models and particularly suited to analyze anomalies.

Let us consider the simplest Kähler potential of a 'minimal' non-homogeneous σ -model (cf. (2.24)),

$$K(\bar{\Phi}, \Phi) = F(z) \quad , \quad (4.23)$$

with

$$F(z) = z \quad , \quad z = v^\dagger L^\dagger(\bar{\Phi}) L(\Phi) v \quad ,$$

$$L(\Phi) = e^{iX \cdot \Phi} \quad , \quad X \cdot \Phi = X_a \Phi^a \delta_a^\alpha \quad .$$

The bilinear fermionic part of the Lagrangian is easily obtained by using ($y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$)

$$\phi(y) = \varphi(y) + \sqrt{2} \theta \psi(y) + \theta \theta F(y) ,$$

and

$$dL(\phi) = L(\phi) E_a(\phi) d\phi^a ,$$

where E_a is the $\bar{G}/\tilde{H}$ vielbein defined through (2.15). This yields the expansion

$$K(\bar{\phi}, \phi) = v^\dagger l^\dagger l v + 2 v^\dagger e_a^\dagger \bar{\chi}^a l^\dagger l e_a \chi^a v + \dots , \quad (4.24)$$

$$l = L(\varphi) , \quad e_a = E_a(\varphi) .$$

A remarkable feature of this Lagrangian becomes manifest, if we redefine [16] the fermion fields according to

$$\tilde{\psi} = e_a \chi^a v , \quad (4.25a)$$

or, more explicitly,

$$\tilde{\psi}^a = \kappa_a E_a^b(\varphi) (X_b v)^a \chi^a \quad (4.25b)$$

where κ_a is an appropriate normalization factor. Then, under isometry transformations belonging to the global group G , one has

$$l(\varphi) \rightarrow l(\varphi') = g l(\varphi) \tilde{h}^{-1}(\varphi, g) , \quad (4.26a)$$

$$e_a(\varphi) \rightarrow e_a(\varphi') = \tilde{h}(\varphi, g) e_a(\varphi) \tilde{h}^{-1}(\varphi, g) , \quad (4.26b)$$

$$\tilde{h}^{-1}(\varphi, g) v = v , \quad (4.26c)$$

which implies that the fermions $\tilde{\psi}$ transform under the induced, field dependent $\tilde{H}$ transformations

$$\tilde{\psi}(\varphi) \rightarrow \tilde{\psi}(\varphi') = \tilde{h}(\varphi, g) \tilde{\psi} , \quad (4.27a)$$

whereas $l^\dagger(\varphi) l(\varphi)$ transforms like a composite gauge connection,

$$l^\dagger(\varphi') l(\varphi) \rightarrow l^\dagger(\varphi') l(\varphi') = (\tilde{h}^{-1})^\dagger(\varphi', g) l^\dagger(\varphi) l(\varphi) \tilde{h}^{-1}(\varphi, g) . \quad (4.27b)$$

In terms of the variables $\tilde{\psi}$ and $1^\dagger 1$ the Lagrangian (4.24) has the structure of a ' $\tilde{H}$ gauge theory', which will be defined more precisely in the following section. This feature is in complete analogy to the non-supersymmetric case; the local H compensating transformations are simply replaced by local holomorphic $\tilde{H}$ transformations related to the complex coset space $\tilde{M} = \tilde{G}/\tilde{H}$. The advantage of our choice of fermion basis (4.25), as compared to the conventional one (4.6), is that it preserves the symmetry of the manifestly supersymmetric Kähler potential (4.23) which is invariant under non-unitary, holomorphic $\tilde{H}$ transformations rather than unitary, non-holomorphic H transformations. Indeed, the Kähler potential (4.23) is nothing but the vacuum expectation values of the composite $\tilde{H}$ gauge connection $L^\dagger(\tilde{\phi})L(\phi)$.

The bilinear fermionic part of the Lagrangian reads in terms of the new variables $\tilde{\psi}$ and $1^\dagger 1$,

$$\mathcal{L}_f = -i \tilde{\bar{\psi}} \tilde{\sigma}^\mu \tilde{D}_\mu (\mathcal{Q}^\dagger \mathcal{Q}, \tilde{\omega}) \tilde{\psi} \quad , \quad (4.28a)$$

with the (non-canonical) covariant derivative

$$\tilde{D}_\mu (\mathcal{Q}^\dagger \mathcal{Q}, \tilde{\omega}) = (\mathcal{Q}^\dagger \mathcal{Q}) (\partial_\mu + \tilde{\omega}_\mu) \quad , \quad \tilde{\omega}_\mu = e_a \partial_\mu \varphi^a \quad . \quad (4.28b)$$

$\tilde{\omega}_\mu$ plays the role of a gauge connection of the non-compact group $\tilde{H}$,

$$\tilde{\omega}_\mu \xrightarrow{g} \tilde{h} \tilde{\omega}_\mu \tilde{h}^{-1} + \tilde{h} \partial_\mu \tilde{h}^{-1} \quad . \quad (4.29)$$

Contrary to the conventional tangent space fermions ψ^a , the $\tilde{H}$ fermions $\tilde{\psi}^a$ do not have canonical kinetic terms. The factor $1^\dagger 1$ in the covariant derivative (4.28) acts as a compensator which allows for a non-compact 'tangent frame' group $\tilde{H}$. It is just the scalar component of the (composite) vector superfield $L^\dagger(\tilde{\phi})L(\phi)$. Indeed, the gauged σ -model

$$\begin{aligned} K(\bar{\phi}, \phi; V) &= v^\dagger L^\dagger(\bar{\phi}) e^{2V} L(\phi) v \\ &\equiv v^\dagger (e^{2V})^{L^{-1}} v \end{aligned} \quad (4.30)$$

contains the bilinear fermionic part

$$L_f = -i \tilde{\Psi} \tilde{\sigma}^r \mathcal{D}_r ((e^{2C})^{l^{-1}}, (A_r)^{l^{-1}}) \tilde{\Psi} \quad , \quad (4.31)$$

with

$$C = V|_{\theta=0} \quad , \quad A_r = V|_{\theta\bar{\theta}} \quad ,$$

$$\mathcal{D}_r(e^{2C}, A_r) = e^{2C} \partial_r - i e^{2C} \left(\frac{1 - e^{\Delta_{2C}}}{\Delta_{2C}} \right) A_r \quad , \quad \Delta_{2C} A_r = 2 [C, A_r] \quad ,$$

where the superscript l^{-1} denotes 'transformed by $g = l^{-1}$ ', e.g., $(e^{2C})^{l^{-1}} = l^\dagger e^{2C} l$. Hence the covariant $\tilde{H}$ derivative (4.28) is simply the ordinary covariant derivative for vanishing external gauge fields:

$$\tilde{\mathcal{D}}_r(l^\dagger l, \tilde{\omega}) = \mathcal{D}_r((e^{2C})^{l^{-1}}, (A_r)^{l^{-1}}) \Big|_{C, A_r = 0} \quad . \quad (4.32)$$

