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AMBIGUITY IN THE EFFECTIVE ACTION IN STRING
THEORIES

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AMBIGUITY IN THE EFFECTIVE ACTION IN STRING THEORIES

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Abstract

We point out that there is an ambiguity in some of coefficients in the "low-energy" effective action (EA) for the massless string modes (in particular, the coefficients of the $R_{\mu\nu}^2$ - and R^2 -terms in the gravitational EA's are ambiguous). In the S-matrix approach this ambiguity is a consequence of an off-shell ambiguity in subtraction of the massless exchanges (manifesting itself as a freedom of field redefinitions which leave the S-matrix invariant because of the equivalence theorem). In the world-sheet path integral approach the ambiguity is due to a freedom in choice of a 2-d renormalization scheme and a Weyl gauge. We also comment on the relation with second-quantized string approach.

The Appendix contains the revised version of sections 7 and 8 of the author's paper "Vector field effective action in the open superstring theory" (Lebedev Institute preprint N 6 (1986)). Sect. 7 deals with the leading $O(\alpha'^2)$ terms in the non-abelian vector EA in the open string theories. Sect. 8 discusses the gravitational EA in the closed string theories. Many points touched upon in the main part of the paper are discussed in more details in the Appendix.

1. Second-quantized approach

We shall start with a qualitative discussion of how the concept of effective action (EA) for the massless string modes naturally arises in a second-quantized string field theory thus clarifying why it is important. Let $S = \frac{1}{g^2} \int \left(\frac{1}{2} \Phi \Delta \Phi + \frac{1}{3} \Phi^3 \right)$ be a classical action of a (closed) string field theory [1]^{1/}.

Let $\Phi = \{ \varphi(x), \psi(x) \}$ be the standard parametrization of Φ in terms of local fields based on the normal mode expansion [1] (φ will generically denote all massless fields and ψ - all massive fields). Substituting this parametrization in S we get, symbolically ^{2/}

$$S = \frac{1}{g^2} \int d^2x \left\{ \frac{1}{2} \varphi(-\square) \varphi + \frac{1}{2} \psi(-\square + m^2) \psi + \varphi^3 + \varphi^2 \psi + \varphi \psi^2 + \psi^3 \right\}. \quad (1)$$

Although one "component" of ψ is the graviton, (1) is not manifestly generally covariant (it contains only 3-graviton vertex).

A clue to a resolution of this paradox is in observation that because of the presence of the $\psi \varphi^2$ -interaction [4] a massive particle can decay into the massless ones. To eliminate this

instability we are to redefine the massive fields: $\psi \rightarrow \xi$. Let

$\psi_0(\varphi) = -(-\square + m^2 + 2\varphi)^{-1} \varphi^2 + \dots$ be the solution of $\delta S / \delta \psi = 0$ on an arbitrary φ -background. Substituting $\psi = \psi_0(\varphi) + \xi$

in (1) we get

$$S = \Gamma_0[\varphi] + \tilde{S}[\varphi, \xi], \quad \tilde{S} = \frac{1}{g^2} \int d^2x \left\{ \frac{1}{2} \xi(-\square + m^2) \xi + (\varphi + \psi_0(\varphi)) \xi^2 + \xi^3 \right\} \quad (2)$$

Γ_0 is a manifestly covariant action for the massless fields,

^{1/} The arguments of Φ are coordinates, 2-d ghosts and (or) 1-d metric defined on a closed contour; the account of the latter argument [2] is crucial in construction of the interaction term in S [3].

^{2/} To make the basic points as clear as possible we oversimplify the discussion by considering φ and ψ as single scalar fields, omitting complicated derivative-dependent factors in the interaction vertices, etc.

i.e. a tree-level EA. It contains all higher-order vertices generated by the "tree" exchanges of the massive modes. The redefinition $\psi \rightarrow \varphi$ is a necessary step in the path-integral quantization of (1), $Z[J] = \int [d\psi] [d\varphi] \exp(-S[\psi, \varphi] + \varphi \cdot J)$: to integrate over ψ we are as usual to expand the integrand near a stationary point, i.e. $\psi = \psi_0(\varphi) + \xi$. Integrating over ξ we get: $Z[J] = \int [d\varphi] \exp(-\Omega[\varphi] + \varphi \cdot J)$, where $\Omega = \Gamma_0 + (\text{contributions of massive loops})$. Since it seems difficult to separate the massive loops from the massless ones without problems with modular invariance we prefer to define the full (loop-corrected) EA not as Ω but as a usual Legendre transform of $W[J] = -\ln Z[J]$, $\Gamma[\varphi] = \Gamma_0[\varphi] + (\text{contributions of all kinds of loops})$.

We see that the standard choice of the massive fields ψ is not a proper one: ψ 's are certainly nonvanishing in any non-trivial compactification (while $\varphi = 0$ in (2) is consistent with non-zero vacuum values of φ). The problem of finding a manifestly covariant formulation of string field theory (with the classical equations starting directly with the Einstein equations) is thus a "kinematical" one: it is not in the action $S[\Phi]$ itself but in a choice of parametrization of Φ in terms of the local fields. A proper "geometrical" parametrization $\Phi = \{\varphi, \xi\}$ should automatically lead to the covariant action (2) in which the massless fields are "decoupled" from the massive. One attractive possibility is to represent Φ as a path integral over disc-like 2-surfaces (cf. [2])

$$\Phi[C] = \int [dx^\mu] [dg_{mn}] \exp(-I) \quad , \quad (3)$$

$\partial M = C$

$$I = \int d^2x \sqrt{g} \left\{ B(x) + \frac{1}{\sqrt{M}} G_{\mu\nu}^{(x)} \partial_\mu x^\mu \partial_\nu x^\nu + \frac{1}{\sqrt{M}} R^{(x)} \phi(x) + \dots \right. \\ \left. + B_{\mu_1 \dots \mu_n}^{(x)} \partial x^{\mu_1} \dots \partial x^{\mu_n} \right\}, \quad \varphi = \{ G_{\mu\nu}, \phi, A_{\mu\nu} \}, \quad \xi = \{ B_{\mu_1 \dots \mu_n} \}$$

where I is a most general action of a local 2-d theory defined on a curved 2-surface [2] and the path integral is computed (at D=26) using a Weyl gauge. The classical solution of $\frac{\delta S}{\delta \Phi} = 0$

is very likely (3) with I corresponding to a Weyl-invariant 2-d theory (the trivial "tachyonic" vacuum state is known to be (3) with $\varphi = \{ G_{\mu\nu} = g_{\mu\nu} \}$ and $\xi = 0$). Substituting (3) in the action $S[\Phi]$ (defined in [2] so as to lead to perturbation theory representable as covariant sums over surfaces [5]) it is easy to see that if $\int \Phi \Delta \Phi \sim \int \Phi^3$, e.g.

if $\delta S / \delta \Phi = 0$, S' takes the form [6,7]

$$S[\varphi, \xi] = c \int_{\partial M=0} [dx^\mu] [dg_{mn}] \exp(-I) \quad (4)$$

where now the integration goes over sphere-like surfaces. Eq.(4)

is probably true also in the general case at least for

$\Gamma[\varphi] = S[\varphi, \xi=0]$, i.e. when ξ is put equal to zero in I .

(analogous representation is possible also for loop contributions to Γ [2]).

A "proper" choice of parametrization of Φ may of course be non-unique, different parametrizations being related through field transformations: $\varphi \rightarrow \varphi(\tilde{\varphi}, \tilde{\xi})$, $\xi \rightarrow \xi(\tilde{\varphi}, \tilde{\xi})$. Redefinitions in the massive sector $\xi \rightarrow \xi(\tilde{\xi})$ obviously do not influence the EA $\Gamma[\varphi]$ (ξ 's are simply integrated out in the path integral over Φ). A less trivial "asymmetry" to be discussed below is related to field redefinitions in the massless sector: $\varphi \rightarrow \varphi(\tilde{\varphi})$. We shall see how the corresponding ambiguity appears in the two approaches that can be used for

computation of the EA: the S-matrix approach and the (world-sheet) path integral approach.

2. S-matrix approach. The idea of the S-matrix approach [8] is to define the (tree) EA as a local gauge invariant action producing the (tree) S-matrix which coincides with the massless sector of the (tree) string theory S-matrix. To find such an action we are first to subtract contributions of massless exchanges from a string-theory S-matrix (and to expand the contributions of massive exchanges in powers of α'). Then we are to extend the resulting local vertices off the mass shell. The result of this procedure turns out to be nonunique since a separation of a massless exchange contribution in an amplitude depends on a choice of off-shell continuation of lower-point amplitudes and thus is ambiguous (we can subtract a pure massless pole but can also subtract it together with a local polynomial in momenta). Equivalently, we may start with a most general local gauge-invariant action $\Gamma_0[\varphi]$ and try to fix the unknown coefficients in it by comparing the full (unsubtracted) field-theory and string-theory (tree) S-matrices. Then an ambiguity in the resulting Γ_0 appears as a consequence of the equivalence theorem (see, e.g., [9]). The generating functional for the (tree) field theory S-matrix is (see, e.g.,

$$[10]): \hat{\Gamma}[\varphi_{in}] = W[J_{in}], \quad J_{in} = \varphi_{in} \bar{\Delta}, \quad \Gamma_0 = \frac{i}{2} \int \bar{\varphi} \Delta \varphi + \mathcal{V}(\varphi),$$

$$\hat{\Gamma}[\varphi_{in}] = \Gamma_0[\varphi_0] - \int \varphi_{in} \Delta \varphi_0 = \frac{i}{2} \int \bar{\varphi} \Delta \bar{\varphi} + \mathcal{V}(\varphi_0), \quad (5)$$

where φ_0 is the perturbative solution of $\frac{\delta \Gamma_0}{\delta \varphi} = 0$, $\varphi_0 = \varphi_{in} + \bar{\varphi} = \varphi_{in} - \Delta^{-1} \left(\frac{\delta \mathcal{V}}{\delta \varphi} \right)_{\varphi_{in}}$. Let $\varphi = \bar{\varphi} + T(\bar{\varphi})$, where T is a local gauge invariant combination of φ and its derivatives, such that it does not influence the mass shell conditions and that the correlators of T with T or φ do not have massless poles. Then

the generating functional for the S-matrix in the "transformed" theory with the action $\tilde{\Gamma}_0[\tilde{\varphi}] = \Gamma_0[\varphi]$ is the same as in the original theory (5) [9] (the reason is that $\int \varphi_{;\mu} \Delta T(\tilde{\varphi}_0)$ vanishes) ^{3/}. The consequence of this equivalence is that all the coefficients in Γ_0 which can be changed by a transformation $\varphi = \tilde{\varphi} + T(\tilde{\varphi})$ can not be determined from the comparison of the S-matrices. The corresponding ambiguity in the structure of EA does not imply any problem of principle since, e.g., the classical equations for $\Gamma_0[\varphi]$ and $\tilde{\Gamma}_0[\tilde{\varphi}]$ are equivalent up to the factor $\delta\varphi/\delta\tilde{\varphi}$ which is non-vanishing in perturbation theory.

