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BRS INVARIANCE OF THE INTERACTING POLYAKOV STRING

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A B S T R A C T

We calculate the modification to the BRS symmetry of Polyakov's string required to ensure the invariance of the generating functional and the decoupling of spurious states in the presence of interactions.

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BRS invariance appears to be crucial to the second quantization of string theories¹⁾. In Polyakov's first quantized model²⁾ this invariance is the symmetry of the Faddeev-Popov ghost representation of Jacobians resulting from the integration over metrics. The Riemann-Roch theorem implies that there are $6h-6$ real ghost zero-modes for a string world-sheet that is a closed surface with h handles ($h>1$, i.e., no conformal Killing vectors and the only case we consider), corresponding to an interacting closed string with h loops. These zero-modes are deformations of the metric orthogonal to diffeomorphisms and conformal changes and are a basis for the tangent space to Teichmüller space. These deformations are included in the partition function by integrating over the moduli of different conformal structures, the Teichmüller parameters. We will show that these zero-modes result in an acceptable modification of the BRS symmetry that ensures the invariance of the generating functional and the decoupling of spurious states.

Polyakov's partition function for the bosonic string is

$$Z_h = \int_{\mathcal{M} \times \mathcal{E}} \mathcal{D}X \mathcal{D}g \frac{1}{V} \exp - \frac{i}{2} \int_{\Sigma} d^2z \sqrt{g} (g^{ab} \partial_a X^\mu \partial_b X^\mu + \lambda) \quad (1)$$

We take the world-sheet parameter space Σ to have h handles and no boundary. $\mathcal{M}$ is the space of metrics on Σ and $\mathcal{E}$ the embeddings of Σ in the target space-time. V is the volume of diffeomorphisms and conformal transformations acting on g fixed by the metric on variations of g that induces the measure $\mathcal{D}g$

$$(\delta_1 g, \delta_2 g) = \int_{\Sigma} d^2z \sqrt{g} (g^{ab} g^{rs} + c g^{ar} g^{bs}) \delta_1 g_a \delta_2 g_b \quad (2)$$

with c a constant. An arbitrary metric may be parametrized as a diffeomorphism acting on a conformal transformation of a metric $g(y)$ which is an element of $R \equiv \mathcal{M} / (\text{conformal transformations} \times \text{diffeomorphisms})$. The y^A are $6h-6$ real co-ordinates on R which is the moduli space of the Riemann surface Σ , that is the space of inequivalent conformal structures on Σ . Two sets of complex co-ordinates z, z' define different complex structures if there is no analytic map m such that $z.m = z'.f$ for $f \in \text{Diff}(\Sigma)$, the diffeomorphisms defined on Σ . Teichmüller space $T(\Sigma)$ differs from R only by restricting the diffeomorphisms to be connected to the identity, $\text{Diff}_0(\Sigma)$, thus $R = T(\Sigma)/\text{Mod}(\Sigma)$, $\text{Mod}(\Sigma) = \text{Diff}(\Sigma)/\text{Diff}_0(\Sigma)$. The uniformization theorem identifies Σ with the fundamental region on the upper half complex plane, H , of some Fuchsian group Γ , itself a subgroup of the Möbius transformations. This is naturally equipped with the Poincaré metric

$$ds^2 = dz d\bar{z} / (\operatorname{Im} z)^2 \quad (3)$$

A map m between the two inequivalent complex structures which preserves the Poincaré metric modulo conformal transformations is quasi-conformal³⁾ and satisfies the Beltrami equation

$$\frac{\partial m}{\partial \bar{z}} = \mu(z) \frac{\partial m}{\partial z} \quad (4)$$

for some Beltrami differential μ with $|\mu| < 1$ and

$$\mu(\gamma(z)) \overline{\gamma'(z)} / \gamma'(z) = \mu(z), \text{ all } \gamma \in \Gamma \quad (5)$$