We have seen that a non-homogeneous supersymmetric σ -model can be formulated as a $\tilde{H}$ gauge theory. Before we discuss this peculiar structure in more detail in the next section, let us first investigate how the above construction is related to the usual CCWZ formalism. For simplicity, we consider only the Kähler potential (4.23), i.e., $F(z) = z$. It is useful to separate the compact (φ_R) and non-compact (φ_0) coordinates,

$$l = e^{i(\varphi_R \cdot X + \varphi_0 \tilde{z})} = e^{i\tilde{\varphi}_R \cdot X} e^{i\varphi_0 \tilde{z}} \quad (4.33a)$$

$$= l^{\text{BKMU}}(\tilde{\varphi}_R) l_z(\varphi_0)$$

by means of the field redefinition

$$\tilde{\varphi}_R = \frac{1 - e^{-\varphi_0}}{\varphi_0} \varphi_R \quad . \quad (4.33b)$$

Here $l^{\text{BKMU}}(\tilde{\varphi}_R)$ is, à la Ref. [11], a representative of the coset $\bar{G}/\tilde{H} \times GL(1, \mathbb{C})$, which is isomorphic to the homogeneous Kählerian coset $G/H' \times U(1)$. In fact, as shown in Ref. [12], there exists a non-unitary and non-holomorphic transformation $U(\tilde{\varphi}^*, \varphi) \in \tilde{H}$ which maps the non-unitary $l^{\text{BKMU}}(\varphi) \in \bar{G}/\tilde{H} \times GL(1, \mathbb{C})$ into the usual unitary but non-holomorphic CCWZ variable $l^{\text{CCWZ}}(\tilde{\varphi}^*, \varphi) \in G/H' \times U(1)$; specifically,

$$l^{\text{BKM}}(\varphi) = l^{\text{CCWZ}}(\varphi^*, \varphi) U(\varphi^*, \varphi) , \quad (4.34)$$

$$U(\varphi^*, \varphi) \in \tilde{H} .$$

The $\tilde{H}$ group element $U(\varphi^*, \varphi)$, whose explicit form [12] need not concern us here, provides the connection between the two types of tangent space fermions, which reads

$$l(\varphi) = l^{\text{CCWZ}}(\tilde{\varphi}_R^*, \tilde{\varphi}_R) U(\tilde{\varphi}_R^*, \tilde{\varphi}_R) l_z(\varphi_0) , \quad (4.35a)$$

$$\tilde{\psi} = (l_z(\varphi))^{-1} U^{-1}(\tilde{\varphi}^*, \tilde{\varphi}) \psi . \quad (4.35b)$$

Under G-transformations,

$$U l_z(\varphi) \rightarrow h(\varphi^*, \varphi; g) U l_z(\varphi) \tilde{h}^{-1}(\varphi, g) , \quad (4.36a)$$

$$\psi \rightarrow h(\varphi^*, \varphi; g) \psi , \quad (4.36b)$$

with $h(\varphi^*, \varphi; g)$ given by

$$l^{\text{CCWZ}}(\varphi^*, \varphi) \rightarrow g l^{\text{CCWZ}}(\varphi^*, \varphi) h^{-1}(\varphi^*, \varphi; g) . \quad (4.37)$$

The fermions ψ are the usual CCWZ fields which transform under the induced unitary H transformations. In terms of these fields, the Lagrangian (4.31) has the canonical form,

$$\mathcal{L}_f = -i \bar{\psi}_b \bar{\sigma}^\mu D_\mu(\omega)_a^b \psi^a , \quad (4.38a)$$

with

$$\omega_a^b = ((l^{\text{CCWZ}})^{-1} d l^{\text{CCWZ}})_a^b . \quad (4.38b)$$

Since (4.38c) does not contain a projection to $\mathfrak{k}(H)$, ω takes values also in the set of broken generators and differs therefore from the conventional CCWZ connection. Hence, as expected, the holonomy group $\mathcal{H}$ is larger than the isotropy group H .

4.4 $\tilde{H}$ gauge theories

Let us now study the structure of ' $\tilde{H}$ gauge theories' in more detail. The form of the covariant derivative $\tilde{D}_\mu$ (4.32) shows that our $\tilde{H}$ gauge theory may be defined as a conventional gauge theory, but with a composite gauge connection $L^\dagger L$. In fact, the same bilinear fermionic part as in (4.28) arises (among others), when one couples non-linear Goldstone superfields Φ to external, independent matter superfields $\tilde{\chi}$,

$$\mathcal{L}_M = \int d^4\theta \tilde{\Phi}^\dagger L^\dagger(\Phi) L(\Phi) \tilde{\Phi} = -i \tilde{\chi}^\dagger \gamma^\mu D_\mu (\ell^\dagger \ell, \tilde{\omega}) \tilde{\chi} + \dots ; \quad (4.39)$$

$\tilde{\chi}$ can be thought of as a truncated G -representation (cf. (4.25)), and $L(\Phi)$ is in the G -representation of the corresponding vacuum expectation value. (4.39) has the form of an ordinary gauge theory with $L^\dagger L$ playing the role of a gauge connection,

$$L^\dagger L = e^{2\tilde{V}} \quad (4.40)$$

However, the group theoretical structure of a $\tilde{H}$ gauge theory is quite different from an ordinary G gauge theory. The reason is that (4.39) is invariant under transformations

$$e^{2\tilde{V}} \rightarrow (\tilde{h}^{-1}(\phi, g))^\dagger e^{2\tilde{V}} \tilde{h}^{-1}(\phi, g) , \quad (4.41)$$

which do not belong to the complex extension of an unitary subgroup of G . $\tilde{\chi}$ necessarily transforms as some $\tilde{H}$ -representation, which need not be a full G -representation. In other words, although $L^\dagger L$ carries values in the full group G ($L \in \tilde{G}/\tilde{H}$), all of G cannot, in general, be realized on $\tilde{\chi}$ since its dimension is too small. Hence (4.39) can be viewed as G gauge theory with a truncated representation, $\rho_G \rightarrow \rho_{\tilde{H}}$, such that it remains invariant under $\tilde{H}$ transformations acting on $\tilde{\chi}$ but not under linear G transformations.

To give an intuitive understanding of how $\tilde{H}$ acts on $\tilde{\chi}$, recall that, because of the obligate 'novino' H -singlet field, the representation content of the Goldstone fields under H is always reducible:

$$\rho_H = \underline{R} + \underline{1} \quad (4.42)$$

Under $\tilde{H}$ transformations, however, the fields transform irreducibly,

$$\rho_{\tilde{H}} = \underline{R+1} , \quad (4.43)$$

because the fields get mixed by the $\tilde{H}/\bar{H}$ part of the semi-direct product group $\tilde{H}$, i.e., schematically,

$$\begin{pmatrix} \phi_R \\ \phi_0 \end{pmatrix}' = \begin{pmatrix} * & 0 \\ * & 1 \end{pmatrix} \begin{pmatrix} \phi_R \\ \phi_0 \end{pmatrix} \quad (4.44)$$

In general, in non-homogeneous σ -models irreducible $\tilde{H}$ -representations are reducible with respect to H . The $\tilde{H}$ invariance of the σ -models is also important for the coupling of matter fields. It implies that it is not possible to couple arbitrary H -representations to such models in a CCWZ like way [17], but only complete $\tilde{H}$ -representations.