Let us now illustrate the above discussion on the example of EA for the massless vector field in the open superstring theory. The most general gauge-invariant local lagrangian is (to the order α'^3) [11]

$$\begin{aligned} \mathcal{L} = a_0 \{ F_{\mu\nu}^2 + 2\alpha' (a_2 F_{\mu\nu} F_{\nu\rho} F_{\rho\mu} + a_2 (\partial_\rho F_{\rho\mu})^2) + (2\alpha')^2 [a_3 F_{\mu\nu} F_{\nu\lambda} F_{\lambda\mu} F_{\rho\sigma} \\ + a_4 F_{\mu\nu} F_{\nu\lambda} F_{\lambda\mu} F_{\rho\sigma} + a_5 (F_{\mu\nu} F_{\mu\nu})^2 + a_6 F_{\mu\nu} F_{\lambda\rho} F_{\rho\mu} F_{\lambda\rho} + \\ + a_7 F_{\mu\nu} \partial_\lambda F_{\mu\nu} \partial_\rho F_{\rho\lambda} + a_8 \partial_\lambda F_{\lambda\mu} \partial_\rho F_{\rho\nu} F_{\mu\nu} + a_9 (\partial_\rho \partial_\lambda F_{\lambda\mu})^2] + O(\alpha'^3) \}. \end{aligned} \quad (6)$$

Computing the 3- and 4-point massless vector scattering amplitudes in the theory (6) and comparing them with the known superstring-theory amplitudes [12,13], we have found that [11]:

$$a_1 = 0, a_3 = 2a_4 = \frac{\pi^2}{3}, a_5 = 2a_6 = -\frac{\pi^2}{12}, a_2, a_7, a_8, a_9 = \text{arbitrary.}$$

To understand why the coefficients a_2, a_7, a_8 and a_9 are not

^{3/} It is possible under the above conditions to consider $T(\varphi) = c_1 \partial\varphi + c_2 \partial^2\varphi + \dots$, starting with higher-derivative terms which are linear in φ . In this case the transformed theory formally has additional (massive) asymptotic states. Still the S-matrix computed in a perturbation theory has the massless sector equivalent to the original S-matrix. To maintain the equivalence at $\ell \geq 1$ -loop level it is necessary to account for the jacobian corresponding to the transformation $\varphi \rightarrow \tilde{\varphi}$ [9].

fixed by the S-matrix let us consider the transformation

$$A_\mu = \tilde{A}_\mu + T_\mu(\tilde{A}) \quad , \quad T_\mu(A) = 2\alpha' b_1 \partial_\rho F_{\rho\mu} + \\ + (2\alpha')^2 [b_2 F_{\rho\nu} \partial_\mu F_{\rho\nu} + b_3 \partial_\mu F_{\rho\nu} F_{\rho\nu} + b_4 \partial_\rho F_{\rho\nu} F_{\mu\nu} + \\ + b_5 F_{\mu\nu} \partial_\rho F_{\rho\nu} + b_6 \partial^\lambda \partial_\rho F_{\rho\mu}] + O(\alpha'^3), \quad (7)$$

where T_μ is the most general gauge-covariant vector constructed of A_μ . Substituting (7) into (6) we find that the only

coefficients that change are a_2, a_7, a_8, a_9 : $\tilde{a}_2 = a_2 - 4b_1$,
 $\tilde{a}_7 = a_7 - 4b_2 + 4b_3 - 3a_1 b_1$, $\tilde{a}_8 = a_8 - 4b_4 + 4b_5 + 8b_1^2 - 6a_1 b_1 - 8a_2 b_1$,
 $\tilde{a}_9 = a_9 + 4b_6 + 2b_1^2 - 2a_1 b_1$.

In view of the equivalence theorem, these coefficients can thus be made arbitrary without changing the S-matrix. The freedom of making the transformation (7) implies that the coefficients (e.g. a_2 and a_9) of all of the invariants in (6) which contain terms bilinear in A_μ ("propagator corrections") are ambiguous, i.e. can not be fixed from the S-matrix. This freedom may simplify the problem of determining EA to all orders in α' ;

it is sufficient to compute only the "unambiguous" coefficients in (6) (all ambiguous structures in EA can then be restored by the transformation (7)). Thus the Born-Infeld -type action [14, 11] (all the terms in summing the abelian limit of (6) which depend only on $F_{\mu\nu}$ and not on $\partial_\rho F_{\rho\mu}$ and which thus have unambiguous coefficients [II]) may look very like the exact EA with all ambiguous coefficients set equal to zero.

Our second example is the gravitational part of the (tree) EA in closed string theories. It has the following general

$$\Gamma_0 = a_0 \int d^D x \sqrt{G} e^{-2\phi} \left\{ \alpha' (R + 4(\partial_\mu \phi)^2) + \right. \\ \left. + \alpha'^2 (a_1 R_{\mu\nu\rho\sigma}^2 + a_2 R_{\mu\nu}^2 + a_3 R^2 + \dots) + O(\alpha'^3) \right\} \quad (8)$$

where ϕ is the dilaton and we ignore higher-order terms depending on derivatives of ϕ and $R_{\mu\nu\rho}^{\lambda}$. The coefficient a_1 is fixed by the 3-graviton string-theory amplitude to be $1/4$ in the Bose string theory [15, 16], $1/8$ in the heterotic string theory [17] and 0 in the type II superstring theory [13]. As for a_2 and a_3 , one could a priori think that they are fixed by the 4-point amplitude. However, it is easy to see that (like a_2, a_3, a_8 and a_9 in (6)) these coefficients principally cannot be determined from the S-matrix. The crucial observation is that the field transformations

$$G_{\mu\nu} = \tilde{G}_{\mu\nu} + T_{\mu\nu}(\tilde{G}, \tilde{\phi}), \quad \phi = \tilde{\phi} + T(\tilde{\phi}, \tilde{G}),$$

$$T_{\mu\nu} = \alpha'(\ell_1 R_{\mu\nu} + \ell_2 G_{\mu\nu} R) + \dots, \quad T = \alpha'\ell_3 R + \dots, \quad (9)$$

which do not disturb the perturbative S-matrix because of the equivalence theorem, change, however, the values of a_2 and a_3 : $\tilde{a}_2 = a_2 - \ell_1$, $\tilde{a}_3 = a_3 + \frac{1}{2}\ell_1 + (\frac{1}{2}D-1)\ell_2 - 2\ell_3$. Thus the question [15] whether the curvature squared terms in (9) form the Gauss-Bonnet combination ($a_1 = -4a_2$, $a_3 = a_2$) cannot be answered using only the S-matrix. The additional requirement ^{4/} of the absence of "propagator corrections" does not fix all the freedom in (8) implied by (9). It is clear that all the terms in (8) which cannot be represented as a product of two covariant second rank tensors (e.g. $R_{\lambda\mu\nu\rho} R_{\nu\rho\alpha\beta} R_{\alpha\beta\lambda\mu}$, etc.)

^{4/} The status of this requirement is not clear to us. It may be considered as a condition on an off-shell continuation of a string theory, i.e. on the structure of the kinetic operator in a string field theory [1]. However, it seems that the physical consequences of the theories related by (9) are the same and hence such a condition is no more important than a choice of a gauge. Note that the ambiguity is present also in superstring theory case (the results of refs. [18] are inconclusive).

certainly have unambiguous coefficients. The recognition of the freedom (9) may thus be important for establishing the structure of EA (8) to all orders in α' .

3. Path-integral approach. The starting observation of this approach [6,7] is that the functional (cf. (3), (4))

$$\hat{\Gamma}[\varphi] = c \int [dx^\mu] [dg_{mn}] \exp(-I), \quad I = I_0 + \varphi^i V_i \quad (10)$$

(where I describes the interaction of a (closed) string with the massless fields φ^i) with φ put on-shell, $\varphi = \varphi_{in}$ ($\square \varphi_{in} = 0$, etc.) is just the (tree) generating functional for covariant [5] on-shell string-theory scattering amplitudes. There are two possible procedures that can be used for a computation of (10). Expanding the integrand in powers of φ_{in} and computing the correlators $\langle V \dots V \rangle$ we find that the integral over the conformal factor of the metric decouples (at $D = 26$) [19] (and hence the result is independent of the Weyl gauge used). The final expression $\hat{\Gamma}_{(2)}[\varphi_{in}] \sim \sum_n \int d^D p_1 \dots d^D p_n A_n(p_1, \dots, p_n) \cdot \varphi_{in}(p_1) \dots \varphi_{in}(p_n)$ is non-local in coordinate space, i.e. contains $\square^{-1}$ -factors corresponding to the massless poles in the amplitudes. The alternative procedure is to consider I as an action of a 2-d field theory and to compute $\hat{\Gamma}$ -perturbatively in loop expansion. Expanding $\chi(z)$ near its constant value, $\chi'' = \gamma'' + \zeta''(z)$, we then get a manifestly local result:

$$\hat{\Gamma}_{(2)} = \int d^2 y \mathcal{L}(\gamma), \quad \text{where } \mathcal{L} \text{ is a power series in } \varphi_{in} \text{ and their derivatives at } \gamma.$$

The apparent puzzle of how the results of the two procedures can be reconciled is resolved by the remark that $\hat{\Gamma}_{(2)}$ is formally infinite, i.e. depends on a 2-d UV cut-off ϵ . Hence in order to be able to compare $\hat{\Gamma}_{(2)}$ with $\hat{\Gamma}_{(1)}$ we are

to use the regularized 2-d propagator (e.g. $\sim \ell_n(1/|z-z'| + \epsilon)$) also in computing $\hat{\Gamma}_{(1)}$. Then all the massless poles of the amplitudes become regularized ^{5/} and the result for $\hat{\Gamma}_{(1)}[\varphi_n, \epsilon]$ (expanded in powers of loop counting parameter α') becomes local and coincides with $\hat{\Gamma}_{(2)}[\varphi_n, \epsilon]$.

Interpreting (10) as a "partition function" of a renormalizable (φ_i are massless) 2-d theory with the action I the natural next step is to absorb the ϵ -dependence in a renormalization of the 2-d "coupling constants" φ_i and to define the renormalized partition function $\Gamma_c[\varphi_i] = \hat{\Gamma}[\varphi_n, \epsilon]$. Here $\varphi_0 = \varphi_n + \overline{\varphi}(\varphi_n, \epsilon)$ are the renormalized "couplings". It is clear from the above that the 2-d renormalization corresponds to subtraction of the massless exchanges in the amplitudes [20] (see also [21]). This suggests that $\Gamma_c[\varphi_i]$ is just the effective action evaluated on the classical solution of $\frac{\delta I_c}{\delta \varphi} = 0$ (cf. (5))^{6,7/}.