If one point is chosen as the origin then every point of $T(\Sigma)$ may be associated with a Beltrami differential in this way. This can be done uniquely by taking as the solution to (4) the element in the homotopy class $[m]$ of m with the smallest dilatation defined as $(1+K)/(1-K)$, $K < |\partial m / \partial \bar{z}| / |\partial m / \partial z|$. Extending $\mu(z)$ to the whole complex plane by taking it to vanish on the lower half L generates a solution to the Beltrami equation m_μ which is quasi-conformal on H and conformal on L . The Schwarzian derivative of $m_\mu(z)$ on L

$$\phi_\mu(z) = \{m_\mu(z), z\} \equiv \frac{d}{dz} \left(\frac{\frac{\partial^2 m_\mu}{\partial z^2}}{\frac{\partial m_\mu}{\partial z}} \right) - \frac{1}{2} \left(\frac{\frac{\partial^2 m_\mu}{\partial z^2}}{\frac{\partial m_\mu}{\partial z}} \right)^2 \quad (6)$$

defines a holomorphic mapping from $T(\Sigma)$ into the Bers space $B_2(\Gamma)$ of quadratic differentials $g \rightarrow \mu(g) \rightarrow \phi_\mu(g)$. Co-ordinates y^A on $T(\Sigma)$ can be taken to be co-ordinates of ϕ_μ with respect to some basis of $B(\Gamma)$, note $B(\Gamma) \approx C^{3h-3}$. The measure on $T(\Sigma)$ is induced by a scalar product on its tangent space which is the space of quadratic differentials $Q(\Gamma)$ consisting of holomorphic functions such that

$$\varphi(\gamma(z)) [\gamma'(z)]^2 = \varphi(z), \quad z \in H, \gamma \in \Gamma. \quad (7)$$

The Weyl-Petersson scalar product

$$\langle \varphi, \psi \rangle_{WP} = \int_{H/\Gamma} d^2 z \varphi \bar{\psi} \cdot (\operatorname{Im} z)^2 \quad (8)$$

ensures modular invariance of the measure.

Taking into account diffeomorphism invariance and the absence of a conformal anomaly in 26 space-time dimensions^{2),4)} enables the partition function (1) to be written as an integral over moduli⁵⁾

$$Z_h = \frac{1}{M} \int \prod_{A=1}^{6h-6} dy^A \mu (\text{Det}'^{1/2} P^+P)_{\hat{g}} \left(\frac{2\pi}{\int d^2z \sqrt{\hat{g}}} \text{Det}' \Delta_{\hat{g}} \right)^{-13} \quad (9)$$

The measure on $T(\Sigma)$,

$$\mu = \det'^{1/2} \langle \varphi_A, \varphi_B \rangle \quad (10)$$

and $\text{Det}'^{1/2} P^+P$ are Jacobians for the transformation of the integral over g to an integral over $\{y^A\}$ and vectors $\{V_a\}$ parametrizing diffeomorphisms. P is the operator

$$P_{ab}(V) = \nabla_a V_b + \nabla_b V_a - g_{ab} \nabla^c V_c \quad (11)$$

and P^+ is defined with respect to the scalar product (2). The kernel of P^+ is the set of quadratic differentials. The last factor of (9) results from the integration over x^μ . M is the volume of $\text{Mod}(\Sigma)$ and the subscript g denotes that this is the metric used throughout (9).

The Faddeev-Popov representation of $1/\text{Det} P^+P$ is contained in the integral

$$G_g(\beta, \gamma) = \int \mathcal{D}(b, c) \exp - \int d^2z \sqrt{g} (b_{ab} \nabla^a c^b + b_{ab} \beta^{ab} - \gamma_a c^a) \quad (12)$$

b and c are real Fermi fields, β and γ are Grassmannian sources and b is traceless with respect to the metric g . Decomposing b and β into pieces in $\text{Ker} P^+$ and orthogonal parts

$$b = b^A \varphi_A + b_\perp, \quad \beta = \beta_A \eta^A + \beta_\perp$$

(with η^A dual to ϕ_A) enables the b integration to be transformed to one over the variables b^A, p where