As an explicit example for $\tilde{H}$ -representations, let us consider the model of Section 3 which is based on the coset $G/H = U(n+2)/U(n) \times SU(2)$ with $\rho_H = (\bar{n}, 2) + (1, 1)$. The corresponding $\tilde{H}$ -representation is given by $\rho_{\tilde{H}} = ((\bar{n}, 2) + (1, 1)) = (\overline{2n+1})$, as it occurs in the branching of $\rho_G = \bar{\square} = \overline{(n+2)(n+1)/2}$ to $\tilde{H}$. More explicitly, from (4.25) and (3.8) one obtains

$$\tilde{\Psi}_{ai} = -\frac{i}{\sqrt{2}} e^{-i\varphi} \varepsilon_{ij} \chi_a^j, \quad (4.45a)$$

$$\tilde{\Psi}_{ij} = \frac{i}{\sqrt{2}} \varepsilon_{ij} \chi, \quad (4.45b)$$

which transform under $\tilde{H}/\bar{H}$ as

$$[Y_a^i, \tilde{\Psi}_{bj}] = 0, \quad (4.46a)$$

$$[Y_a^i, \tilde{\Psi}_{jk}] = 2\delta_j^i \tilde{\Psi}_{ak}. \quad (4.46b)$$

Clearly, (4.45) is not a complete G -representation. Note that because of the semi-direct product structure of $\tilde{H}$, $\tilde{\Psi}_{ai}$ and $\tilde{\Psi}_{ij}$ may be rescaled independently.

Another, inequivalent $\tilde{H}$ -representation of matter fields, which can be coupled to the σ -model, can be obtained by decomposing ρ_G into H -representations: $\bar{\square} \rightarrow \bar{\square}_{[ab]} \oplus \bar{\square}_{[ai]} \oplus \bar{\square}_{[ij]}$ ($a, b = 1 \dots n; i, j = n+1, n+2$), according to $\overline{(n+2)(n+1)/2} = \overline{(n(n-1)/2, 1)} + (\bar{n}, 2) + (1, 1)$. By inspecting the transformation properties one finds that besides $\overline{2n+1} = (\bar{\square}_{[ai]} \oplus \bar{\square}_{[ij]})$ also $\overline{1/2(n^2-3n)} = (\bar{\square}_{[ab]} \oplus \bar{\square}_{[ai]})$ forms an irreducible $\tilde{H}$ -multiplet. Both $\tilde{H}$ -multiplets also form representations of the group $\bar{G}/\tilde{H}$ which, however, does not commute with $\tilde{H}$. They are not representations of the full group $\bar{G}$.

Besides under $\bar{H}$, (4.39) is also invariant under transformations belonging to $\bar{G}/\tilde{H}$. This follows from the peculiar group property of $\bar{G}/\tilde{H}$, i.e.,

$$L \xrightarrow{g} L_g = L' \quad , \quad g \in \bar{G}/\tilde{H} \quad . \quad (4.47)$$

This $\bar{G}/\tilde{H}$ invariance is related to general diffeomorphisms

$$\delta \phi_R = \phi_R U(\phi_0) \quad , \quad (4.48a)$$

$$\delta \phi_0 = \phi_0 V(\phi_0) \quad , \quad (4.48b)$$

(where U and V are arbitrary functions) which are not isometries of the metric. Since the non-linear fields are chiral with respect to the isotropy group H , these transformations are the most general holomorphic, H -covariant diffeomorphisms besides the isometry transformations. Hence $\bar{G}/\tilde{H}$ transformations (4.47) belong to the part of the (complexified) holonomy group which does not contain $\tilde{H}$.

The Lagrangian (4.39) is not the most general $\tilde{H}$ gauge theory which is bilinear in the matter fields $\tilde{\chi}$. Instead, one may consider

$$\mathcal{L}_H = \int d^4\theta \tilde{\Phi} (g(z) L^\dagger L + h(z) L^\dagger L v v^\dagger L^\dagger L) \tilde{\Phi} \quad , \quad (4.49)$$

$$z = v^\dagger L^\dagger L v \quad ,$$

where g and h are arbitrary functions. (4.49) contains the most general $\tilde{H}$ gauge connection. In fact, an expression of this form is obtained for the bilinear fermionic part (4.28) of the σ -model if we choose for the Kähler potential an arbitrary function $F(z)$ instead of $F(z) = z$. One finds that in (4.28b) $L^\dagger L$ is replaced by

$$F'(z) L^\dagger L + F''(z) L^\dagger L v v^\dagger L^\dagger L \quad , \quad (4.50)$$

i.e., $g(z) = F'(z)$, $h(z) = F''(z)$.

5. ANOMALIES OF SUPERSYMMETRIC σ -models

σ -models with chiral fermions have anomalies, i.e., the symmetries of the classical Lagrangian are not preserved by quantum corrections. Sometimes it is possible to add a local Wess-Zumino (WZ) type counter term to the effective action which restores the classical symmetries. This section addresses the problem of finding conditions for the existence of appropriate supersymmetric WZ counter terms. After a brief repetition of some general properties of σ -model anomalies we will give a proof that such WZ counter terms do not exist for

homogeneous Kähler manifolds which implies that these models are inconsistent. Next we will discuss the class of non-homogeneous models, and argue that the conditions for the existence of anomaly cancelling counter terms are stronger than those usually discussed.

5.1 Isometry anomalies

If the σ -model is to describe an effective low energy theory of Goldstone fields which result from a spontaneous symmetry breaking of G to a subgroup H , the G -currents of the σ -model must be conserved beyond the classical level. As global G -transformations induce local transformations on the σ -model fermions, the classical conservation of the G -currents is in general spoiled if the fermions are chiral.

In principle anomalies of supersymmetric σ -models could be discussed directly in terms of the fermions χ^α which transform as vector fields under coordinate transformations. As the kinetic terms of the χ^α are not canonical, however, one usually switches to tangent space fermions, as described in Section 4.1, which transform under the holonomy group $\mathcal{H}$, and, in particular, under the subgroup H . The fermionic action^{*)} is then given by

$$\Gamma_f(\omega) = \int d^4x \bar{\Psi} \mathcal{D}(\omega) \Psi + \dots, \quad (5.1)$$

with the spin connection $\omega \in \mathfrak{E}(\mathcal{H})$.

Let us first review the construction of the WZ counter term in the non-supersymmetric case. Here ω is not fixed, and one may always choose $\omega \in \mathfrak{E}(H)$. Global G -transformations $g = 1 + \epsilon^A T_A$ induce local, field dependent, non-holomorphic H -transformations $h(\phi^*, \varphi; g) = 1 + \varrho(\phi^*, \varphi; g)$, $\varrho(\phi^*, \varphi; g) = \epsilon^A \varrho_A^i(\phi^*, \varphi; g) H_i$, on the fermions:

$$\Psi \rightarrow \Psi + \varrho \Psi, \quad (5.2a)$$

$$\omega_\mu \rightarrow \omega_\mu + [\varrho, \omega_\mu] - \partial_\mu \varrho. \quad (5.2b)$$

If the fermions are chiral, the fermion effective action $W_f(\omega)$

*) We consider only the renormalizable part of Γ_f .

$$e^{-W_f(w)} = \int d\psi d\bar{\psi} e^{-\Gamma_f(w)} , \quad (5.3)$$

is generally not invariant,

$$\delta W_f(w) = \mathcal{A}(\varphi, w) , \quad (5.4)$$

i.e., the σ -model has an *isometry anomaly* [37, 38, 25, 39]. As usual $\mathcal{A}(\varphi, w)$ is proportional to the symmetric trace

$$d_{ijk}^{(\varrho_H)} = \text{tr}_{\varrho_H} [H_i \{H_j, H_k\}] , \quad (5.5)$$

where ϱ_H denotes the H -representation content of the fermions.