5/ The poles of the amplitudes are known to correspond to the region of the Koba-Nielsen integration where some (or all) points coincide. The 2-d regularization thus "smears" the poles (e.g. $\int_0^1 d\bar{z} z^{-s-1} (1-z)^{-t-1} \rightarrow \int_0^1 d\bar{z} (z+\epsilon)^{-s-1} (1-z-\epsilon)^{-t-1}$ and hence near $z=0$: $\int_0^1 d\bar{z} z^{-s-1} \dots \rightarrow -\ln \epsilon + \frac{1}{2} \ln^2 \epsilon + \dots$)

6/ Thus the Legendre transform is in certain sense "trivial" in a string theory [2], i.e. is done by a 2-d renormalization [20]. Note that we assumed that the "source term" in (5) $\int \varphi_n \Delta \varphi$ can be ignored in a string theory. Though this is quite plausible this point deserves further clarification.

7/ Note that φ_0 is constrained by the condition that $\varphi_n(\varphi_0) = \varphi_0 + \overline{\varphi}(\varphi_0, \epsilon)$ satisfies the on-shell restrictions. Recognition of this point solves the following apparent puzzle: the condition of the Weyl invariance of (10) is, on one hand, that $\varphi = \varphi_n$ in I and on the other hand, that the " β -functions" vanish, $\beta(\varphi) = 0$; however $\beta(\varphi_n) \neq 0$. The point is that the arguments of β 's are the renormalized couplings and that $\beta(\varphi_0) = 0$ and $\beta \varphi_n = 0$ are equivalent ($\beta \varphi_n = 0$ is simply the vanishing of " β -functions" expressed in terms of the bare couplings).

To find an off-shell value of EA we are to relax the condition that $\mathcal{Y} = \mathcal{Y}_{i,n}$ in (10) and to take $\mathcal{Y}$ in I to be an arbitrary "bare" coupling $\mathcal{Y}_B$. The (tree) EA is then the renormalized value of the partition function (10), $\Gamma_0[\mathcal{Y}] \equiv \hat{\Gamma}[\mathcal{Y}_B, \varepsilon]$ (here $\mathcal{Y}$ is the renormalized coupling and we do not explicitly indicate the dependence of Γ_0 on renormalization mass μ). It is assumed that a Weyl gauge (WG) is used in computing (10). The off-shell ($\mathcal{Y}_B \neq \mathcal{Y}_{i,n}$) result for Γ_0 is in general WG-dependent (compare with a gauge dependence of an off-shell EA in gauge theories). The crucial consistency requirement (which as we shall see is likely to be satisfied) is that the dependence on WG should drop out of the effective equations

$$\delta \Gamma_0 / \delta \mathcal{Y} = 0.$$

After the renormalization of $\hat{\Gamma}$ is done there remains a freedom of finite renormalizations, $\mathcal{Y} \rightarrow \mathcal{Y}(\tilde{\mathcal{Y}})$ (corresponding to changes of a renormalization scheme (RS)), $\Gamma_0[\mathcal{Y}] = \tilde{\Gamma}_0[\tilde{\mathcal{Y}}]$. Though the exact result for EA is RS-independent the coefficients in the expansion of Γ_0 in powers of $\partial \mathcal{Y}$ (or α') are RS-dependent. This of course is the well-known RS-dependence problem of a perturbation theory in a renormalizable QFT [22] (a peculiarity of the string theory is that here the fields are themselves coupling constants of a 2-d QFT and hence already the classical action has ambiguous "perturbative" expansion). The field redefinition ambiguity of the EA observed above in the S-matrix approach is recognized in the path-integral approach as the RS- and WG-dependence of derivative expansion of EA ^{8/}. In particular, the coefficient α_2 in (8) is RS-de-

^{8/} A change of WG (as well as a change of RS) can be represented as a particular field transformation $\mathcal{Y} \rightarrow \mathcal{Y}(\tilde{\mathcal{Y}})$. In fact, a change of WG is a change of a conformal factor ρ of the metric, $g_{mn} = \exp(2\rho) \tilde{g}_{mn}$, but $\delta \Gamma_0 / \delta \rho \sim \int \delta \Gamma_0 / \delta \rho$. This suggests that the effective equations are RS- and WG-invariant.

pendent, while α_3 is WG-dependent [11].

Now let us discuss the connection of the path integral approach with the " β -function" approach of refs. [20, 23, 24].

It is plausible that the (tree level) effective equations

$\frac{\delta \Gamma_0}{\delta \varphi} = 0$ are equivalent to the condition of Weyl invariance of the 2-d theory, i.e. to the vanishing of the "Weyl anomaly" coefficients: $\bar{\beta}^i = 0$. If the 2-d theory is Weyl-invariant then [25, 23] the massless tadpoles vanish $\langle V_i \rangle_\varphi = 0$ and hence (see (10)) $\delta \Gamma_0 / \delta \varphi^i = 0$ are satisfied. Though

Here the important clarification is needed. Let $I = \frac{1}{4\pi\alpha'} \int d^2x \sqrt{g} [G_{\mu\nu}(x) \partial_\mu x^\mu \partial_\nu x^\nu + \alpha' R^{(2)} \phi(x)]$. Consider $\exp(-W[g]) = \int [dx] \exp(-I)$ and define the renormalized energy-momentum $T_{\mu\nu}^m = \frac{2}{\sqrt{g}} g^{\mu\nu} \frac{\delta W}{\delta g^{\mu\nu}}$. Then it is possible to prove that [11] (cf. [23]): $T_{\mu\nu}^m = \frac{1}{4\pi\alpha'} \langle \bar{\beta}_{\mu\nu}(x) \partial_\mu x^\mu \partial_\nu x^\nu \rangle + \frac{1}{4\pi} R^{(2)} \langle \beta_\phi(x) \rangle$, where $\bar{\beta}_{\mu\nu} = \beta_{\mu\nu} + 2\alpha' \partial_\mu \partial_\nu \phi$, $\bar{\beta}_\phi = \beta_\phi + \alpha' (\partial_\mu \phi)^2$, $\beta_{\mu\nu} = \mu \frac{d}{d\mu} G_{\mu\nu}$, $\beta_\phi = \mu \frac{d}{d\mu} \phi$, and $\langle \dots \rangle$ is the averaging with the measure $\sim e^{-I}$, $\langle 1 \rangle = 1$. β 's (and not $\bar{\beta}$'s) are the ordinary RG functions directly related to the UV infinities. The condition of Weyl invariance, $\bar{\beta}_{\mu\nu} = 0$, thus is not ^{sufficient} equivalent to the UV finiteness condition, $\beta_{\mu\nu} = 0$, $\beta_\phi = 0$ unless $\phi = \text{const.}$ ^{Let} a string theory G -model is finite ~~only~~ for a solution of $\bar{\beta}$'s = 0 for which $\phi \neq \text{const.}$ Computing the correlators in $T_{\mu\nu}^m$ by expanding near a stationary point $\bar{x}$ of I we get: $T_{\mu\nu}^m = \frac{1}{4\pi\alpha'} \beta_{\mu\nu}^{(1)}(\bar{x}) \partial \bar{x}^\mu \partial \bar{x}^\nu + \frac{1}{4\pi} R^{(2)} \beta_\phi^{(1)}(\bar{x}) + \beta_{\mu\nu}^{(2)}(\bar{x}) \partial \bar{x}^\mu \partial \bar{x}^\nu \Delta^{-1} R^{(2)} + R^{(2)} \Delta^{-1} R^{(2)} \beta_\phi^{(2)}$ where $\beta^{(n)}$ are functions of $\bar{\beta}$'s, $G_{\mu\nu}$, ϕ and their derivatives (non-local terms in $T_{\mu\nu}^m$ were missed in [6, 23]). The " β -functions" that appeared in ref. [23] are $\beta_{\mu\nu}^{(1)}$ and $\beta_\phi^{(1)}$ (note that $\beta_\phi^{(1)} = \bar{\beta}_\phi - \frac{1}{4} \bar{\beta}_{\mu\nu} G^{\mu\nu} + \dots$ [11]). It is probably true, however, that $\bar{\beta}_{\mu\nu} = 0$, $\bar{\beta}_\phi = 0$ are equivalent to $\beta_{\mu\nu}^{(1)} = 0$, $\beta_\phi^{(1)} = 0$ and hence the conclusions of ref. [23] are justified.

*/ see p. 36

we do not know the proof of the converse statement it is likely to be true, given that the RS-independence of Γ_0 implies that

$$\frac{\partial \Gamma_0}{\partial \ell_{\mu}} + \int d^3x \beta^i(\varphi(x)) \frac{\delta \Gamma_0}{\delta \varphi^i(x)} = 0$$

and hence that Γ_0 is scale-invariant if $\delta \Gamma_0 / \delta \varphi = 0$ (another intuitive argument (ignoring

the complication of $\phi \neq \text{const}$) is that since the "Legendre transform" $\tilde{\Gamma} \rightarrow \Gamma$ is done by the 2-d renormalization governed by β 's, β 's should vanish on the effective mass shell. A naive form of the equivalence statement (cf. [20]) $\tilde{\beta}^i \sim \delta \Gamma_0 / \delta \varphi^i$ can be objected as "scheme" (RS-and WG-) dependent ($\tilde{\beta}^i$ is a contravariant and $\delta \Gamma_0 / \delta \varphi^i$ is a covariant vector under $\varphi \rightarrow \varphi(\tilde{\varphi})$). Moreover, no "scheme" (or choice of φ^i) exists in which this relation can be satisfied to the order α'^2 in the closed string theory [11]. It is possible, however, to satisfy the following relation [11] 10/ : $\frac{\delta \Gamma_0}{\delta \varphi^i} = \mathcal{X}_{ij} \tilde{\beta}^j$, where $\mathcal{X}_{ij}(\varphi)$ (transforming as a tensor) is a non-trivial power series in derivatives of φ and α' . If $\mathcal{X}_{ij}$ is non-degenerate (this is true in the perturbation theory in α') $\frac{\delta \Gamma_0}{\delta \varphi^i} = 0$ are equivalent to $\tilde{\beta}^i = 0$. However, this equivalence does not hold to each particular order in α' because of the α' -dependence of $\mathcal{X}_{ij}$. The above relation implies that the WG-dependence of Γ_0 factorizes on-shell (: $\tilde{\beta}$'s are certainly WG-independent) and hence the WG-fixing procedure assumed in computing Γ_0 is justified.

So far we discussed the tree approximation. Let us now remark on loop corrections. It is obvious that the equations

10/ Analogous relation was recently discussed by A.B. Zamolodchikov in the context of conformal QFT and by A.M. Polyakov in connection with string theory. It is possible to check that given such a relation the "anomalous dimension" matrix $\gamma^i_j \sim \delta \beta^i / \delta \varphi^j$ is symmetric only on the mass shell, $\delta \Gamma_0 / \delta \varphi^i = 0$; thus resolving the problem raised in [20].

