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D. Sahdev

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R.B. Zhang

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A COMMENT ON THE COVARIANT SPECTRUM OF THE OPEN BOSONIC STRING *

D. Sahdev and R.B. Zhang
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

We organize the spectrum of the open bosonic string in such a way that the counting of component fields (for the covariant formulation based on BRST invariance) can be extended to all mass levels.

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A number of groups [1,2] have recently developed the second quantized covariant theory of free bosonic strings, in formulations which are believed to be closely related [3]. In view of this, we shall restrict all our considerations to the approach of Siegel and Zwiebach [1,1a] who construct a BRST invariant string Lagrangian in terms of a string functional $\phi(x(\sigma), c(\sigma), \bar{c}(\sigma))$ dependent on two ghost variables $c(\sigma)$ and $\bar{c}(\sigma)$, in addition to the usual string co-ordinate $x_\mu(\sigma)$. This BRST-invariant Lagrangian, when elaborated in terms of component fields produces a whole sequence of gauge invariant Lagrangians for point particles of progressively increasing spins. The fact that these Lagrangiangs when worked out for the first few mass levels correctly reproduce those derived by earlier approaches to higher spin actions [4] has greatly reinforced one's confidence in the correctness of the results.

The purpose of this note is to extend the counting of states, for this framework, from the first few mass levels to arbitrarily higher ones. This counting which at first sight seems to be a horrendous exercise in group-theoretic combinatorics actually turns out to be rather simple, thanks to certain regularities in the string spectrum which we shall discuss below.

The essential idea behind the counting is that, at any mass level, the various states can be assigned to specific sequences of Young diagrams — sequences which can be indefinitely extended as we climb up the mass ladder. This permits one to adopt an inductive approach. To see how this works out in detail, we first recall a series of facts on the spectrum of the string, as also on the structure of Stuckelberg fields needed to describe tensors of arbitrary rank.

The functional $\phi(x(\sigma))$ of the old formalism [5] contains sufficient degrees of freedom to describe the propagating modes of the string but not enough to describe them covariantly [1]. For the latter purpose, we need to work in terms of $\phi(x(\sigma), c(\sigma), \bar{c}(\sigma))$ which includes a dependence on two extra anticommuting variables $c(\sigma)$, $\bar{c}(\sigma)$. $\phi(x(\sigma), c(\sigma), \bar{c}(\sigma))$ contains, in addition to the required physical degrees of freedom, an infinite set of non-physical ones. The former can be singled out by defining a certain SU(1,1) algebra, in terms of $c(\sigma)$ and $\bar{c}(\sigma)$ and requiring that one retain only those combinations which are singlets under this algebra. The fact that we are thus always dealing with bilinears prompts us, in turn, to bosonize the two fermionic co-ordinates into a single bosonic one, $\chi(\sigma)$, and to generate all physical states by the action, on the vacuum, of the creation operators $\hat{a}_n^\dagger (n \geqslant 2)$ corresponding to this bosonic co-ordinate [1]. We need not

concern ourselves with the details of the above procedure. We need only the fact that the physical states of the string are generated by applying to the vacuum, arbitrary products of the bosonic creation operators, $\alpha_m^{\mu\dagger}$ $(m \geqslant 0)$ contained in $x_\mu(\sigma)$:

$$X_\mu(\sigma) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{\sqrt{2\alpha'}}\,X_\mu + i\sum_{n\neq 0}\frac{\alpha_n}{n}\cos n\sigma\right) \tag{1}$$

and of the operators $\hat{a}_n^\dagger(n \geqslant 2)$ corresponding to the bosonic ghost co-ordinate $\chi(\sigma)$. All states involving the $\hat{a}_n^\dagger(n \geqslant 2)$ belong to the ghost sector. The mass level of the state $\alpha_{m_1}^{\mu_1\dagger}\ldots\alpha_{m_n}^{\mu_n\dagger}\,\hat{a}_{n_1}\ldots\hat{a}_{n_p}|0\rangle$ is $\sum_{i=1}^{n} m_i + \sum_{j=1}^{p} n_j$, and the state corresponds to a tensor of rank n. The number, C_N, of singlet states at level N is clearly given by the partition function:

$$\frac{1}{\prod_{n=2}^{\infty}(1-x^n)} = \sum_{n=2}^{\infty}\sum_{m=1}^{\infty}\left(1+(x^n)^m\right) = \sum_{N=1}^{\infty} C_N x^N \tag{2}$$