In order to find a local counter term which cancels the isometry anomaly one considers the integrated anomaly 1-cocycle associated with the finite group element $\hat{g} = \hat{1}^{-1}(\hat{\psi}, \varphi)$,

$$\begin{aligned} \Gamma^{WZ}(\hat{w}, \hat{\ell}) &= \int_0^1 ds \mathcal{A}((\partial_s \hat{\ell}_s) \hat{\ell}_s^{-1}, \hat{w}^{\ell_s^{-1}}) \\ &= W_f(\hat{w}^{\ell^{-1}}) - W_f(\hat{w}) , \quad \hat{\ell}_s|_{s=0} = 1 , \quad \hat{\ell}_s|_{s=1} = \hat{\ell} , \end{aligned} \quad (5.6)$$

where the hat denotes some yet unspecified G -representation ϱ_G . Γ^{WZ} is essentially the Wess-Zumino term [40, 41]. Under the G -transformation $g = 1 + i\epsilon^A T_A$, the variation of Γ^{WZ} reads

$$\delta \Gamma^{WZ}(\hat{w}, \hat{\ell}) = - \mathcal{A}(\hat{\varphi}, \hat{w}) . \quad (5.7)$$

(5.7) has precisely the form of the anomalous variation (5.4), the only difference lies in the fermion representation. $\mathcal{A}(\hat{\varphi}, \hat{w})$ is proportional to

$$d_{ijk}^{(\varrho_{G|H})} = \text{tr}_{\varrho_G} [H_i \{H_j, H_k\}] , \quad (5.8)$$

where $\varrho_{G|H}$ denotes ϱ_G restricted to H . Clearly, the counter term Γ^{WZ} will cancel the isometry anomaly (5.4) if ϱ_G satisfies the equality

$$d_{ijk}^{(\varrho_{G|H})} = d_{ijk}^{(\varrho_H)} . \quad (5.9)$$

This is nothing but 't Hooft's anomaly matching condition [42].

5.2 Homogeneous models

We will now show that the matching condition (5.9) can never be satisfied for homogeneous, irreducible Kähler manifolds $\tilde{M} = \tilde{G}/\tilde{H} \cong G/H$. This is already known for symmetric Kählerian spaces [25, 39]; here we give a proof for general Kählerian coset spaces [10].

Homogeneous Kähler manifolds have the form $M = G/H$, $H = H' \times U(1)_Q$, where H' is the centralizer of the torus $U(1)_Q$ of G . The complex structure on M , and thereby the H -representation ϱ_H of the Goldstone superfields, is defined by the requirement that all fields have positive Q -charge [10]. This implies

$$\text{tr}_{\varrho_H} [Q_H^2 Q] \geq q_{\min} \text{tr}_{\varrho_H} [Q_H^2] > 0, \quad (5.10)$$

where Q_H is the generator of any $U(1)$ subgroup of H , and $q_{\min}$ the smallest Q -charge occurring in ϱ_H . The last inequality follows because ϱ_H is a non-trivial H -representation. As all representations of the groups $SO(n)$, $Sp(n)$ and E_n are anomaly free (except for those values of n for which they are equivalent to some $SU(m)$ group), (5.10) implies that for these groups the matching condition (5.9) can never be fulfilled.

Let us now consider the remaining case $G = SU(n)$ and assume the existence of a representation ϱ_G with anomaly coefficient $a(\varrho_G)$,

$$\text{tr}_{\varrho_G} [H_i \{H_j, H_k\}] = a(\varrho_G) \text{tr}_F [H_i \{H_j, H_k\}], \quad (5.11)$$

which satisfies (5.9); F denotes the fundamental representation of $SU(n)$. Let $\tau_3^{(ij)}$ be the diagonal generator of the $SU(2)$ subgroup of $SU(n)$ which acts on the i th and j th rows of F . We then have, with $Q = \text{diag}(\alpha_1, \dots, \alpha_n)$,

$$\text{tr}_{\varrho_G} [(\tau_3^{ij})^2 Q] = a(\varrho_G) (\alpha_i + \alpha_j). \quad (5.12)$$

For n even, we obtain from (5.10) and (5.12) as a necessary condition for anomaly matching ($n = 2m$)

$$0 < a(\varrho_G) \sum_{i=1}^m (\alpha_i + \alpha_{i+m}) = a(\varrho_G) \text{tr}_F [Q], \quad (5.13)$$

which is in contradiction with the tracelessness of all $SU(n)$ generators. A contradiction between the positivity of (5.12) and the tracelessness of Q can be obtained in the same way for n odd.

We conclude that for all homogeneous (coset) σ -models the anomaly matching conditions (5.9) cannot be satisfied. Therefore no local counter terms exist which could restore the required G -invariance at the one-loop level, and these σ -models are inconsistent. This follows because the above construction for

anomaly cancellation is the only one for this kind of (non-parallelizable) manifolds, unless one introduces additional fields [25, 39]. Even if the necessary conditions (5.9) could have been satisfied, this would not have been sufficient for the existence of a *supersymmetric* Wess-Zumino term, because this has been shown to vanish on any compact manifold [43].

5.3 Holonomy and mixed isometry anomalies

So far we have discussed essentially the non-supersymmetric case. This was already sufficient to show that homogeneous Kählerian σ -models, with a fermionic representation content fixed by supersymmetry, are not consistent. Let us now consider the supersymmetric, non-homogeneous case. Here we are not free to choose the spin connection ω as in (4.16); rather ω is constrained by supersymmetry to be torsion free, as discussed in Section (4.1). This implies (since we are not dealing with Kählerian symmetric cosets) that ω takes values not only in $\mathfrak{e}(H)$, but in the larger Lie algebra of the holonomy group $\mathcal{H}$. This is equivalent to saying that there are non zero constants g_A which characterize the coupling of the fermions to broken currents^{*)}.

This fact is important in the context of anomalies. More precisely, under an infinitesimal local tangent frame rotation belonging to $\mathcal{H}$, the fermionic effective action (5.3) changes by the holonomy anomaly,

$$\delta W_f = \mathcal{A}(\xi, \omega) \quad , \quad \xi, \omega \in \mathfrak{e}(\mathcal{H}) \quad . \quad (5.14)$$

Since $\mathcal{H}$ is larger than H , the conditions to cancel this anomaly are stronger than the usual H matching conditions. (This is in contrast to the non-supersymmetric case where $\mathcal{H}$ can always be chosen to be equal to H , i.e., $g_A = 0$.) In particular the obligate 'novino' H -singlet field does not contribute to H -anomalies, but it does contribute to holonomy anomalies. The procedure to cancel the $\mathcal{H}$ -anomaly (5.14) is analogous to the cancellation of the H -anomaly (5.4) [25, 39]. It leads to 't Hooft type matching conditions associated with $\mathcal{H}$.