$\bar{\beta}^i = 0$ which are insensitive to a world sheet topology cannot represent loop-corrected equations of motion. Since the generating functional for loop-corrected string-theory S-matrix is given by the sum of contributions (10), one for each particular topology, the analogous expression is true for the off-shell EA [2,6]. Computing the partition function (10), e.g., on a torus, we find two types of infinities: local UV infinities (due to "trees" with massless exchanges) and modular infinities (due to a tree exchange in which the dilaton goes to the vacuum [26]). Introducing on a world sheet a geodesic distance cut-off ($|l_g| \geq \epsilon$) we automatically regularize both kinds of infinities (modular infinities correspond to some of geodesic lengths shrinking to zero, see, e.g., [27]). Then the full renormalization $\Psi = \Psi(\varphi, \epsilon)$ eliminating all the ϵ -dependence in $\hat{\Gamma}$ will produce finite EA with contributions of all kinds of massless exchanges subtracted out (as required by the qualitative definition of EA given in sect. 1). In analogy with the "tree" case it is natural to anticipate that $\frac{\delta \Gamma}{\delta \varphi^i} = 0$ are equivalent to the vanishing of some generalized " $\hat{\beta}$ -functions", $\hat{\beta}^i = 0$, i.e. to the vanishing (at least for $\phi = \text{const}$, see footnote 9) of the sum of local UV and modular infinities (an idea of a cancellation between local and modular infinities was suggested in [20] and implemented on a particular example in [28]). ^{11/} The necessity of renormalizations

^{11/} In the boson string theory the tree-level equations $\bar{\beta}^i = 0$ certainly do not imply $\hat{\beta}^i = 0$ (there are modular infinities even for a flat-spac. vacuum). It is very likely that in the superstring theories the vanishing of $\bar{\beta}$ -functions (i.e. the classical Weyl invariance) implies the absence of the modular infinities [29], i.e. $\bar{\beta}^i = 0$ imply $\hat{\beta}^i = 0$. The converse implication, however, is not true. If $\bar{\beta}^i \neq 0$ there are modular (as well as local UV) infinities which may compensate for a non-trivial solution of $\hat{\beta}^i = 0$. This suggests that $\hat{\beta}^i = \bar{\beta}^i + t_j^i$; the vanishing of t_j^i corresponds to non-trivial solutions.

in loop corrections implies that the corresponding contributions to EA will depend on a choice of a renormalization scheme. This ambiguity is again a subclass of the field redefinition ambiguity of the S-matrix approach.

In conclusion, we note that understanding of the massless field redefinition ambiguity (see (7), (9)) may be important in a search for a manifestly gauge-covariant string field theory.

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Appendix. Erratum to the paper "Vector field effective action in the open superstring theory" by A.A.Tseytlin, Lebedev Institute preprint N 6 (1986)

Sect. 2 should be revised along the lines of the present paper. New sections 7 and 8 are given below.

7. Non-abelian effective action deduced from the on-shell tree amplitudes

Our aim here is to find the first several terms in the $\alpha' \rightarrow 0$ expansion of the non-abelian vector field effective action and to compare their abelian limits with the lowest order terms (29), (30) in the actions (25), (27). Our starting point will be the expressions [9,10] for the 3- and 4-point on-mass-shell scattering amplitudes discussed in the previous section. The general strategy of the S-matrix approach was already sketched in sect. 2. We first write down the most general gauge-invariant ^{13/}local lagrangian containing all possible independent invariants of increasing dimensions

$$\begin{aligned} \mathcal{L} = a_0 \text{tr} \{ & F_{\mu\nu}^2 + 2\alpha' (a_1 F_{\mu\lambda} F_{\lambda\nu} F_{\nu\mu} + a_2 \partial_\lambda F_{\lambda\mu} \partial_\rho F_{\rho\mu}) + \\ & + (2\alpha')^2 (a_3 F_{\mu\lambda} F_{\nu\lambda} F_{\mu\rho} F_{\nu\rho} + a_4 F_{\mu\lambda} F_{\nu\lambda} F_{\nu\rho} F_{\mu\rho} + \\ & + a_5 F_{\mu\nu} F_{\mu\nu} F_{\lambda\rho} F_{\lambda\rho} + a_6 F_{\mu\nu} F_{\lambda\rho} F_{\mu\nu} F_{\lambda\rho} + \\ & + a_7 F_{\mu\nu} \partial_\lambda F_{\mu\nu} \partial_\rho F_{\rho\lambda} + a_8 \partial_\lambda F_{\lambda\mu} \partial_\rho F_{\rho\nu} F_{\mu\nu} + (56) \\ & + a_9 \partial_\rho \partial_\lambda F_{\lambda\mu} \partial_\rho \partial_\sigma F_{\sigma\mu}) + O(\alpha'^3) \} \end{aligned}$$

^{13/} gauge invariance of the amplitudes and hence of the effective lagrangian is guaranteed by the gauge invariance of the generating functional (4), (9).

where $a_0 = -\frac{1}{8g^2} (2\pi\alpha')^2$,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \quad \partial_\mu = \partial_\mu + [A_\mu,]$$

and A_μ is an antisymmetric or an antihermitean matrix in the fundamental representation of $SO(N)$ or $SU(N)$. To prove that (56) contains all independent invariants to the order α'^3 one is to use the Bianchi identity implying that

$$\begin{aligned} \partial^2 F_{\mu\nu} &= \partial_\mu N_\nu - \partial_\nu N_\mu + 2[F_{\mu\lambda}, F_{\nu\lambda}], \quad N_\mu \equiv \partial_\rho F_{\rho\mu}, \\ \text{tr}(F_{\mu\nu} F_{\nu\rho} \partial^2 F_{\rho\mu}) &= \text{tr} \{ 2 F_{\mu\lambda} F_{\nu\lambda} [F_{\mu\rho}, F_{\nu\rho}] - \\ &\quad - 2 F_{\mu\nu} N_\mu N_\nu - F_{\lambda\rho} \partial_\mu F_{\lambda\rho} N_\mu \} + \text{total derivative}, \end{aligned} \quad (57)$$

$$\begin{aligned} \text{tr}(\partial^2 F_{\mu\nu} \partial^2 F_{\mu\nu}) &= \text{tr} \{ 2 (\partial_\mu N_\nu)^2 - 12 F_{\mu\nu} N_\mu N_\nu - \\ &\quad - 8 F_{\lambda\rho} \partial_\mu F_{\lambda\rho} N_\mu + 8 F_{\mu\lambda} F_{\nu\lambda} [F_{\mu\rho}, F_{\nu\rho}] \} + \text{total derivative}. \end{aligned}$$

Then we compute the tree amplitudes in the theory (56) (assuming the "in" - states to satisfy $\square A_\mu^{\text{in}} = 0$, $\partial_\mu A_\mu^{\text{in}} = 0$) and finally compare them with the string-theory amplitudes.

It serves much labor to understand from the very beginning which coefficients in (56) cannot be fixed from the comparison of the S-matrices. The point is that because of the "equivalence theorem" (see sect. 2) it is possible to change the field variable A_μ in (56) without influencing the S-matrix. To preserve the gauge invariance and locality of the lagrangian and also the mass-shell condition we are to make the following transformation

$$\begin{aligned} A_\mu &= \tilde{A}_\mu + T_\mu(\tilde{A}) \\ T_\mu(A) &= 2\alpha' b_1 \partial_\rho F_{\rho\mu} + (2\alpha')^2 [b_2 F_{\rho\nu} \partial_\mu F_{\rho\nu} + \\ &\quad + b_3 (\partial_\mu F_{\rho\nu}) F_{\rho\nu} + b_4 (\partial_\rho F_{\rho\nu}) F_{\mu\nu} + \\ &\quad + b_5 F_{\mu\nu} \partial_\rho F_{\rho\nu} + b_6 \partial^2 \partial_\rho F_{\rho\mu}] + O(\alpha'^3) \end{aligned} \quad (58)$$

The conditions of the equivalence theorem (that the correlators $\langle T \dots T \rangle$ and $\langle T \dots A \rangle$ must not have single-particle massless poles [38], ϵk) are satisfied for (58) (within the perturbation theory in α'). Substituting (58) into (56) we find that (56) retains its form but the coefficients a_2, a_7, a_8 and a_9 change :

$$\tilde{a}_2 = a_2 - 4b_1, \quad \tilde{a}_7 = a_7 - 4b_2 + 4b_3 - 3a_1 b_1, \quad \tilde{a}_8 = a_8 - 4b_4 + 4b_5 + 8b_1^2 - 6a_1 b_1 - 8a_2 b_1, \quad \tilde{a}_9 = a_9 + 4b_6 + 2b_1^2 - 2a_2 b_1.$$

Hence these coefficients can be given arbitrary values without changing the S-matrix (the "propagator corrections" produced by the a_2 - and a_9 -terms can thus be also made arbitrary, cf. [12]).

Now let us establish the values of a_1, a_3, a_4, a_5, a_6 . Computing the 3-point amplitude and comparing it with (48), (52) it is easy to check that the linear in momentum term in (52) corresponds [2] to the standard Yang-Mills 3-vertex while the cubic in momentum term corresponds [3] to the F^3 term in (56). We find that $a_1 = \frac{4}{3}$ in the Bose string case ^{14/} and $a_1 = 0$ in the superstring case.

To fix $a_3, \dots, a_6$ it is sufficient to consider only the $\zeta \cdot \zeta \cdot \zeta \cdot \zeta$ - terms in the 4-point amplitudes. Using that $\Gamma(1+\epsilon) = 1 - \gamma\epsilon + \frac{1}{2}(\gamma^2 + \zeta(2))\epsilon^2 + O(\epsilon^3)$, where $\zeta(2) = \frac{\pi^2}{6}$ and γ is the Euler constant, we get the following expansion of the function $\Gamma\left(\frac{\zeta}{z}\right)$ in (54)

$$\Gamma\left(\frac{\zeta}{z}\right) = \frac{1}{\zeta z} - \frac{\pi^2}{6} + O(\alpha') \quad (59)$$

^{14/} The presence of the F^3 -term in the Bose-string effective lagrangian was first pointed out in [3] (the coefficient of this term given in [3] differs from ours). This reference contains also a remarkable provision that the full effective lagrangian may be similar to the Born-Infeld lagrangian.

The expansion in powers of momenta of the $\varphi\varphi\varphi\varphi$ -part of the bosonic amplitude (53) then takes the form

$$A_{4B} = (2\alpha')^2 [B_0 + B_1 + B_2 + B_2' + O(\alpha'^3)] , \quad (69)$$

$$B_0 = \varphi_{1324} + \frac{2}{x} \varphi_{1234} + \frac{2}{y} \varphi_{1423} ,$$

$$B_1 = -2 \left(\varphi_{1324} + \varphi_{1234} + \varphi_{1423} \right) ,$$

$$B_2 = -\frac{\pi^2}{6} \left(xy \varphi_{1324} + yz \varphi_{1234} + xz \varphi_{1423} \right) ,$$

$$B_2' = z^2 \varphi_{1324} + xz \varphi_{1234} + yz \varphi_{1423} , \quad \varphi_{pqr s} \equiv \varphi_p \cdot \varphi_q \cdot \varphi_r \cdot \varphi_s ,$$

$x = \alpha'^2 s, \text{ etc.}$

As is clear from (55), only the terms B_0 and B_1 are present in (60) in the superstring case (the terms B_2 and B_2' originate from the expansion of the tachyonic pole factors $1/(1+x)$, etc., in (53)). B_0 is recognized as the standard Yang-Mills

4-point amplitude [2]: the 4-vertex $\bar{B}_0 = \varphi_{1324} - \frac{1}{2} \varphi_{1234} - \frac{1}{2} \varphi_{1423}$ plus the contribution of the exchange with the two 3-vertices from the F^2 -term in (56). The B_1 -term is reproduced by the sum of the contact 4-vertex coming from the F^3 -term in (56) and the exchange diagrams with one F^2 -3-vertex and one F^3 -3-vertex (this provides an independent check of $a_1 = \frac{4}{3}$). The

B_2 -term in (60) is reproduced by the contact A^4 -terms in (56) if $a_3 = 2a_4 = \frac{\pi^2}{3}$, $a_5 = 2a_6 = -\frac{\pi^2}{12}$. 15/.