$$P(p) = b_\perp \quad (13)$$

with Jacobian $\mu^{-1} \text{Det}^{-1/2} P^+P$. Note that for $h > 1$ P has no zero-modes and (13) is invertible. The integration over P and c produces another factor of $\text{Det} P^+P$ so G_g may be evaluated as

$$G_g(\beta, \gamma) =$$

$$\prod_A \beta_A \mu^{-1} \text{Det}'^{1/2}(P^+P) \exp \int_{\Sigma} d^2z d^2z' \sqrt{g g'} \gamma_a(z) N_{bc}^d(z, z') \beta^{bc}(z') \quad (14)$$

with N a Green's function for P and P^+ traceless in the lower indices and orthogonal to $\text{Ker } P^+$. Thus G contains $\text{Det}'^{1/2} P^+P$ and we conclude that any h -loop scattering amplitude may be obtained by integrating over $T(\Sigma)$ appropriate derivatives of

$$\begin{aligned} Z_{\text{eff}}^{\hat{g}}(\xi, \beta, \gamma) &\equiv \int \mathcal{D}(\chi, b, c) \exp - (S(\chi, b, c) + S_J) \\ &= \left(\prod_A \beta_A \right) \mu^{-1} \exp(-S_0 + W) \end{aligned} \quad (15)$$

where

$$S(\chi, b, c) = \int d^2z \sqrt{\hat{g}} \left(\frac{1}{2} \hat{g}^{ab} \partial_a \chi^\mu \partial_b \chi^\mu + \frac{1}{2} \lambda + b_{ab} \nabla^a c^b \right)$$

$$S_J = - \int d^2z \sqrt{\hat{g}} \left(\xi^\mu \chi^\mu - b_{ab} \beta^{ab} + \gamma_a c^a \right)$$

$$\exp - S_0(\hat{g}) = \left(\frac{2\pi}{\int d^2z \sqrt{\hat{g}}} \text{Det}' \Delta_{\hat{g}} \right)^{-13} \left(\text{Det}'^{1/2} P^+P \right)_{\hat{g}}$$

$$\begin{aligned} W(\xi, \beta, \gamma) &= \int d^2z d^2z' \sqrt{g g'} \left(\xi^\mu(z) G(z, z') \xi^\mu(z') \right. \\ &\quad \left. + \gamma_a(z) N_{bc}^d(z, z') \beta^{bc}(z') \right) \end{aligned} \quad (16)$$

and G is the Green's function for Δ_g .

The action $S(\chi, b, c)$ has the classically nilpotent BRS symmetry corresponding to diffeomorphism invariance

$$\delta_\lambda \chi^\mu = -\lambda c^a \partial_a \chi^\mu \quad \delta_\lambda c^a = -\lambda c^b \partial_b c^a$$

$$\delta_\lambda b_{ab} = 2\lambda \frac{1}{\sqrt{g}} \frac{\delta S(\chi, b, c)}{\delta g^{ab}} = 2\lambda \left(T_{ab}(\chi) + T_{ab}^{gh}(b, c) \right)$$

$$2 T_{ab}(X) = \partial_a X^\mu \partial_b X^\mu - \frac{1}{2} g_{ab} g^{\mu\nu} \partial_\mu X^\mu \partial_\nu X^\mu$$

$$2 T_{ab}^{gh}(b, c) = b_{pa} \nabla_b c^p + b_{pb} \nabla_a c^p + (\nabla_p b_{ab}) c^p - g_{ab} b_{rs} \nabla^r c^s \quad (17)$$

where λ is a Grassmannian parameter. BRS invariance⁶⁾ has been used to prove the no-ghost theorem for the free string⁷⁾ and is central to recent attempts to second-quantize string theories¹⁾, so it is important to check that multiloop amplitudes obtained from (15) respect this invariance. If the measure in Z_{eff} respects this symmetry then a shift in integration variables corresponding to (17) would yield the Ward identity

$$\Omega Z_{\text{eff}} = 0$$

$$\Omega = \int d^2x \sqrt{g} \left\{ -\xi^\mu \frac{\delta}{\delta r_\mu} \frac{1}{\sqrt{g}} \partial_a \frac{1}{\sqrt{g}} \frac{\delta}{\delta \xi^\mu} - \gamma_a \frac{1}{\sqrt{g}} \frac{\delta}{\delta r_a} \partial_b \frac{1}{\sqrt{g}} \frac{\delta}{\delta r_b} + 2\beta^{ab} \left(T_{ab} \left(\frac{\delta}{\delta \xi} \right) + T_{ab}^{gh} \left(\frac{\delta}{\delta \beta}, \frac{\delta}{\delta r} \right) \right) \right\} \quad (18)$$