It is not clear whether the cancellation of holonomy anomalies is physically required. Therefore we restrict our discussion to the cancellation of isometry anomalies in the case of non-vanishing axial coupling constants g_A . In the non-supersymmetric case one proceeds as follows [38]. In general the spin connection has the form

$$\omega = \omega^0 + g_A T \quad , \quad (5.15)$$

*) Physical consequences of such couplings (in models with 'full doubling', i.e., $H = \bar{H}$) have been discussed in Ref. [44].

where $w^0 \in \mathcal{E}(H)$ and τ is a H covariant contorsion tensor, valued in $\mathcal{E}(\mathcal{H})$. Clearly, the symmetric trace $d_{ABC}^{(\mathcal{H})}$, as computed from the triangle diagram, now also contains contributions from broken generators, i.e., it is of 'mixed' type. This, however, does not lead to stronger matching conditions, because anomalies are defined only modulo local counter terms. In particular there exists always a Bardeen type counter term $R^B(w, w')$ [45] with the property

$$\delta R^B(w, w') = \mathcal{A}(\mathcal{G}, w) - \mathcal{A}(\mathcal{G}, w') \quad (5.16)$$

for every pair of connections w, w' which transform equally under H . (This means that the cohomology class of $\mathcal{A}$ does not depend on the choice of the connection.) Since w^0 and w are equivalent H -connections, we can use two such terms to switch the (anomalous) variations of W_f and Γ^{WZ} to anomalies associated with purely $\mathcal{E}(H)$ valued connections:

$$\delta W_f(w) + \delta R^B(w^0, w) = \mathcal{A}(\mathcal{G}, w^0), \quad (5.17a)$$

$$\delta \Gamma^{WZ}(\hat{w}) + \delta R^B(\hat{w}, \hat{w}^0) = -\mathcal{A}(\hat{\mathcal{G}}, \hat{w}^0), \quad (5.17b)$$

where the hat denotes the representation ρ_G . Equation (5.17) implies the usual matching conditions (5.9), i.e., all mixed anomalies are cancelled by the counter terms R^B .

We emphasize, however, that the above construction does not work for the cancellation of holonomy anomalies, because w and w^0 are not equivalent with respect to $\mathcal{H}$. Furthermore, the procedure is not supersymmetric. As we will argue in the following section, it is in general not possible to reduce the cancellation of isometry anomalies to a matching of H -anomalies in a manifestly supersymmetric way. This seems related to the fact that in non-homogeneous supersymmetric σ -models there exist no (torsion free) connections which are valued in $\mathcal{E}(H)$ only.

5.4 Non-homogeneous models

In this section we want to establish conditions for the existence of a supersymmetric WZ counter term which cancels the isometry anomalies in the case of non-homogeneous σ -models. First let us briefly recall some properties of the supersymmetric chiral anomaly. Consider a theory with linear global G invariance, coupled to external G -gauge fields $V = V^A T_A$,

$$\Gamma^{lin}(V) = \int d^8z \, \bar{\Phi} e^{2V} \Phi \quad (5.18)$$

The action Γ^{lin} is invariant not only under local unitary G -transformations, but also under local non-unitary complexified $\bar{G}$ -transformations,

$$\begin{aligned}\Phi &\rightarrow \bar{q} \Phi, & \bar{\Phi} &\rightarrow \bar{\Phi} \bar{q}^\dagger, \\ e^{2V} &\rightarrow (\bar{q}^{-1})^\dagger e^{2V} \bar{q}^{-1},\end{aligned}\tag{5.19}$$

with

$$q = e^{i\Lambda^A T_A}, \quad \bar{q} = e^{-i\bar{\Lambda}^A T_A},$$

where $\Lambda(\bar{\Lambda})$ are chiral (anti-chiral) superfields. The classical invariance of Γ^{lin} under the transformations (5.19) is spoiled by quantum corrections. The variation of the effective action

$$e^{-W^{\text{lin}}[V]} = \int d\bar{\Phi} d\Phi e^{-\Gamma^{\text{lin}}[\bar{\Phi}, \Phi; V]},\tag{5.20}$$

is given by the supersymmetric chiral anomaly,

$$\begin{aligned}\delta_{\Lambda, \bar{\Lambda}} W[V] &= \mathcal{A}(\Lambda, V) + \text{c.c.} \\ &= i \int d^8z \, \text{tr} [\Lambda G(e^{2V})] + \text{c.c.},\end{aligned}\tag{5.21}$$

where $G(e^{2V})$ is the superspace chiral anomaly density, for which explicit expressions can e.g. be found in Refs. [46]. The anomaly (5.21) corresponds to an anomalous covariant divergence of the current superfield (t is the adjoint representation):

$$\bar{D}^2 \left(\frac{V \cdot t}{1 - e^{2V \cdot t}} \right)_A = \bar{D}^2 \text{tr} [T_A G(e^{2V})].\tag{5.22}$$

The anomaly density $G(e^{2V})$ is defined as non-trivial solution of the WZ-consistency conditions,

$$\int d^8z (\delta_{\Lambda_1} \text{tr} [\Lambda_2 G] - \delta_{\Lambda_2} \text{tr} [\Lambda_1 G]) = \int d^8z \text{tr} [[\Lambda_1, \Lambda_2] G].\tag{5.23}$$

It is therefore determined only up to variations of local counter terms, i.e.,

$$\mathcal{A}'(\Lambda, V) = \mathcal{A}(\Lambda, V) + \delta R(V)\tag{5.24}$$

and $\mathcal{A}(\Lambda, V)$ represent equivalent anomalies. It can be shown [47], that by means of such counter terms R the anomaly can always be transformed to a form where it is proportional to the usual d_{ABC} tensor, i.e.,

$$\text{tr}[\Lambda G(e^{2V})] = \text{tr}[\Lambda \{T_A, T_B\}] G_{AB}(V) \quad (5.25)$$

Here Λ belongs to $\mathfrak{f}(\bar{G})$, i.e., it may represent a hermitian or non-hermitian (e.g. step-) generator. As long as we consider $\bar{G}$ -transformations this distinction is irrelevant. It is important to realize, however, that the expressions (5.21), (5.25) also define anomalies for variations belonging to complex subgroups of $\bar{G}$ (like $\tilde{H}$) which are not complex extensions of unitary subgroups of G .

This means that (5.25) provides us automatically with an expression for the anomaly of a $\tilde{H}$ gauge theory, as introduced in Section (4.4). However, since an anomaly is defined as non-trivial solution of the WZ-consistency conditions (5.23), it is clear that a $\tilde{H}$ anomaly ($\Lambda \in \mathfrak{f}(\tilde{H})$) is in general not as much constrained as a $\bar{G}$ anomaly ($\Lambda \in \mathfrak{f}(\bar{G})$); there may be more $\tilde{H}$ - than $\bar{G}$ -invariant counter terms. How do we specify the most general $\tilde{H}$ anomaly? We cannot answer this question rigorously because we do not know all possible counter terms. We can only assume that the situation is like in the non-supersymmetric case, where two anomalies $\mathcal{A}(\varrho, \omega)$, $\mathcal{A}(\varrho, \omega')$ are equivalent as long as ω and ω' transform in the same way. Hence we argue that two anomalies $\mathcal{A}(\Lambda, V)$, $\mathcal{A}(\Lambda, V')$ are equivalent if V and V' transform in the same way,

$$\mathcal{A}(\Lambda, V') = \mathcal{A}(\Lambda, V) + \delta R^B(V', V) \quad , \quad (5.26)$$

where $R^B(V', V)$ is the supersymmetric generalization of Bardeen's counter term R^B . Equation (5.26) yields a formal expression for a $\tilde{H}$ anomaly, if $\mathcal{A}(\Lambda, V)$ is given by the usual $\bar{G}$ anomaly (5.21) with $\Lambda \in \mathfrak{f}(\tilde{H})$, and V' corresponds to the most general $\tilde{H}$ covariant gauge connection (4.49). This follows simply because if one replaces in the expression (5.21) for the $\bar{G}$ anomaly the connection e^{2V} by the most general $\tilde{H}$ connection (4.49), the expression one obtains still satisfies the WZ-conditions corresponding to $\tilde{H}$. Thus it can deviate from (5.21) only by the variation of a local counter term.