The B_2' -term in (60) is reproduced by the sum of the contact A^4 -piece of the a_5 - and a_6 -terms in (56) and the contribution of the exchange diagrams with both 3-vertices corresponding to the F^3 -term if $a_5 = -a_6 = -\frac{1}{2}$. Thus in the Bose string theory case

15/ We substitute $A_n^{in} = \lambda \varphi_n e^{ikx}$ in the a_3, a_4, a_5, a_6 -terms in (56), separate $t_2(\lambda_1 \lambda_2 \lambda_3 \lambda_4)$, make all (four) cyclic permutations in the remaining $\varphi\varphi\varphi\varphi$ -factor and then compare the result with (60) (multiplied by $-\pi$ because of the normalization we have chosen in (56)).

$$a_1 = \frac{4}{3}, a_3 = 2a_4 = \frac{\pi^2}{3}, a_5 = -\frac{\pi^2}{12} - \frac{1}{2}, a_6 = -\frac{\pi^2}{24} + \frac{1}{2}, (61)$$

$$a_2, a_7, a_8, a_9 = \text{arbitrary}$$

Since B_2' is absent in the superstring theory amplitude the final result in this case is

$$a_1 = 0, a_3 = 2a_4 = \frac{\pi^2}{3}, a_5 = 2a_6 = -\frac{\pi^2}{12},$$

$$a_2, a_7, a_8, a_9 = \text{arbitrary} \quad (62)$$

The reason why the a_2, a_7, a_8 and a_9 -terms in (56) do not influence the 4-point amplitude is that the contribution of their contact A'' -parts is cancelled against the contribution of the exchange diagrams with one 3-vertex being the Yang-Mills F^2 -one and the other being the 3-vertex from the corresponding term. The computation of exchange contributions is most easily done using the S-matrix generating functional technique (see eq. (2)). If $S = \int (\frac{1}{2} \varphi \Delta \varphi + \frac{1}{3} \varphi^3 + g \varphi^4) d^D x$ then

$$\hat{\Gamma}_{tree}[\varphi_{in}] = \int d^D x \left[\frac{\lambda}{3} \varphi_{in}^3 + g \varphi_{in}^4 - \frac{1}{2} \lambda^2 \varphi_{in}^2 \Delta^{-1} \varphi_{in}^2 + O(\varphi_{in}^5) \right] \quad (63)$$

As is clear from (2), (63) to find the total contribution of the "mixed" exchange and the "contact" terms to the order we consider we are simply to substitute the classical solution of $\partial_\mu F_{\mu\nu} = 0$ in the corresponding terms in (56). The "ambiguous" (a_2^- , a_7^- , a_8^- and a_9^-) terms vanish under such a substitution. To find the contribution of, e.g., the exchanges with two F^2 -3-vertices we start with $\mathcal{L} = a_0 \text{tr}(F^2 + 2\lambda' a_1 F^3)$ and solve the classical equation: $\partial_\mu F_{\mu\nu} = \frac{3}{2} \lambda' a_1 \partial_\rho [F_{\nu\lambda}, F_{\lambda\rho}]$. Choosing the gauge $\partial_\mu \bar{A}_\mu = 0$ ($A_0 = A_{in} + \bar{A}$) we have, to the

relevant order : $\bar{A}_\nu = \frac{3}{4} \alpha' a_1 \square^{-1} [\partial_\nu f_{\rho\lambda}, f_{\rho\lambda}] + \dots$

$f_{\rho\lambda} = \partial_\rho A_\lambda^{in} - \partial_\lambda A_\rho^{in}$. Substituting A_μ in the F^3 -term (taken with $\frac{1}{2}$ -coefficient, cf. (2), (63)) we

find that the exchange part in B_2' in (50) is

$\frac{1}{2} x(z-y) \zeta_{1234} + \frac{1}{2} y(z-x) \zeta_{1423}$. The $F^2 - F^3$ -exchange part in B_1 is found analogously and is $(x-z) \zeta_{1423} + (y-z) \zeta_{1234}$.

The simplest expression for the effective lagrangian is

found if we put all arbitrary coefficients equal to zero (i.e. if we choose a "minimal scheme")

$$\mathcal{L} = a_0 \text{tr} \left\{ F_{\mu\nu}^2 + \frac{1}{3} (2\pi\alpha')^2 (F_{\mu\nu} F_{\rho\nu} F_{\mu\lambda} F_{\rho\lambda} + \frac{1}{2} F_{\mu\nu} F_{\rho\nu} F_{\rho\lambda} F_{\mu\lambda} - \frac{1}{4} F_{\mu\nu} F_{\mu\nu} F_{\rho\lambda} F_{\rho\lambda} - \frac{1}{8} F_{\mu\nu} F_{\rho\lambda} F_{\mu\lambda} F_{\rho\nu}) + O(\alpha'^3) \right\} \quad (64)$$

The abelian $SO(2)$ -limits of the Bose-string and super-string effective lagrangians coincide (in the "minimal scheme" and to the order α'^3)

$$\mathcal{L} = -2a_0 \left\{ F^2 - \frac{1}{2} (2\pi\alpha')^2 \left[F^4 - \frac{1}{4} (F^2)^2 \right] + O(\alpha'^3) \right\} \quad (65)$$

Eq. (65) is in perfect agreement with the leading terms in the expansion (29) of the Born-Infeld lagrangian (25). In order for (27), (30) to agree with (65) we are to choose the coefficient of the $F_{\mu\nu}^2$ -term in (27) (which as we noted in Sect. 4 cannot be fixed by a computation in the abelian case) to be $1/8$ (we are also to take the normalization constant in (26) equal to $2/3$).

8. Summary and remarks on effective lagrangians in closed string theories.

In this paper we have found the exact expression for the abelian constant field strength limit of the tree-level effective lagrangian for the massless vector field in the open

superstring theory: $\mathcal{L}_S = \frac{1}{4} \tilde{F}_{\mu\nu}^2 + \frac{2}{3} (\mathcal{L}_S - 1)$, where $\mathcal{L}_S$ is defined in (27), (30). The corresponding expression in the Bose string theory is $\mathcal{L}_B = \mathcal{L}_B - 1$, where $\mathcal{L}_B$ (25), (29) is the D=26- analog of the Born-Infeld lagrangian. We discussed one-loop generalizations of these results. It was demonstrated that if the effective action is defined to correspond to processes in which the vector field couples to only one component of the boundary of a world sheet it is "renormalizable" in the Bose-string case (30), (37) and finite in the SO(32)-superstring case (44)-(47). Starting from the known on-shell scattering amplitudes we also determined a number of leading terms in the α' -expansion of the non-abelian effective lagrangians (56), (61), (62). We have found that there is a principal ambiguity in the structure of these lagrangians which reflects a possibility to redefine the field variable (58) without changing the S-matrix. In particular, the coefficients of the terms which contain "propagator corrections" can not be fixed from the S-matrix. We noted that the freedom of field redefinitions (which is related to the freedom of choosing a renormalization scheme in the 2-dimensional theory or a scheme of subtraction^{of} massless poles in the string-theory amplitudes) makes possible to use the simplest "scheme", for example the "minimal scheme" in which all ambiguous coefficients are put equal to zero and thus the "propagator corrections" are absent. The effective lagrangians computed in such a scheme may directly coincide (in the abelian limit) with the above Born-Infeld-type lagrangians (as we have checked (64), (65), this true to the order α'^3).

Let us now discuss the structure of the (tree) gravitational effective actions in closed string theories. We at

present are not aware of an approximation (analogous to $F_{\mu\nu} = \text{const}$ in the open string case) in which an exact (in α') expression for EA can be obtained in a closed string theory.

That is why we shall analyze only the leading terms in the

α' - expansion of EA [6,8]

$$\Gamma = a_0 \int d^D x \sqrt{G} e^{-2\phi} \left\{ \alpha' [R + 4(\partial_\mu \phi)^2] + \right. \\ \left. + \alpha'^2 (a_1 R_{\lambda\mu\nu\rho}^2 + a_2 R_{\mu\nu}^2 + a_3 R^2 + \dots) + \alpha'^3 (a_4 R_{\lambda\mu\nu\rho}^3 + \dots) \right. \\ \left. + O(\alpha'^4) \right\} \quad (66)$$

Here ϕ is the dilaton field and dots stand for terms involving derivatives of ϕ and $R_{\lambda\mu\nu\rho}^2$ and for other curvature invariants. It is important to account for the dilaton dependence in discussing the tree gravitational EA because of the mixing of ϕ with the metric (the terms which depend on the anti-symmetric tensor can be consistently ignored)^{16/}. Let us first try to determine the unknown coefficients in (66) by comparing the gravitational sectors of the field-theory and string-theory S-matrices. Within the S-matrix approach it is more standard to redefine the metric $G_{\mu\nu} \rightarrow G_{\mu\nu} \exp(4\phi/D-2)$, transforming (66) to the form [6,8]

$$\Gamma = a_0 \int d^D x \sqrt{G} \left\{ \alpha' [R + \gamma(\partial_\mu \phi)^2] + \right. \\ \left. + \alpha'^2 e^{\gamma\phi} (a_1 R_{\lambda\mu\nu\rho}^2 + a_2 R_{\mu\nu}^2 + a_3 R^2 + \dots) + \right. \\ \left. + \alpha'^3 e^{2\gamma\phi} (a_4 R_{\lambda\mu\nu\rho}^3 + \dots) + O(\alpha'^4) \right\}, \quad \gamma = -\frac{4}{D-2} \quad (67)$$

To see which of the coefficients in (67) cannot be fixed from the S-matrix let us consider the local transformations of the

^{16/} Dilaton exchanges contribute to the gravitational sector of the S-matrix because of the non-vanishing dilaton-graviton transition vertex. In the G -model approach (see below) there is a mixing of renormalisations of ϕ and $G_{\mu\nu}$.

fields preserving the structure of (67) and satisfying the conditions of the equivalence theorem (see sects. 2 and 7)

$$G_{\mu\nu} = \tilde{G}_{\mu\nu} + T_{\mu\nu}(\tilde{G}, \tilde{\phi}), \quad \phi = \tilde{\phi} + T(\tilde{G}, \tilde{\phi}),$$

$$T_{\mu\nu}(\tilde{G}, \tilde{\phi}) = \alpha' e^{\delta\phi} (\ell_1 R_{\mu\nu} + \ell_2 G_{\mu\nu} R) + \dots, \quad T = \alpha' e^{\delta\phi} \ell_3 R + \dots \quad (68)$$

($e^{\delta\phi}$ -factors are absent if we use the unrescaled $G_{\mu\nu}$ corresponding to (66)). Substituting (68) in (67) we find the transformation of the coefficients ($\delta G_{\mu\nu} = T_{\mu\nu} \rightarrow \delta R = \partial_\mu \partial_\nu T_{\mu\nu} - \partial^2 T_{\mu\mu} - R_{\mu\nu} T_{\mu\nu} + O(T^2)$):

$$\tilde{a}_1 = a_1, \quad \tilde{a}_2 = a_2 - \ell_1, \quad \tilde{a}_3 = a_3 + \frac{1}{2} \ell_1 + \left(\frac{1}{2} D - 1\right) \ell_2 - 2 \ell_3.$$

Thus a_2 and a_3 can be made arbitrary without influencing the S-matrix, i.e. their values are "scheme-dependent". Thus the S-matrix principally does not give the answer to the question [12] whether the curvature squared terms in (67) form the Gauss-Bonnet combination. The terms in (67) with coefficients that can be unambiguously fixed from the S-matrix are the invariants which are not influenced by the transformation (68), for example, the invariants which cannot be represented as a product of two scalars or a product of two second-rank tensors (e.g., $\text{tr}(R_{\mu\nu} R_{\nu\rho} \dots R_{\lambda\mu})$, where $R_{\mu\nu} = (R^\alpha_{\beta\mu\nu})$). To fix all ambiguous coefficients in EA one may try impose some additional off-shell requirements like the absence of "propagator corrections" (we shall call the corresponding choice of coefficients a "minimal scheme").