Putting down the string spectrum up to, say, mass level 6 we get the ghost – free states of Table I and the ghost sector states of Table II, where $\underset{1\;1\;1}{\square\square\square}$ e.g. corresponds to $\alpha_{1\mu}^\dagger\,\alpha_{1\nu}^\dagger\,\alpha_{1\lambda}^\dagger|0\rangle$, $\underset{1}{\square}\times\underset{2}{\square}$ to $\alpha_{1\mu}^\dagger\,\alpha_{2\nu}^\dagger|0\rangle$ and $\underset{1}{\square}\times\underset{2}{\square}$ (2) to $\alpha_{1\mu}^\dagger\,\alpha_{2\nu}^\dagger\,\hat{a}_2^\dagger|0\rangle$ etc. (We never subtract off any traces. Hence all the Young diagrams represent the corresponding <u>traceful</u> tensors.) We see that all states in Table I belong to specific sequences of Young diagrams, constructed as follows:

The leading diagram of a sequence at an arbitrary mass level consists of a product of Young diagrams carrying integer labels in increasing order. The right-most diagram (corresponding to the highest label) contains at least 2 boxes. The sequence progresses by successive elimination of a $\underset{1}{\square}$ box and the concomitant increase in the number assigned to the last box on the right. The sequence terminates when we run out of $\underset{1}{\square}$ boxes. Thus at level 10, say, we have the leading and terminal diagrams given in Table III. We note that the terminal diagrams at mass level N are always in 1-1 correspondence with the combinations of powers of x in Eq.(2) which produce x^N. Thus the number of sequences at level N is exactly C_N. Furthermore, each sequence at level N can be labelled by the particular partitioning of N given by the integers carried by its terminal diagram.

To see how these sequences are completed by fields from the ghost sector, we recall some basic facts about higher spin gauge fields. To describe massive tensor fields covariantly we need a set of auxiliary and Stuckelberg fields, which can most conveniently be derived by starting from the corresponding massless tensor field in (D+1)-dimensions and reducing it down to D-dimensions [1,6]. In particular, a vector is described by $(\square + (\cdot))$, an antisymmetric tensor $A^{(n)}$ of rank n, by the pair $(A^{(n)}, A^{(n-1)})$ and a totally symmetric tensor, $\phi^{(n)}$ of rank n by the 4 fields $\phi^{(n)}, \phi^{(n-1)}, \phi^{(n-2)}, \phi^{(n-3)}$. The structure of fields needed for the covariant description of an arbitrary product of vectors and symmetric tensors (which is what we need) can simply be obtained by multiplying together the fields required by each factor.

Thus, e.g. the fields needed to describe $\underset{n}{\boxed{\vdots}\square}$ are given by

$$\left(\underset{n}{\boxed{\vdots}} + \underset{n-1}{\boxed{\vdots}}\right)\times\left(\square + (\cdot)\right) = \left(\underset{n}{\boxed{\vdots}\square} + \underset{(n-1)}{\boxed{\vdots}\square} + \underset{n}{\boxed{\vdots}} + \underset{(n-1)}{\boxed{\vdots}}\right) + \left(\underset{n+1}{\boxed{\vdots}} + \underset{n}{\boxed{\vdots}}\right) \tag{3}$$