Let us now consider the supersymmetric WZ-term. It is given by the integrated anomaly of a $\bar{G}$ -gauge theory where the $\tilde{H}$ gauge field $L^\dagger L \equiv e^{2V}$ appears in some yet unspecified representation $\varrho_{\bar{G}}$:

$$\begin{aligned} \Gamma^{WZ}[\hat{V}, \hat{L}] &= \int_0^1 ds \mathcal{A}((\partial_s \hat{L}_s) \hat{L}_s^{-1}, \hat{V}^{L_s^{-1}}) + c.c. \\ &= W[\hat{V}^{L^{-1}}] - W[\hat{V}], \quad \hat{L}_s|_{s=0} = 1, \quad \hat{L}_s|_{s=1} = L. \end{aligned} \quad (5.27a)$$

Under infinitesimal global G transformations, $\delta L = i\varepsilon^A T_A L - L\varrho$, $\varrho = \varepsilon^A \varrho_A^i(\varphi) \tilde{H}_i$, $\varepsilon \in \mathfrak{f}(\tilde{H})$, one has

$$\delta \Gamma^{WZ}[\hat{V}, \hat{L}] = -\mathcal{A}(\hat{\varrho}, \hat{V}) + c.c., \quad (5.27b)$$

which is the anomaly of a $\tilde{H}$ gauge theory with matter fields $\hat{\Phi}$ in the representation $\varrho_{\tilde{G}}$

$$\hat{K} = \bar{\hat{\Phi}} e^{2\hat{V}} \hat{\Phi}, \quad (5.28)$$

where the gauge connection transforms as

$$e^{2\hat{V}} \xrightarrow{g \in G} (\tilde{h}^{-1}(\phi, g))^\dagger e^{2\hat{V}} \tilde{h}^{-1}(\phi, g). \quad (5.29)$$

Equation (5.27b) is proportional to $d_{ABC}^{(\varrho_{\tilde{G}})}$, but with one generator restricted to $\mathfrak{f}(\tilde{H})$.

It is crucial to realize that in supersymmetric theories local symmetries are associated with holomorphic (chiral) transformations. Therefore the manifestly supersymmetric WZ-term depends on the $\tilde{G}/\tilde{H}$ valued coset variable $L(\phi)$ rather than on the usual, non-holomorphic CCWZ variable $l(\varphi^*, \varphi)$. Hence the WZ-counter term emphasizes the $\tilde{G}/\tilde{H}$ structure of the theory.

Knowing the variation of the counter term $\delta \Gamma^{WZ}$, the isometry anomaly of the σ -model has to be brought to the same form in order to achieve cancellation. Therefore we do not switch to the usual CCWZ type tangent space fermions ψ^a (c.f. (4.6)), but rather to the fermions $\tilde{\psi}$ (c.f. (4.25)) which transform under the induced holomorphic $\tilde{H}$ transformations. (The transition between the two bases is performed by means of the non-holomorphic anomalous $\tilde{H}$ transformation $U(\varphi^*, \varphi)$ (4.34) which induces a non-trivial Jacobian. This Jacobian, by its nature as 1-cocycle compensates precisely for the change of basis [48].) In the $\tilde{\psi}$ basis the fermionic action has the structure of a $\tilde{H}$ gauge theory. As anomalies are generally determined by the bilinear fermionic part of the Lagrangian it is natural to assume that the supersymmetric σ -model anomaly is equivalent to the anomaly of a $\tilde{H}$ gauge theory (c.f. (4.39); see also Ref. [16])

$$\ell_{\tilde{H}} = \bar{\Phi}_{\tilde{H}} e^{2\tilde{V}} \Phi_{\tilde{H}} \quad , \quad (5.30)$$

where $\tilde{\Phi}_{\tilde{H}}$ are independent matter fields whose $\tilde{H}$ -representation is identical to that of the fermions $\tilde{\psi}$. The cancellation of isometry anomalies then requires the equality of the anomalies of (5.30),

$$\delta W_{\tilde{H}}[\tilde{V}] = \mathcal{A}(\varrho, \tilde{V}) \quad , \quad \varrho \in \mathfrak{k}(\tilde{H}) \quad (5.31)$$

and that of (5.28), i.e. the variation of the WZ-counter term $\delta \Gamma^{WZ}$ as given in (5.27). Such ' $\tilde{H}$ -matching conditions' have first been introduced in Ref. [16].

Since the $\tilde{H}$ anomaly is defined only up to variations of local counter terms, the anomaly matching must occur only modulo the effect of such terms. However, we do not know the general structure of an $\tilde{H}$ anomaly explicitly. Therefore we are not able to relate the requirement of $\tilde{H}$ anomaly matching to a constraint on the usual symmetric traces d_{ABC} in a rigorous way.

The variation of the WZ counter term (5.27) suggests that the $\tilde{H}$ anomaly is proportional to $d_{ABC}^{(\tilde{G}|\tilde{H})}$, i.e., the $\tilde{G}$ -anomaly coefficients, where one index is restricted to generators of $\mathfrak{k}(\tilde{H})$ because $\varrho \in \mathfrak{k}(\tilde{H})$. For the 'minimal' non-homogeneous models, on which we focus our discussion, the H -representation of the Goldstone superfields has the form $\varphi_H = \underline{R} + \underline{1}$, where $\underline{R}$ is chiral. For such models the set of $d_{ABC}^{(\tilde{G}|\tilde{H})}$ involves all $d_{ABC}^{(\tilde{G})}$ coefficients except for $\text{Tr}_{\varphi_H} (Q^3)$, where Q is the broken central charge whose Goldstone superfield is the singlet of φ_H . This seems to indicate very strong matching conditions.

The analogue of the set of anomaly coefficients $d_{ABC}^{(\tilde{G}|\tilde{H})}$ for the σ -model is the set of symmetric traces $d_{ABC}^{(\sigma, H)}$ which can be obtained in the usual way from the anomaly coefficients of the unbroken currents $d_{ijk}^{(\sigma, H)}$ and the 'axial' couplings g_A of fermion pairs of broken generators. Clearly, any anomaly matching beyond the familiar 't Hooft matching of H -anomalies corresponds to restrictions on the 'axial' couplings g_A . Indeed, such restrictions among axial couplings are present in the $U(n+2)/U(n) \times SU(2)$ model described in Section 3. The currents (3.16) contain four axial couplings, three of which are determined by the ratio f_2/f_1 of the two 'decay' constants.

On the other hand, anomaly constraints cannot fully determine the 'axial' couplings g_A . Rather, the general $\tilde{H}$ anomaly is only defined up to Bardeen-type counter terms R^B which effectively change a $\tilde{H}$ -connection to another, equivalent one. The most general $\tilde{H}$ -connection is given by (4.49),

$$q(\tau) L^\dagger L + h(\tau) L^\dagger L v v^\dagger L^\dagger L \quad , \quad L^\dagger L = e^{2\tilde{V}} \quad ,$$

where g and h are arbitrary functions of

$$\tau = v^\dagger L^\dagger L v$$

Equation (4.49) clearly allows for a variety of 'axial' couplings to the $\tilde{H}$ -gauge field $\tilde{V}$ when inserted into (5.28) or (5.30), in the appropriate representation. In other words, we expect that the 'axial' couplings are determined only up to ' $\tilde{H}$ -covariance', i.e., up to the ambiguity of (4.49). This means that in general not all 'axial' couplings can be put effectively to zero^{*} by means of counter terms R^B . This is in contrast to the non-supersymmetric case (c.f. (5.17)), where H -covariance of $\omega = \omega^0 + g_A \tau$ does not restrict any of the g_A .