The 3-point graviton scattering amplitude in the closed Bose string theory is a "square" of the 3-point vector scattering amplitude (52), symbolically: $A_{3B} \sim (k\varphi^3 + 2\alpha' k^3 \varphi^3) \cdot (k\bar{\varphi}^3 + 2\alpha' k^3 \bar{\varphi}^3)$. It predicts the presence of the Einstein [3,5] as well as the $R^2_{\lambda\mu\nu\rho}$ [12, 42] and $R^3_{\lambda\mu\nu\rho}$ -terms

in (7), i.e. fixes the coefficients a_2 and a_4 , $a_1^{(R)} = \frac{1}{4}$.
 In the "minimal scheme" $a_2 = -4a_1$, $a_3 = a_1$. The 3-graviton vertex in the heterotic string theory [43] is the product of the Bose-string 3-vertex and the superstring 3-vertex,
 $A_{3H} \sim (k\varphi^3 + 2\alpha' k^3 \varphi^3) k \bar{\varphi}^3$. Hence the coefficient a_1 here is one-half of the a_1 in the Bose string theory: $a_1^{(H)} = \frac{1}{8} [43]$;
 $a_4^{(H)} = 0$ since there is no α'^2 -term in the amplitude.
 The coefficients a_2 and a_3 are again arbitrary but additional requirement of supersymmetry of EA may fix them to the "Gauss-Bonnet" values, $a_2 = -4a_1$, $a_3 = a_1$ (cf. [44]).
 In type II superstring theory [9,10] $A_{3S} \sim k \varphi^3 k \bar{\varphi}^3$ and hence $a_1^{(S)} = 0$, $a_4^{(S)} = 0$. The "minimal" choice for a_2 and a_3 ($a_2 = a_3 = 0$) here again may be dictated by an off-shell supersymmetry.

Let us now discuss the structure of EA within the path integral approach [6-8]. The action $I_0 + V \cdot \varphi$ in (3) in the closed Bose string theory case reads

$$I = \frac{1}{4\pi\alpha'} \int d^2x \sqrt{g} \left[g^{mn} x''^m x'^n G_{\mu\nu}(x) + \alpha' R^{(2)} \phi(x) \right] \quad (69)$$

($R^{(2)}$ is the scalar curvature of g_{mn}). Up to the antisymmetric tensor term (which we ignore) this is the most general scale-invariant action of a renormalizable bosonic theory defined on a curved two-dimensional space. As discussed in sect. 2, the (tree) EA is a renormalized partition function

17/ The requirement of $D = 10$ supersymmetry ^{does not} resolve the ambiguity in EA we have found ^{(note} in this connection a discrepancy between refs. [44] and [45]).

of the theory (69) defined on a 2-sphere with the standard metric

$$\hat{\Gamma}[\varphi_B, \varepsilon] = \int [dx^\mu] e^{-I[\varphi_B]}, \quad \varphi^i = \{G_{\mu\nu}, \phi\} \quad (70)$$

(ε is an UV cut-off). Renormalization corresponds to subtraction of the massless graviton and dilaton exchanges and thus produces a manifestly local expression for $\hat{\Gamma}$ (66). The exact result for the renormalized partition function is renormalization-scheme (RS) - independent, $\hat{\Gamma}[\varphi_R] = \tilde{\Gamma}[\tilde{\varphi}_R]$. However, the coefficients in the expansion of $\hat{\Gamma}$ in powers of φ or in powers of the loop counting parameter α' may, in general, be RS-dependent. This is the well-known ambiguity of the perturbation theory in a renormalizable field theory (see, e.g., [46]). A peculiarity of the string theory is that here the fields themselves appear as coupling constants of a 2-d quantum field theory and thus already the classical EA has ambiguous "perturbative" expansion.

In general, $\hat{\Gamma} = \exp(-W[\varphi_R, g_{mn}, \mu])$ where φ_R^i are renormalized coupling constants (we omit the subscript "R" in what follows), g_{mn} is a background metric appropriate for a given world-sheet topology and μ is a renormalization mass parameter. In the tree approximation, $\hat{\Gamma} = \hat{\Gamma}[\varphi, \mu a]$, where

a is a radius of a 2-sphere (φ^i are dimensionless). The simplest renormalization group equation is the statement of independence of $\hat{\Gamma}$ on the parameter μ of RS

$$\frac{d\hat{\Gamma}}{dt} = \frac{\partial \hat{\Gamma}}{\partial t} + \int d^2x \beta^i(\varphi(x)) \frac{\delta \hat{\Gamma}}{\delta \varphi^i(x)} = 0, \quad (71)$$

$$\beta^i \equiv \frac{d\varphi^i}{dt}, \quad t = \ln \mu$$

This implies the scale anomaly equation

$$a \frac{\partial \Gamma}{\partial a} = - \int d^D x \beta'(v) \frac{\delta \Gamma}{\delta \varphi^i}, \text{ or}$$

$$\int d^2 z \sqrt{g} T_m^m = \int d^D x \beta'(v) \frac{\delta W}{\delta \varphi^i}, \quad (72)$$

where $T_m^m = \frac{2g^{mn}}{\sqrt{g}} \frac{\delta W}{\delta g^{mn}}$. Using the path integral representation for W (69), (70) we get

$$\int d^2 z \sqrt{g} T_m^m = \frac{1}{4\pi a'} \int d^2 z \sqrt{g} \langle \beta_{\mu\nu}(x) \partial_m x^\mu \partial^m x^\nu + \alpha' \beta_\phi(x) R^{(2)} \rangle \quad (73)$$

where the averaging $\langle \dots \rangle = \int [dx] e^{-I} / \int [dx] e^{-I}$ is done using the renormalized path integral, and $\beta_{\mu\nu} = \frac{dG_{\mu\nu}}{dt}$, $\beta_\phi = \frac{d\phi}{dt}$. The "Weyl anomaly" equation can be obtained by dropping the integral over z (this step can be justified repeating the steps leading to (73) for the explicitly z -dependent couplings [47]) and by accounting for the total derivative term coming from the classical Weyl non-invariance of the dilaton coupling in (69)

$$T_m^m = \frac{1}{4\pi a'} \langle \beta_{\mu\nu}(x) \partial_m x^\mu \partial^m x^\nu \rangle + \frac{1}{4\pi} \langle \beta_\phi(x) \rangle R^{(2)} - \frac{1}{2\pi} \langle \Delta \phi(x) \rangle \quad (74)$$

where $\Delta = -\frac{1}{\sqrt{g}} \partial_m (\sqrt{g} g^{mn} \partial_n)$. Finally, we may write down the corresponding operator "anomaly" equation in terms of renormalized composite operators

$$\hat{T}_m^m = \frac{1}{4\pi a'} [\beta_{\mu\nu}(\hat{x}) \partial_m \hat{x}^\mu \partial^m \hat{x}^\nu] + \frac{1}{4\pi} [\beta_\phi(\hat{x})] R^{(2)} - \frac{1}{2\pi} [\Delta \phi(\hat{x})] \quad (75)$$

where $\hat{x}^\mu$ are the operators and $[\dots]$ denote finite composite operators or "normal products" [48]. Using the "quantum equations of motion" $\langle \frac{\delta I}{\delta x} \rangle = 0$, or

$$\langle \mathcal{D}^m \partial_m x^\mu - \frac{\alpha'}{2} G^{\mu\nu} \partial_\nu \phi R^{(2)} \rangle = 0$$

it is possible to rewrite

(74) as

$$T_m^m = \frac{1}{4\pi\alpha'} \langle \bar{\beta}_{\mu\nu}(x) \partial_m x^\mu \partial^m x^\nu \rangle + \frac{1}{4\pi} \langle \bar{\beta}_\phi(x) \rangle R^{(2)} \quad (76)$$

$$\bar{\beta}_{\mu\nu} = \beta_{\mu\nu} + 2\alpha' \partial_\mu \partial_\nu \phi, \quad \bar{\beta}_\phi = \beta_\phi + \alpha' (\partial_\mu \phi)^2 \quad (77)$$

It should be stressed that $\bar{\beta}$'s differ from the ordinary renormalization group functions and thus are not directly related to UV infinities. Computing the expectation values in (74) by expanding near a stationary point $\bar{x}^\mu$ of the action (69) we get

$$\begin{aligned} T_m^m = & \frac{1}{4\pi\alpha'} \beta_{\mu\nu}^{(0)}(\bar{x}) \partial_m \bar{x}^\mu \partial^m \bar{x}^\nu + \frac{1}{4\pi} \beta_\phi^{(0)}(\bar{x}) R^{(2)} + \\ & + \beta_{\mu\nu}^{(2)}(\bar{x}) \partial_m \bar{x}^\mu \partial^m \bar{x}^\nu \Delta^{-1} R^{(2)} + \beta_\phi^{(2)}(\bar{x}) R^{(2)} \Delta^{-1} R^{(2)} + \dots \\ & + \beta_{\mu\nu}^{(n)}(\bar{x}) \partial_m \bar{x}^\mu \partial^m \bar{x}^\nu \Delta^{-k} R^{(2)} \dots \Delta^{-l} R^{(2)} + \beta_\phi^{(n)}(\bar{x}) R^{(2)} \Delta^{-p} R^{(2)} \dots \Delta^{-q} R^{(2)} + \dots \end{aligned} \quad (78)$$

Eq. (78) a priori contains all "connected" covariant non-local terms of dimension cm^{-2} built of $g_{\mu\nu}$ and $\bar{x}^\mu$ (only $\beta_\phi^{(0)}$ is non-vanishing in the free string case [22]). The coefficients