It is convenient to refer to each of the sets above as "complete". We see that sums and products of complete sets are always complete but they are of course reducible in general. It is important for what follows to note that the set $\boxed{\;n\;}'' = \boxed{\;n\;} + \boxed{n-1} + .. + (\cdot)$ is always complete although for $n > 3$ it is reducible. Indeed if we count diagrams from the left in sets of 4, each of these sets is complete by itself, as is also the remainder (which consists either of $\square\square + \square + (\cdot)$, $\square + (\cdot)$, $(\cdot)$ or nothing). Similarly, $\boxed{\begin{smallmatrix}\square & n\\ \square\end{smallmatrix}}'' = \boxed{\begin{smallmatrix}\square & n\\ \square\end{smallmatrix}} + \left(\square\times\boxed{n-1}''\right)$ is always complete but is reducible for $n > 4$.

The complete set which contains the sequence $\underset{1\qquad 1}{\boxed{\;n\;}} + \left(\underset{1\qquad 1}{\boxed{n-2}}\times\underset{2}{\square}\right) + \ldots\ldots + \left(\underset{1}{\square}\times\underset{n-1}{\square}\right) + \underset{n}{\square}$ can now be easily constructed:

$$\underset{1\qquad 1}{\boxed{\;n\;}}'' = \underset{1\qquad 1}{\boxed{\;n\;}} + \left\{\underset{1\qquad 1}{\boxed{n-2}}\times\underset{2}{\square} - \left(\underset{1\qquad 1}{\boxed{n-2}}\times\underset{2}{\square} - \underset{1\qquad 1}{\boxed{n-1}}\right)\right\} + \boxed{n-2}''$$

$$\Rightarrow \underset{1\qquad 1}{\boxed{\;n\;}}'' + \boxed{\begin{smallmatrix}\square & n-2\\ \square\end{smallmatrix}}'' = \left(\underset{1\qquad 1}{\boxed{\;n\;}} + \sum_{m=2}^{n}\boxed{n-m}\times\underset{m}{\square}\right) + \sum_{m=2}^{n}\boxed{n-m}(m)(4)$$

Similarly, for the sequence $\underset{1\qquad 1}{\boxed{\;n\;}}\times\underset{p\qquad p}{\boxed{\;m\;}} + \sum_{m=1}^{n}\underset{1\qquad 1}{\boxed{n-m}}\times\underset{p\qquad p}{\boxed{\;m\;}}\times\underset{p+m}{\square}$ $(p \geqslant 2)$ we have

$$\underset{1 \qquad 1}{\boxed{\quad n \quad}}'' \times \underset{p \qquad p}{\boxed{\quad m \quad}}'' + \boxed{\; n-1 \;}'' \times \begin{array}{l}\boxed{\; m-1 \;}\\ \square\end{array}'' = \Big(\underset{1 \qquad 1}{\boxed{\; n \;}} \times \underset{p \qquad p}{\boxed{\; m \;}} + \sum_{m=1}^{n} \underset{1 \qquad 1}{\boxed{\; n-m \;}} \times \underset{p \qquad p}{\boxed{\; m \;}} \times \underset{p+m}{\square} \Big)$$

$$+ \Big\{ \underset{p \qquad p}{\boxed{\; m-1 \;}} + \underset{p \qquad p}{\boxed{\; m-2 \;}}(p) + \ldots\ldots + (\cdot)(\underbrace{p\,p \ldots p}_{m-1}) \Big\} \times \Big\{ \underset{1 \qquad 1}{\boxed{\; n \;}}(p) + \underset{1 \qquad 1}{\boxed{\; n-1 \;}}(p+1) + \ldots + (\cdot)(p+n) \Big\}$$

$$+ \Big\{ \underset{p \qquad p}{\boxed{\; m-2 \;}}(p) + \underset{p \qquad p}{\boxed{\; m-1 \;}}(pp) + \ldots\ldots + (\cdot)(\underbrace{p\,p \ldots p}_{m-1}) \Big\} \times \Big\{ \underset{1 \qquad 1}{\boxed{\; n-1 \;}} \times \underset{p+1}{\square} + \underset{1 \qquad 1}{\boxed{\; n-2 \;}} \times \underset{p+2}{\square} + \ldots + (\cdot)\underset{p+n}{\square} \Big\} \tag{5}$$

In each of Eqs.(4) and (5) the left-hand side and hence the right-hand side is complete. Furthermore, the first term on the right-hand side is the sequence of interest while the second one consists of terms from the ghost sector.