This may be checked directly. It is convenient to introduce "classical" fields

$$\tilde{X} \equiv \frac{1}{\sqrt{g}} \frac{\delta W}{\delta \xi}, \quad \tilde{b} \equiv \frac{1}{\sqrt{g}} \frac{\delta W}{\delta \beta} = \frac{1}{\sqrt{g}} \frac{\delta W}{\delta \beta_\perp}, \quad \tilde{c} \equiv \frac{1}{\sqrt{g}} \frac{\delta W}{\delta r} \quad (19)$$

where, corresponding to the decomposition of β

$$\frac{\delta}{\delta \beta} = \sum_A \varphi_A \frac{\partial}{\partial \beta_A} + \frac{\delta}{\delta \beta_\perp} \quad (20)$$

Exploiting the linearity of T^{gh} in b , ΩZ_{eff} may be written as

$$\int d^2x \sqrt{g} \left\{ -\xi^\mu \tilde{c}^a \partial_a \tilde{X}^\mu - \gamma_a \tilde{c}^b \partial_b \tilde{c}^a + 2\beta_\perp^{ab} (T_{ab}(\tilde{X}) + T_{ab}^{gh}(\tilde{b}, \tilde{c})) \right\}$$

$$+ 2\beta_\perp^{ab} T_{ab}^{gh} \left(\sum_A \varphi_A \frac{\partial}{\partial \beta_A}, \tilde{c} \right) + 2\beta^{ab} T_{ab}^0$$

$$+ \sum_A 2\beta_A \eta^{ab} (T_{ab}(\tilde{X}) + T_{ab}(\tilde{b} + \sum_B \varphi_B \frac{\partial}{\partial \beta_B}, \tilde{c}))$$

$$\times (\pi \beta_c) \mu^{-1} \exp(W - S_0) \quad (21)$$

T° is the vacuum expectation value of $T + T^{\text{gh}}$ and stands in need of regularization as does S_0 . Divergences in S_0 are cancelled by the "cosmological constant" λ in (1) and the finite part fixed by requiring the absence of the conformal anomaly for the critical dimension of space-time. This determines the regularized value of T° as

$$T_{ab}^\circ \equiv \frac{1}{\sqrt{g}} \frac{\delta S_0}{\delta g^{ab}} = -\sum_A \eta_{ab}^A \frac{\partial S_0}{\partial y^A} \quad (22)$$

The last part follows if the ϕ_A are chosen as the derivatives of g with respect to the y^A . Thus βT° when integrated over Σ is just a sum of the β_A and cannot contribute to (21) because of $\Pi\beta_c$ in Z_{eff} . The term in braces becomes the BRS transform of $S(\tilde{\chi}, \tilde{b}, \tilde{c})$ when the sources are expressed in terms of "classical" fields and so vanishes. The next term may be put into the form

$$\sum_A \int d^2x \sqrt{g} \varphi_{Aab} \nabla^a (\tilde{c}^b \nabla_c \tilde{c}^c) \frac{\partial}{\partial \beta_A}$$

which is zero by partial integration. In the remaining terms β_A picks out the $\partial/\partial \beta_A$ part of T^{gh} , again because of $\Pi\beta_c$ in Z_{eff} . After a little algebra (21) becomes

$$\Omega Z_{\text{eff}} = \left(\sum_A \frac{1}{2} \int d^2x \sqrt{g} \eta^{rs} \varphi_{Ars} \nabla \cdot \tilde{c} \right) Z_{\text{eff}} \quad (23)$$

and in general this will not vanish.