How could ' $\tilde{H}$ -matching conditions' explicitly look like? The simplest conjecture is that there have to exist σ -model anomaly coefficients $d_{ABC}^{(\sigma, \tilde{H})}$, calculated for a certain set of 'axial' couplings compatible with ' $\tilde{H}$ -covariance', which are equal to anomaly coefficients $d_{ABC}^{(\bar{G}|\tilde{H})}$ for a certain $\bar{G}$ -representation $\rho_{\bar{G}}$.

For the $U(n+2)/U(n) \times SU(2)$ example of Section 3 we have listed the $d_{ABC}^{(\sigma, \tilde{H})}$ coefficients which follow from the currents (3.16) in the table. They depend on the parameters f_2/f_1 , λ , and κ which are independent for the most general $\tilde{H}$ -connection (4.49). We have kept only the $U(1)$ charge $U = 2Q - nZ$ which, unlike Q and Z , is not affected by the $SU(2)_{HC}$ gauge anomaly (c.f. (3.18)).

Clearly, the $\bar{G}$ -representation $\rho_{\bar{G}}$ is given by the 'preon' fields $\hat{\phi}_\alpha^p$. It turns out, however, that the corresponding coefficients $d_{ABC}^{(\bar{G}|\tilde{H})}$ do not match the σ -model $d_{ABC}^{(\sigma, \tilde{H})}$ for any values f_2/f_1 , λ , and κ . Currents, whose anomaly coefficients $d_{ABC}^{(\bar{G}|\tilde{H})}$ do match the $d_{ABC}^{(\sigma, \tilde{H})}$, can be obtained from the ordinary G -currents by inverting the sign of the $U(2)$ currents J_j^i and J_Z , which is suggested by the 'Higgs phase' of the model (c.f. Ref. [23]). These currents read $(\hat{\phi}_\alpha^p = (\phi_\alpha^p, \psi_\alpha^p); \alpha = (a, i); a = 1 \dots n; i = n+1, n+2)$:

$$\hat{J}_{b\mu}^a = \bar{\psi}_p^a \bar{\sigma}_\mu \psi_p^p - \frac{1}{n} \delta_b^a \bar{\psi}_p^c \bar{\sigma}_\mu \psi_p^c, \quad (5.32a)$$

$$\hat{J}_{Q\mu} = \bar{\psi}_p^a \bar{\sigma}_\mu \psi_p^p, \quad (5.32b)$$

$$\hat{J}_{j\mu}^i = -(\bar{\psi}_p^i \bar{\sigma}_\mu \psi_p^p - \frac{1}{2} \delta_j^i \bar{\psi}_p^k \bar{\sigma}_\mu \psi_p^k), \quad (5.32c)$$

$$\hat{J}_{i\mu}^a = \bar{\psi}_p^a \bar{\sigma}_\mu \psi_p^p, \quad (5.32d)$$

^{*}) For instance, if the representation of $L^\dagger L$ does not contain a $\tilde{H}$ singlet.

$$\hat{j}_{a\mu}^i = \bar{\Psi}_\mu^i \bar{\psi}_\mu^p \psi_a^p \quad , \quad (5.32e)$$

$$\hat{j}_{\tau\mu}^i = - \bar{\Psi}_\mu^i \bar{\psi}_\mu^p \psi_i^p \quad . \quad (5.32f)$$

Matching, i.e., $d_{ABC}^{(\bar{G}|\tilde{H})} = d_{ABC}^{(\sigma,\tilde{H})}$ is obtained for $f_2/f_1 = 2$, $\lambda = 0$ and $\kappa = 1$, which corresponds to a specific choice of the general $\tilde{H}$ -connection (4.49) in the Bardeen type counter term R^B (this does *not* mean that in the σ -model $2f_1 = f_2$, $\kappa = 1$). As expected, the U^3 anomaly is not matched.

What can we conclude from our discussion of ' $\tilde{H}$ -anomaly matching'? The general arguments given above suggest that in supersymmetric σ -models the matching of isometry anomalies imposes restrictions on the possible sets of 'axial' coupling constants. The symmetry, which correlates the various 'axial' couplings, is $\tilde{H}$, the complex subgroup of $\bar{G}$ which also determines the complex structure and the form of the Kähler potential. For chiral representations the resulting ' $\tilde{H}$ -matching conditions' appear non-trivial, as illustrated by the $U(n+2)/U(n) \times SU(2)$ example. However, a rigorous formulation of the ' $\tilde{H}$ -matching conditions' requires a deeper understanding of the $\tilde{H}$ -anomaly and the related (supersymmetric) Bardeen-type counter terms which is beyond the scope of this paper.

5.5 Further anomalies

In this section we present briefly additional arguments which show that homogeneous ($\bar{G}/\tilde{H} \cong G/H$) models cannot arise as effective theories; these arguments do not apply for the non-homogeneous ($\dim(\bar{G}/\tilde{H}) > \dim(G/H)$) σ -models.

First, let us couple the σ -model to (background super-) gravity. Then the global current associated with the $U(1)$ factor in H which defines the complex structure is not conserved [36], due to an anomalous diagram involving the $U(1)$ current and two energy momentum tensors. This follows since for such $U(1)$ factors necessarily $\text{Tr}(Q) \neq 0$. On the other hand, in any underlying theory where Q is embedded into some semi-simple group G , $\text{Tr}(Q) = 0$, implying the absence of this anomaly. Thus, by a 't Hooft like argument it follows that a homogeneous model can never be a low energy approximation of a linear theory. This is in contrast to non-homogeneous models where Q is necessarily spontaneously broken when there are chiral fermions.

Secondly we give a similar argument leading to the same conclusion. Again we couple the σ -model to supergravity. This is achieved (in minimal supergravity) by taking

$$\mathcal{L} = \int d^4x \, E \, e^{-\frac{1}{3}K(\Phi, \bar{\Phi})} \quad , \quad (5.33)$$

where E is the super vielbein determinant and K the Kähler potential. Let K describe a homogeneous σ -model. Then, under G transformations, K is not completely invariant, but only invariant up to holomorphic functions (c.f. (2.26)).

To maintain invariance under of (5.33), E has to transform in a certain way which is equivalent to a super-Weyl transformation. On appropriately redefined fermion fields, it acts as a chiral rotation [49]:

$$\begin{aligned} \chi^a &\rightarrow e^{\frac{1}{4}(\gamma - \gamma^*)\gamma_5} \chi^a \quad , \\ \psi_\mu &\rightarrow e^{-\frac{1}{4}(\gamma - \gamma^*)\gamma_5} \psi_\mu \end{aligned} \quad (5.34)$$

(we use a 4-component notation here; ψ_μ is the Rapita-Schwinger vector-spinor). As shown in [49], a global definition of (5.42) over the whole Kähler manifold restricts it to be a Hodge manifold. Our point here is that the transformations (5.42) are anomalous [36]; since they are induced by (global) G transformations, we encounter a new type of (supergravity induced) isometry anomaly. In any underlying theory where G is realized linearly, the Lagrangian is completely invariant, so that there is no such type of anomaly. Again, employing 't Hooft's argumentation we can conclude that homogeneous models do not represent low energy approximations of linear theories. On the other hand, for non-homogeneous models, there exist always completely invariant Kähler potentials: for 'non-doubled' models, the holomorphic variation of K can be absorbed by the 'novino' field [26]. For 'full doubled' models, which need not contain such a 'novino', it was shown in [50] that there also exist completely invariant Kähler potentials. Hence, non-homogeneous models do not suffer from the above topological restriction and isometry anomaly.