We can now pass to the most general sequence which is obtained by simply multiplying the sequence in Eq.(5) by an arbitrary product $\boxed{\text{ARB}}$ of Young diagrams carrying integer labels q $(1 < q < p)$. We simply multiply both sides of Eq.(5) by $\boxed{\text{ARB}}$ where $\boxed{\text{ARB}}''$ is obtained by replacing each factor $\boxed{\text{F}}$ in $\boxed{\text{ARB}}$ by $\boxed{\text{F}}''$. $\boxed{\text{F}}''$ or $\underset{p \qquad p}{\boxed{\; m \;}}''$ (for $p \geq 2$, m arbitrary) is of course

$$\underset{p \qquad p}{\boxed{\; m \;}}'' = \underset{p \qquad p}{\boxed{\; m \;}} + \underset{p \qquad p}{\boxed{\; m-1 \;}}(p) + \ \ldots\ldots + (\cdot)(\underbrace{p\,p \ldots p}_{m}) \tag{6}$$

All products are to be taken according to the rule

$$\underset{p \qquad p}{\boxed{\; m \;}}(\underbrace{pp \ldots p}_{r}) \times \underset{q \qquad q}{\boxed{\; n \;}}(\underbrace{qq \ldots q}_{s}) = \underset{p \qquad p}{\boxed{\; m \;}} \times \underset{q \qquad q}{\boxed{\; n \;}}(\underbrace{p \ldots p}_{r}\underbrace{q \ldots q}_{s}) \tag{7}$$

With these assignments of integer labels to the fields in Eqs.(4)-(7) coming from the ghost sector (1) each product on the right-hand side of these equations is at the same mass level and (2) any diagram in the ghost sector can be unambiguously assigned to a given sequence. The latter is done by arranging all the labels (including the ones carried by ghost oscillators) in an increasing sequence. If the diagram begins with a series of $\underset{1}{\square}$ boxes, we evolve it, as explained above (by successively eliminating a $\underset{1}{\square}$ box and simultaneously increasing the value of the right-most label, until we run out of $\underset{1}{\square}$ boxes.

If at the right we have two equal labels - one ghost and the other not - we can increase the values of either). Our original diagram and each one that follows upto the terminal one will belong to the sequence carrying the same signature as the terminal diagram. Thus, e.g. $\underset{1}{\square} \times \underset{4}{\square}\,(3) = \underset{1}{\square} \times (3) \times \underset{4}{\square}$ evolves into $(3) \times \underset{5}{\square}$ and belongs to the sequence ending in $\underset{3}{\square} \times \underset{5}{\square}$ (at level 8). Similarly $\underset{11}{\square\square}\,(33)$ also belongs to the $\underset{3}{\square} \times \underset{5}{\square}$ sequence while $\underset{2}{\square} \times \underset{3}{\square}\,(3)$ belongs to the $\underset{2}{\square} \times \underset{33}{\square\square}$ sequence.

From this it straightforwardly follows that (1) all the fields needed to complete the sequences at level N actually occur among the physical fields in the ghost sector at that level and (2) they are the only ones that do so. Hence the counting works out just right.

The question of whether this assignment of integer labels is merely a covenient book-keeping device for the counting of states or has a deeper significance is currently under investigation.

In conclusion, we have arranged the component fields lying in the string spectrum into well-defined sequences and have picked out the general pattern through which the ghost sector provides the auxiliary and Stuckelberg fields needed for a covariant description of these sequences. The prescription is precise, surprisingly simple and applicable at all mass levels. It will doubtlessly prove to be extremely useful for the various checks that will need to be made at the component level as the non-linear aspects of string theory undergo a fuller development.