The right-hand side of (23) may be absorbed into a new BRS operator, $\tilde{\Omega}$, by writing $\tilde{c}$ in terms of the source β

$$\begin{aligned} \tilde{\Omega} &= \\ \Omega &= \frac{1}{2} \int d^2x \sqrt{g} \beta^{ab}(x) \left(\sum_A \int d^2x' \sqrt{g'} \eta^{rs}(x') \varphi_{Ars}(x') \nabla'_c N_{ab}^c(x', x) \right) \\ \tilde{\Omega} Z_{\text{eff}} &= 0 \end{aligned} \quad (24)$$

This amounts to modifying the BRS transformation (17) by adding a c-number term to $\delta_\lambda b$. This is acceptable because it is only the transformation of the "geometrical ghost" c that is fixed by cohomology. $\delta_\lambda b$ is the only part of (17) to contain operators that require regularization so it is not surprising that we should have to add a c-number to secure BRS invariance when the finite parts of those operators have been fixed by a different physical requirement. A c-number modification of $\delta_\lambda b$ is also necessary in Kato and Ogawa's treatment of the free string⁷⁾.

The decoupling of spurious states from physical processes follows immediately from the (modified) BRS invariance of Z_{eff} . The expectation value of an operator is obtained, as usual, by applying to Z_{eff} the same operator expressed in terms of derivatives with respect to sources, setting the sources to zero, and then integrating over $T(\Sigma)$. The BRS transform of an operator V can be written as

$$\delta_\lambda V \left(\frac{\delta}{\delta \xi}, \frac{\delta}{\delta \beta}, \frac{\delta}{\delta \gamma} \right) = \left[V \left(\frac{\delta}{\delta \xi}, \frac{\delta}{\delta \beta}, \frac{\delta}{\delta \gamma} \right), \lambda \tilde{\Omega} \right] \quad (25)$$

Physical states are created by BRS invariant operators, V_{phys} , which therefore commute with $\lambda \tilde{\Omega}$. A spurious state is created by an operator that is the BRS transform of another and is thus physical although cohomologous to zero. They decouple because

$$\begin{aligned} & (V_{\text{phys}} [V, \lambda \tilde{\Omega}] V'_{\text{phys}} Z_{\text{eff}})_{\text{sources} = 0} \\ &= ([V_{\text{phys}}, V V'_{\text{phys}}, \lambda \tilde{\Omega}] Z_{\text{eff}})_{\text{sources} = 0} \end{aligned} \quad (26)$$

and both terms in the commutator vanish, the first because of the invariance of Z_{eff} and the second because the sources are set to zero at the end.

Our approach differs from that of Friedan et al.⁷⁾ who work with the equivalent of Z_{eff} with the zero-modes β_A cancelled. For them spurious states decouple if

$$\left(\int dy \mu^2 [V, \lambda \Omega] \pi \frac{\partial}{\partial \beta_A} Z_{\text{eff}} \right)_{\text{sources} = 0} \quad (27)$$

is zero. Ω does not commute with $\Pi \partial / \partial \beta_A$ so the integrand does not in general vanish; however, they argue that it is a total derivative and so (27) may be transformed to an integral over the boundary of R . R may be compactified in such a way that the Weyl-Peterson measure extends continuously to the boundary allowing the use of Stokes theorem⁵⁾. This involves adding to the space of moduli points corresponding to manifolds with nodes, that is degenerate configurations corresponding to emission of tachyons and tadpoles obtained by shrinking a geodesic on the surface to zero and thus changing the topology. However, it is difficult to see that this will work for a general V with an arbitrary dependence on the y^A . In any case the integrand has been shown to diverge on the boundary⁹⁾. This is because even in the critical dimension $D = 26$ the determinant fibre bundle $\exp(-S)$ does not define a holomorphic global section on the compactified R . Instead we have insisted on the (modified) BRS invariance of Z_{eff} and require

that the cancellation of zero-modes, which is necessary for a non-vanishing expectation value, be effected by the operators V_{phys} themselves. Of course the construction of physical operators remains an open problem, as does the demonstration of nilpotency of the BRS transformation.

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