Finally, recall that we have shown in Section (5.2) that for homogeneous models the isometry anomalies cannot be eliminated. What we like to point out is that these models are not even supersymmetric. This is because the supersymmetry transformation law reads in terms of the usual tangent space fermions

$$\begin{aligned} \delta_\epsilon \psi^a &= i\sqrt{2} \, \sigma^\mu \bar{\epsilon} \, e^a_\alpha \partial_\mu \psi^\alpha - \Lambda^a_b \psi^b \quad , \\ \Lambda^a_b &= e^a_\gamma \partial_\alpha (e^{-1})^\gamma_b \delta_\epsilon \psi^{\alpha\bar{a}} - \omega^a_b \delta_\epsilon \psi^a \quad , \end{aligned} \quad (5.35)$$

where Λ_b^a is a local transformation valued in $\mathfrak{E}(\mathcal{H})$. Thus an uncanceled holonomy anomaly leads also to a supersymmetry anomaly. For non-homogeneous models, the presence of the WZ-term changes the transformation law (5.43) because it contains derivatives acting on the auxiliary F-component [43]. It is therefore conceivable that in non-homogeneous models a supersymmetry anomaly does not appear.

6. SUMMARY AND CONCLUSIONS

In the foregoing chapters we have presented a systematic analysis of the simplest class of consistent supersymmetric σ -models. Such models, which describe the Goldstone superfields arising from the spontaneous breaking of a symmetry G to a subgroup H , are necessarily associated with non-compact non-homogeneous Kähler manifolds $\tilde{M} = \tilde{G}/\tilde{H}$. The main properties of those σ -models are determined by $\tilde{H}$, a complex subgroup of $\tilde{G}$, which may be larger than the complex extension $\bar{H}$ of H . The non-compactness of the σ -model manifold is related to flat directions in the underlying theory. Particularly interesting are σ -models which deviate 'minimally', i.e. just by one (or more) H -singlet 'novino' superfields from the familiar compact models; the representation of the Goldstone superfields of these models has maximal chirality with respect to H . As an instructive example for this class of minimal non-homogeneous σ -models we have discussed the coset space $U((n+2)/U(n) \times SU(2)$.

The geometry of the supersymmetric non-homogeneous σ -models is essentially different from that of conventional, non-supersymmetric coset models. In the familiar case of homogeneous coset spaces G/H fermions can always be coupled by means of the canonical CCWZ spin connection, and the holonomy group $\mathcal{H}$ of these σ -models can always be chosen to be identical with the isotropy group H . The situation is different for supersymmetric non-homogeneous σ -models. Here the spin connection is necessarily valued in a Lie algebra larger than $\mathfrak{E}(H)$, implying a holonomy group larger than the isotropy group.

The conventional transition from 'curved' to 'flat' indices by means of the canonical CCWZ vielbein is a non-holomorphic transformation. As supersymmetric σ -models are associated with complex coset spaces $\tilde{M} = \tilde{G}/\tilde{H}$, there exists an alternative formulation of these theories in which the transition from 'curved' to 'flat' indices is performed by means of a holomorphic vielbein related to $\tilde{G}/\tilde{H}$. In this formulation the σ -models appear as $\tilde{H}$ gauge theories, i.e., they are invariant under local complex $\tilde{H}$ transformations of the fermions, where the role of the gauge connection is played by a composite vector field constructed from the coset representative $l(\varphi)$ of $\tilde{G}/\tilde{H}$.

The natural appearance of $\tilde{H}$ gauge theories in the context of supersymmetric σ -models is one of the main points of our paper. The structure of these complex

gauge theories is rather intriguing. Their new feature compared to ordinary supersymmetric gauge theories is that the local complex $\tilde{H}$ invariance is not the complex extension of a unitary symmetry group.

The fermion content of a supersymmetric σ -model is fixed by the coset $\bar{G}/\tilde{H}$. If this fermion representation is chiral with respect to H , the G -invariance of the theory will in general be spoiled by isometry anomalies, and the question arises, under what conditions these anomalies can be cancelled by local Wess-Zumino-type counter terms. We have shown that such counter terms cannot exist for homogeneous σ -models (unless one introduces additional fields), and that furthermore these models have supersymmetry as well as gravity induced anomalies.

On the contrary, these problems do not exist for the class of non-homogeneous supersymmetric σ -models. Here our starting point is the hidden local $\tilde{H}$ symmetry of these models. This symmetry is emphasized by the supersymmetric WZ-type counter term, and it also allows, under certain assumptions, a manifestly supersymmetric formulation of the σ -model isometry anomaly. This then suggests extended 't Hooft type matching conditions for the cancellation of isometry anomalies. Although we have not been able to fully understand the ambiguity of $\tilde{H}$ anomalies, the extended ' $\tilde{H}$ matching conditions' represent a reasonable conjecture. Their main physical implication is that the cancellation of isometry anomalies is only possible if the σ -model can be obtained from spontaneous symmetry breaking. We have illustrated these extended matching conditions by means of the $U(n+2)/U(n) \times SU(2)$ example.

Stronger conditions than usually considered also arise if one demands the cancellation of holonomy anomalies, because for non-homogeneous models the holonomy group is always larger than the isotropy group.

In conclusion, in contrast to compact, homogeneous models, the non-compact, non-homogeneous supersymmetric σ -models do not suffer from any theoretical inconsistency. Hence they provide an attractive framework for supersymmetric low energy effective Lagrangians.

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Figure caption

The complex structures $\Omega_I(a)$ and $\Omega_{II}(b)$ (cf. (2.12)). The dotted area corresponds to $\bar{H}$, the dashed one to $\tilde{H}/\bar{H}$, the white one to the group $\bar{G}/\tilde{H}$.

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Table 1

Anomaly coefficients of the linear G-representation $e_G(d_{ABC}^{(\tilde{G}|\tilde{H})})$
 and the σ -model $\tilde{H}$ -representation $e_{\tilde{H}}(d_{ABC}^{(\sigma,\tilde{H})})$, as
 described in Section 5.4; $U = 2Q - nZ$

Currents	$d_{ABC}^{(\tilde{G} \tilde{H})}, e_G$	$d_{ABC}^{(\sigma,\tilde{H})}, e_{\tilde{H}}$	
		$f_2/f_1, \lambda, \kappa$ arbitrary	$f_2/f_1 = 2, \lambda = 0, \kappa = 1$
$(L_b^a)^3$	2	2	2
$L_{b j}^{a c} Y_d^i$	2	$f_2^2/2f_1^2$	2
$L_{j k}^{i a} Y_b^l$	-2	$-f_2^2/2f_1^2$	-2
$(L_b^a)^2 U$	4	$4 - 2n\lambda$	4
$(L_j^i)^2 U$	$2n$	$2n - n^2\lambda$	$2n$
$X_j^a Y_b^i U$	$4 + 2n$	$f_2^2(2 + n(\kappa - \lambda))/2f_1^2$	$4 + 2n$
U^3	$4n^3 + 16n$	$\kappa^3 n^3 + 2n(2 - n\lambda)^3$	$n^3 + 16n$

a)

b)