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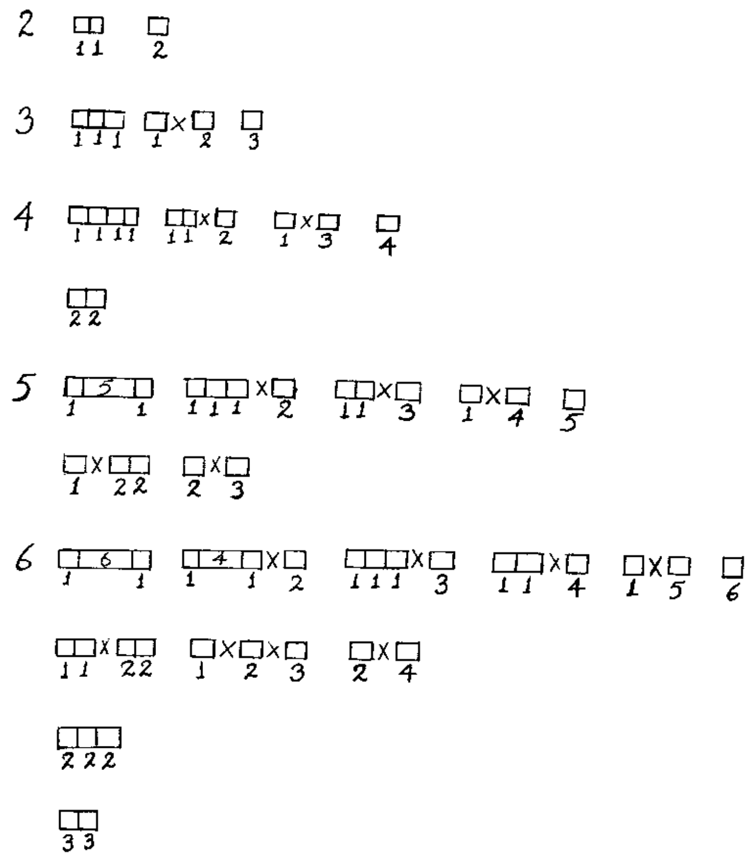
Table I

Level



States upto level 6 in the non-ghost sector of the open bosonic string.