$\beta_{\mu\nu}^{(n)}, \beta_\phi^{(n)}, n=1,2,\dots$ are of course the functions of $\bar{\beta}_{\mu\nu}, \bar{\beta}_\phi, G_{\mu\nu}, \phi$ and their derivatives, $\beta_{\mu\nu}^{(1)} = \beta_{\mu\nu}^{(0)} + \dots, \beta_\phi^{(1)} = \beta_\phi^{(0)} - \frac{1}{4} \beta_{\mu\nu} G^{\mu\nu} + \dots$

(note that $\langle \partial_m x^\mu \partial^m x^\nu \rangle = -\frac{\alpha'}{4} G^{\mu\nu} R^{(2)} + \dots$, see, e.g., de Alwis [19]). We stress again that $\beta^{(n)}$'s in (78)

cannot be directly interpreted as the ordinary β -functions, i.e. $\mu \frac{d}{d\mu}$ - derivatives of renormalized couplings. The " β -functions" that appeared (as β^G and β^ϕ) in ref. [17b] are our $\beta_{\mu\nu}^{(1)}$ and $\beta_\phi^{(1)}$ and not $\bar{\beta}_{\mu\nu}$ and $\bar{\beta}_\phi$ (the terms with $\beta^{(n)}, n \geq 2$, in (78) were missed in refs. [6, 17b]). While $\bar{\beta}_{\mu\nu}$ and $\beta_{\mu\nu}^{(2)}$ coincide to the order α'^3 [49, 17b]

$$\begin{aligned}\bar{\beta}_{\mu\nu} &= \alpha' R_{\mu\nu} + 2\alpha' \partial_\mu \partial_\nu \phi + 2a_1 \alpha'^2 R_{\mu\lambda\rho\sigma} R_{\nu}{}^{\lambda\rho\sigma} + O(\alpha'^3) \\ &= \beta_{\mu\nu}^{(1)} + O(\alpha'^3)\end{aligned}\quad (79)$$

the functions $\bar{\beta}_\phi$ and $\beta_\phi^{(1)}$ differ already at the first non-trivial order ^{18/}.

$$\begin{aligned}\bar{\beta}_\phi &= c_0 - \frac{\alpha'}{2} \partial^2 \phi + \alpha' (\partial_\mu \phi)^2 + \frac{\alpha'^2}{4} (a_1 R_{\mu\nu\rho\sigma}^2 + \tilde{a}_1 R_{\mu\nu}^2 + \dots) + \\ &\quad + O(\alpha'^3),\end{aligned}\quad (80)$$

$$\begin{aligned}\beta_\phi^{(1)} &= c_0 - \frac{\alpha'}{4} \left\{ R + 4\partial^2 \phi - 4(\partial_\mu \phi)^2 + \right. \\ &\quad \left. + \alpha' (a_1 R_{\mu\nu\rho\sigma}^2 + \tilde{a}_1 R_{\mu\nu}^2 + \dots) \right\} + O(\alpha'^3),\end{aligned}\quad (81)$$

$$c_0 = \frac{1}{6}(D-26), \quad a_1 = \frac{1}{4},$$

where dots stand for terms involving derivatives of ϕ and R .

The result for the $O(\alpha')$ -term in the Weyl anomaly coefficient $\beta_\phi^{(1)}$ was first found in [6,8]; it agrees with the result given in the preprint version of ref. [17 b] after accounting for the difference in definitions of α' used (in the published version of [17b] the definition of α' was made standard in the string action but not in the results for the " β -functions"). We have found that no (two-loop) $\alpha'R$ -term is present in the renormalisation of ϕ and hence in (80) (this is to be compared with the known fact [48] that renormalisation due to interaction of the Euler number coupling in four dimensions starts at rather high number of loops).

As is clear from (76), the vanishing of $\bar{\beta}_{\mu\nu}$ and $\bar{\beta}_\phi$ implies the vanishing of all of the $\beta^{(n)}$ -coefficients in (78).

^{18/} It is easy to see, e.g. that for $D=26$
 $\phi_B = \int^\epsilon [\phi + \frac{\alpha'}{2\epsilon} \partial^2 \phi + O(\alpha'^2)]$, $\epsilon=d-2$ and thus $\beta_\phi = \frac{d\phi}{d\epsilon} =$
 $= -\frac{\alpha'}{2} \partial^2 \phi + O(\alpha'^2).$

It is very plausible, that we can "trade" the two functions $\bar{\beta}_{\mu\nu}$ and $\bar{\beta}_\phi$ for the two functions $\beta_{\mu\nu}^{(1)}$ and $\beta_\phi^{(1)}$ so that the conditions $\bar{\beta}_{\mu\nu}, \bar{\beta}_\phi = 0$ and $\beta_{\mu\nu}^{(1)}, \beta_\phi^{(1)} = 0$ are equivalent. This provides justification for the conclusions of ref. [17 b]. It is interesting to remark that the conditions of Weyl invariance ($\bar{\beta}=0$) of the ϕ -model are in general different from the ^{sufficient} conditions of its finiteness ($\beta=0$). ~~the resulting ϕ -model is UV finite.~~ Yet if ~~the resulting ϕ -model is UV finite.~~ $\bar{\beta}=0$ the resulting ϕ -model is UV finite. *on-shell* *1

It is easy to prove (using the normal coordinate expansion of (69)) that there are no other terms in the "central charge" coefficient (81) which depend on the curvature scalar R except the $\alpha' R$ -term (in particular, no R^2 -term is present [50]). The explicitly indicated structure of (79)-(81) is RS-invariant; the only RS-dependent coefficient is $\tilde{\alpha}_2$ ^{19/}. This coefficient can be found by a 3-loop computation [51]: $\tilde{\alpha}_2 = -\frac{1}{\epsilon} + 2 + \frac{3}{2}(\gamma - \ln 4\pi)$, ($\gamma = \text{Euler's constant}$), where dimensional regularisation was used. The UV infinity is exactly canceled after the renormalisation, corresponding to (79): $G_{\mu\nu}^R = \mu^\epsilon [G_{\mu\nu} - \frac{\alpha'}{\epsilon} R_{\mu\nu} + \dots]$. The renormalised value of $\tilde{\alpha}_2$ is clearly RS-dependent.

The computation of the effective action $\Gamma(70)$ in general proceeds as follows: we first expand x^μ near the constant value, $x^\mu = y^\mu + \xi^\mu$, $\xi^\mu \neq \text{const}$ (this is possible even for $\phi \neq \text{const}$ (when $x^\mu = \text{const}$ is not a solution of $\frac{\delta L}{\delta x} = 0$) since the path integral (70) is independent of an expansion point). Then $\Gamma \sim \int d^D y \sqrt{G} e^{-2\phi(y)} [\exp(-\bar{W}) - 1]$, where

19/ A change of RS corresponds to the following transformation $G_{\mu\nu} \rightarrow G_{\mu\nu} + T_{\mu\nu}$, $T_{\mu\nu} = \alpha' b_1 R_{\mu\nu} + O(\alpha'^2)$, under which $\beta_{\mu\nu} \rightarrow \beta_{\mu\nu} - \alpha' \int \left(\frac{\delta T_{\mu\nu}}{\delta G_{\alpha\beta}} R_{\alpha\beta} - \frac{\delta R_{\mu\nu}}{\delta G_{\alpha\beta}} T_{\alpha\beta} \right) = \beta_{\mu\nu} + O(\alpha'^3)$.

$\overline{W}[\phi, y]$ is a functional of the background 2-metric and a function of y . $\overline{W}$ is given by the sum of all connected diagrams on the background (-1 in the brackets is inserted to cancel the cosmological term at $D = 26$; this cancellation could be made automatic had we understood how to fix the Möbius gauge in computing (70), cf. [29]). The dependence of $\overline{W}$ on the Weyl factor of the metric g_{mn} can be obtained by integrating the expression for $T_m^m = \frac{2g^{mn}}{\sqrt{g}} \frac{\delta \overline{W}}{\delta g^{mn}}$ (now $\bar{x} = y = \text{const}$ in (78))

$$\overline{W} = \gamma_0 \frac{1}{\sqrt{\pi}} \int d^2 \bar{z} \sqrt{g} R^{(2)} + \gamma_1 \int R^{(2)} \Delta^{-1} R^{(2)} (1 + \lambda_1 \ell_n \Delta / \mu^2 + \dots) + \gamma_2 \int R^{(2)} \Delta^{-1} R^{(2)} \Delta^{-1} R^{(2)} (1 + \lambda_2 \ell_n \Delta / \mu^2 + \dots) + \dots \quad (82)$$

In general (for $\phi \neq \text{const}$) (82) contains also contributions like $\int R^{(2)} \Delta^{-1} R^{(2)} (\ell_n \frac{\Delta}{\mu^2})^k$, $\int R^{(2)} \Delta^{-1} R^{(2)} \ell_n \Delta$, etc. (higher order non-local terms in $\overline{W}$ were missed in [6,8]). The coefficients $\gamma_n, \lambda_n, \dots$ are functions of $G_{\mu\nu}, R_{\mu\nu}^\lambda, \phi$ and of their covariant derivatives at the point y . They are combinations of the coefficients $\beta_\phi^{(n)}$ in (78) (e.g. $\gamma_2 = -\frac{1}{16\pi} \beta_\phi^{(2)}$) and thus are determined by the $\overline{\beta}_{\mu\nu}$ and $\overline{\beta}_\phi$ functions (76), (77).

Expressed in terms of renormalized $G_{\mu\nu}$ and ϕ all $\gamma_n, \lambda_n, \dots$ are finite. If ϕ is put constant only contributions of irreducible graphs survive in (82) and, e.g., $\gamma_n = c_n (\alpha' R_{\mu\nu})^n + \dots$ ($(R_{\mu\nu})^n$ is the trace of the n -th matrix power of the Ricci tensor). The first contribution to γ_n thus comes from the $(n+1)$ -loop graphs

(for γ_1 this is true if $D = 26$, see. (81))

In particular, at the 3-loop order, $\gamma_2 = c_2 (\alpha' R_{\mu\nu})^2$, $\lambda_2 = d_2 (\alpha' R_{\mu\nu})^2$, $\gamma_k = 0, k \geq 2$ and γ_1 (proportional to $\beta_\phi^{(1)}$ (81)) is:

$$\gamma_1 = \frac{\alpha'}{64\pi} (R + \alpha' a_1 R_{\mu\nu}^2 + \alpha' \tilde{a}_2 R_{\mu\nu}^2) \quad (\text{we put } D = 26). \text{ Assuming that } \gamma_0 \text{ can be absorbed in a finite renormalization of } \phi \text{ we get after expanding } \exp(-\bar{W})$$

$$\Gamma \sim \int d^D y \sqrt{G} e^{-2\phi} (\gamma_1 \omega_1 + \gamma_2 \omega_2 + \frac{1}{2} \gamma_1^2 \omega_1^2 + \dots) \sim$$

$$\sim \int d^D y \sqrt{G} e^{-2\phi} [R + \alpha' (a_1 R_{\mu\nu}^2 + a_2 R_{\mu\nu}^2 + a_3 R^2) + \dots] \quad (83)$$

$$\omega_k \equiv - \int R^{(2)} (\Delta^{-1} R^{(2)})^k,$$

where $a_2 = \tilde{a}_2 + 64\pi c_2(\omega_1/\omega_2)$, $a_3 = \omega_1/128\pi$. In view of RS-dependence of $\tilde{a}_2$, a_2 is also RS-dependent. Since the values of ω_k depend on the choice of the 2-metric (for example for the metric of the round 2-sphere $\omega_1 = 16\pi$, $\omega_2 = -64\pi$), i.e. on the choice of the Weyl gauge, a_2 and a_3 are in general Weyl gauge dependent. We conclude, that the coefficients a_1 and a_3 in (66), (83) are renormalization scheme and (or) Weyl gauge dependent, i.e. they are ambiguous ("scheme"-dependent). This is the same conclusion we were led to above in the S-matrix approach.