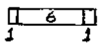
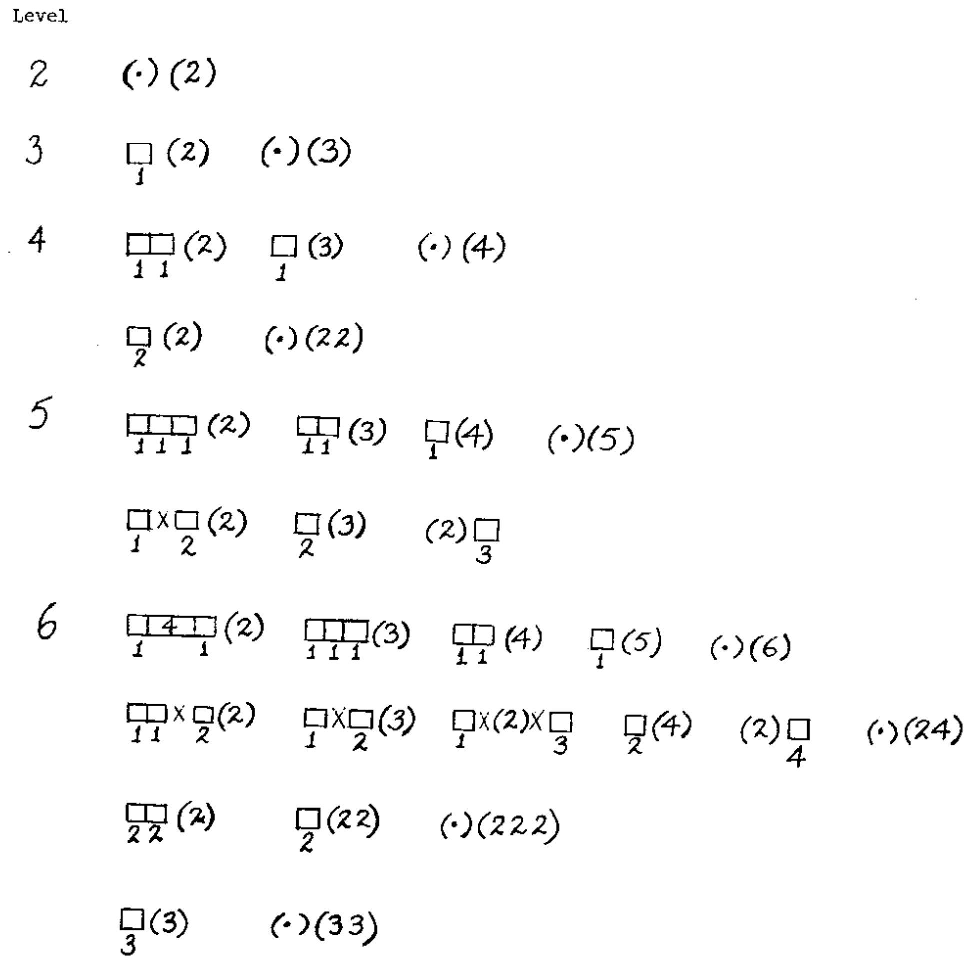
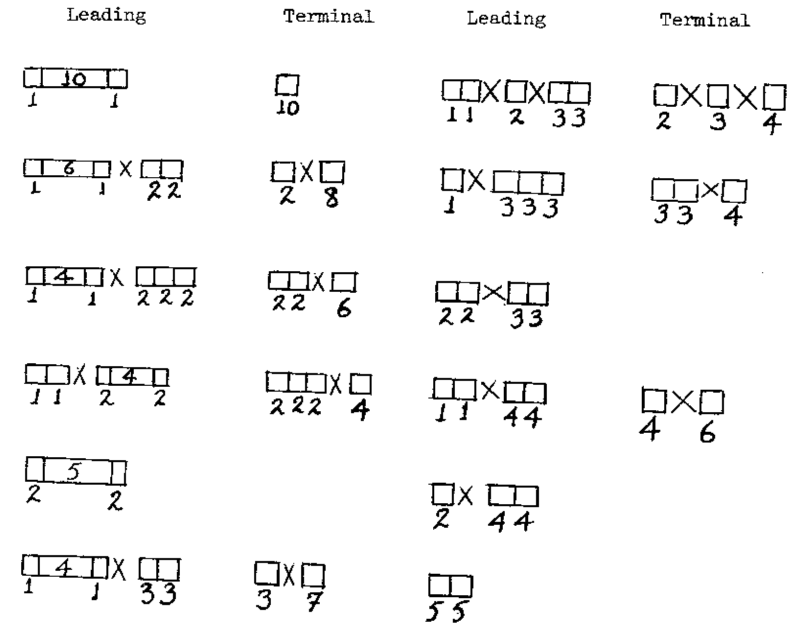
 e.g. stands for a totally symmetric Young diagram with 6 boxes, each carrying the label '1'.

Table II



States upto level 6 in the ghost sector of the open bosonic string. (•) represents a singlet. The numbers enclosed by round brackets refer to ghost oscillators.

Table III



The leading and terminal diagrams at level 10. For each of $\underset{2\qquad 2}{\boxed{\square\ \ 5\ \ \square}}$, $\underset{22}{\square\square}\times\underset{33}{\square\square}$, $\underset{2}{\square}\times\underset{44}{\square\square}$ and $\underset{55}{\square\square}$ the sequence contains only one diagram.