An interesting consequence of eqs. (82), (83) is that all scheme-independent terms (i.e. terms having RS- and WG-independent coefficients) in the effective lagrangian are contained in the

γ_1 -term in (83). Thus a "scheme" may exist in which Γ is exactly determined by the γ_1 -coefficient, $\Gamma \sim \int d^D y \sqrt{G} e^{-2\phi} \gamma_1$, where $\gamma_1 = -\frac{1}{16\pi} \beta_{\phi}^{(1)}$ is proportional to the "central charge" coefficient (81) (as is clear from (83) ^{relation between Γ and γ_1} this is obviously true

at the two-loop order [6, 17b]). This relation is remarkable because $\frac{\delta \Gamma}{\delta \phi} = 0$ then implies (at least for $\phi = \text{const}$) that $\beta_{\phi}^{(1)} = 0$ and hence that the effective mass shell is "Weyl invariant" (thus lending support to the use of the Weyl gauge in computing Γ). Note however, that the corresponding "scheme" is

certainly not the scheme in which the curvature-squared terms in (66), (83) form the Gauss-Bonnet combination in the Bose and heterotic string theories, because the R^2 -term is certainly absent in $\chi_{\phi}^{(1)}$ (81).

Now let us comment on the connection between the effective action approach and the " β -function" approach of ref. [17] (see also sect. 2). Though $\tilde{\beta}^i = 0$ ($\tilde{\beta} = \bar{\beta}$ or $\beta^{(1)}$) are most certainly equivalent to the effective equations $\delta\Gamma/\delta\varphi^i = 0$ (where Γ is EA computed either in the S-matrix or in the path integral approach) no "scheme" exists in which $\frac{\delta\Gamma}{\delta\varphi^i} \sim \tilde{\beta}^i$ can be true. The scheme-covariant relation that is possible to satisfy is: $\frac{\delta\Gamma}{\delta\varphi^i} = \mathcal{X}_i \tilde{\beta}^i$, where $\mathcal{X}(\varphi)$ is non-trivial (α' -dependent) non-degenerate "tensor". This implies, in particular, that $\tilde{\beta}^i = 0$ do not represent equations of motion derivable from an action when considered to any finite order in α' . To prove our statement let us compare the $\tilde{\beta}$ -functions (which we shall choose to be the $\beta^{(1)}$ -functions given in (79), (81)) with $\frac{\delta\Gamma}{\delta G_{\mu\nu}}$ and $\frac{\delta\Gamma}{\delta\phi}$ following from the most general order α'^2 action (66) (we assume

$D = 26$ and $\phi = \text{const}$)

$$\begin{aligned} \frac{\delta\Gamma}{\delta G_{\mu\nu}} = & -a_0 \sqrt{G} e^{-2\phi} \left\{ \alpha' R_{\mu\nu} + \alpha'^2 \left[2(a_1 R_{\mu\lambda\rho\sigma} R_{\nu\lambda\rho\sigma} + a_2 R_{\mu\lambda} R_{\nu\lambda} + \right. \right. \\ & + a_3 R_{\mu\nu} R) + (a_2 + 4a_1) \partial^2 R_{\mu\nu} - (a_2 + 2a_1 + 2a_3) \partial_\mu \partial_\nu R + \\ & + (2a_3 + \frac{1}{2}a_2) G_{\mu\nu} \partial^2 R - (4a_1 + 2a_2) (R_{\mu\lambda} R_{\nu\lambda} - \\ & \left. - R_{\mu\lambda\nu\rho} R_{\lambda\rho}) \right] - \frac{1}{2} G_{\mu\nu} \mathcal{L} \left. \right\} + O(\alpha'^3), \end{aligned} \quad (84)$$

$$\begin{aligned} \frac{\delta\Gamma}{\delta\phi} = & -2a_0 \sqrt{G} e^{-2\phi} \left[\alpha' R + \alpha'^2 (a_1 R_{\lambda\mu\nu\rho}^2 + a_2 R_{\mu\nu}^2 + a_3 R^2) + \right. \\ & \left. + O(\alpha'^3) \right] = -2a_0 \sqrt{G} e^{-2\phi} \mathcal{L} + O(\alpha'^3). \end{aligned} \quad (85)$$

We see that (84) disagrees with (79) even at the leading order because of the absence in (79) of the $-\frac{1}{2} G_{\mu\nu} \mathcal{L}$ -term. Even leaving aside this difficulty, no choice of a_1 , a_3 and $\tilde{a}_2$ exists

for which (79), (81) can agree with (84), (85) at the α'^2 -order.

It is, however, possible to satisfy the following relations:

$$\begin{aligned}\frac{\delta\Gamma}{\delta G_{\mu\nu}} &= \alpha^{\phi\phi}_{\mu\nu\phi\phi} \tilde{\beta}_{\phi\phi} + \alpha^{\phi\phi}_{\mu\nu} \tilde{\beta}_{\phi\phi} \\ \frac{\delta\Gamma}{\delta\phi} &= \alpha^{\phi\phi}_{\phi\phi} \tilde{\beta}_{\phi\phi} + \alpha^{\phi\phi}_{\phi\phi} \tilde{\beta}_{\phi\phi}\end{aligned}\quad (86)$$

The tensor α_{ij} has the simplest derivative independent form in the "scheme" corresponding to the "Gauss-Bonnet case":

$a_2 = -4a_1$, $a_3 = a_1$. Since α_{ij} may contain many invariant terms, its structure is not fixed uniquely by (86). However, it is possible to show that (86) cannot be satisfied by a symmetric α_{ij} . A possible choice of α_{ij} is (symmetrisation over μ, ν and ϕ, ψ is assumed)

$$\alpha^{\phi\phi}_{\mu\nu\phi\phi} = A \left\{ G_{\mu\nu} G_{\phi\phi} + \alpha' [f_1 (R_{\mu\rho\nu\phi} + R_{\rho\mu\phi\nu}) + f_2 (R_{\mu\nu} G_{\phi\phi} + R_{\phi\phi} G_{\mu\nu})] + O(\alpha'^2) \right\}, \quad (87)$$

$$\alpha^{\phi\phi}_{\mu\nu} = 4A \left\{ \frac{1}{2} G_{\mu\nu} (1 + f_3 \alpha' R) + f_4 \alpha' R_{\mu\nu} \right\} + O(\alpha'^2),$$

$$\alpha^{\phi\phi}_{\phi\phi} = -8A (1 + f_5 \alpha' R) + O(\alpha'^2), \quad \alpha^{\phi\phi}_{\phi\phi} = 2A [f_6 \alpha' R_{\phi\phi} + O(\alpha'^2)],$$

where $A = -\alpha \sqrt{G} \exp(-2\phi)$ and

$$f_1 = -4a_1, f_2 = 2a_1 + \frac{1}{2}\tilde{a}_2, f_3 = a_1, f_4 = \frac{1}{2}\tilde{a}_2, f_5 = a_1, f_6 = -\tilde{a}_2 - 4a_1$$

(further simplification occurs if $\tilde{a}_2 = a_2 = -4a_1$: $f_2 = f_6 = 0$).

The above analysis can be repeated also for the heterotic and type II closed superstring theories. The corresponding

generalizations of (69) are the actions of the $N = \frac{1}{2}$

and $N = 1$ locally supersymmetric $\mathcal{G}$ -models [43, 17 b, a] with the " β -functions" of the form (79), (81) with [17 b, c, 52]:

$$c_0^{(H)} = \frac{5}{24}(D-10), \quad a^{(H)} = \frac{1}{8}, \quad c_0^{(S)} = \frac{1}{4}(D-10), \quad q_1^{(S)} = 0$$

(the values of q_1 agree with the results following from the

3-point amplitudes). The coefficient $\tilde{a}_2$ in (81) computed in

the minimal subtraction scheme is the same in all ($N=0, \frac{1}{2}, 1$)

σ -models [47] (this is related to the fact that the $\alpha' R_{\mu\nu}$ -term in (79) and the $\alpha' R$ -term in (84) receive no corrections from fermions [6]). The coefficients a_2 and a_3 in (66) are again "scheme dependent".

Note added.

To discuss the Weyl invariance of a σ -model we are to consider it on a curved 2-d background. Then it is necessary to introduce the dilaton coupling $R^{(2)} \phi(x)$ (69) in order to preserve renormalizability.

Then the finiteness condition is the vanishing of the $\beta_{\mu\nu}$ - and β_ϕ -functions up to the terms that can be eliminated by a diffeomorphism of a target space [49], $\beta_{\mu\nu}^0 = \partial_\mu \bar{z}_\nu + \partial_\nu \bar{z}_\mu$, $\beta_\phi^0 = \bar{z}^\mu \partial_\mu \phi$. Equivalently, $\beta_{\mu\nu}^0$ and β_ϕ^0 do not correspond to infinities in on-shell quantities since the counter-term $\sim \frac{1}{\epsilon} [\beta_{\mu\nu}^0 \partial \bar{x}^\mu \partial \bar{x}^\nu + \alpha' R^{(2)} \beta_\phi^0]$ vanishes after the use of the classical equation

$\partial^\mu \partial_\mu \bar{x}^\mu = \frac{\alpha'}{2} R^{(2)} \partial^\mu \phi$. Thus the Weyl-invariant σ -model having $\bar{\beta}_{\mu\nu} = \beta_{\mu\nu} + 2\alpha' \partial_\mu \partial_\nu \phi =$

$\bar{\beta}_\phi = \beta_\phi + \alpha' (\partial_\mu \phi)^2 = 0$ is finite on-shell ($\bar{z}_\mu \sim \partial_\mu \phi$). We would like to stress that even if a $(N=2)$ σ -model naively is infinite in flat 2-space ($\beta_{\mu\nu} \neq 0$) it may turn out to be finite and Weyl-invariant in curved 2-space for a particular dilaton field. Note in this connection that compactifications with a non-constant (on a compact space) ϕ -background (and non-vanishing torsion) were recently considered by A. Strominger (Santa Barbara preprint NSF-ITP-86-16) (see also C.M. Hull, DAMTP preprint, 1986)

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