Conformal subalgebras of Kac-Moody algebras

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We define a conformal subalgebra of a Kac-Moody algebra as a subalgebra which has the same Sugawara conformal generator as the algebra in which it is embedded. This guarantees the preservation of conformal invariance, for instance, in the recently proposed construction of bosonic strings with nonsimply laced gauge groups. We present a complete list of all such conformal subalgebras.

A crucial property of Frenkel-Kac compactification¹ of bosonic strings is that it respects conformal invariance. To discuss this in general we consider a Kac-Moody algebra G_{KM}

$$[J_n^i, J_m^j] = if^{ijk} J_{n+m}^k + k\delta^{ij}. \quad (1)$$

One can define a group-invariant conformal generator, the Sugawara tensor²

$$T_n^G = \frac{1}{2k + C_G} \sum_{m,i} :J_{m+n}^i J_{-m}^i: \quad (2)$$

(the expression is normal ordered so that positive modes, J_m^i , $m > 0$ are to the right of negative ones). It satisfies the commutation relations

$$[T_n^G, J_m^i] = -mJ_{n+m}^i, \quad (3)$$

$$[T_n^G, T_m^G] = (n-m)T_{n+m}^G + D(G) \frac{1}{12} (n^3 - n) \delta_{m+n,0} \quad (4)$$

with

$$D(G) = \frac{2k \dim G}{2k + C_G}. \quad (5)$$

In Ref. 3 the preservation of conformal invariance for subalgebra H_{KM} of G_{KM} was discussed. Instead of considering states created from the vacuum by the operators J_{-n}^i , one can try to truncate by keeping only states created by a set of subalgebra generators J_m^a , which are linear combinations of those of G_{KM} (Ref. 4):

$$J_m^a = M_i^a J_m^i. \quad (6)$$

In the zero-mode sector the coefficients define an ordinary Lie-algebra embedding, and their normalization is the embedding index $I_2(G/H)$:

$$\sum_i M_i^a M_i^b = I_2(G/H) \delta^{ab} \quad (7)$$

[the embedding index is defined via the second index sum rule $I_2^H(R) = I_2(G/H) I_2^G(R)$; see, e.g., Refs. 5 and 6]. It follows directly from (6) and (7) that the subalgebra generators satisfy a Kac-Moody algebra with central charge $kI_2(G/H)$, while the Virasoro algebra of the Sugawara tensors of H_{KM} has a central charge

$$D(H) = \frac{2kI_2(G/H) \dim H}{2kI_2(G/H) + C_H}. \quad (8)$$

If G is semisimple, $G = \prod G_A$ with Kac-Moody central charges k_A , one defines

$$D(G) = \sum_A [2k_A \dim G_A / (2k_A + C_{G_A})].$$

Furthermore, if $H = \prod H_\alpha$, (8) generalizes to

$$D(H) = \sum_\alpha \frac{\sum_A 2k_A I_2(G_A/H_\alpha) \dim H_\alpha}{\sum_A 2k_A I_2(G_A/H_\alpha) + C_{H_\alpha}}. \quad (9)$$

The central charge of the Virasoro algebra plays an essential role in establishing conformal invariance in the critical dimension, as well as in the related problem of closure of the Lorentz algebra. Obviously, the truncation to the Kac-Moody subalgebra H_{KM} is still conformally invariant if one uses the original Sugawara tensors, T_n^G . However, if $T_n^G \neq T_n^H$, then T_n^G creates states that are not in the truncated subspace.³ Therefore, it only makes sense to define conformal invariance in terms of T_n^H , which respects the truncation. If the truncation changes the central charge of the Virasoro algebra, then the truncated theory ceases to be conformally invariant. In other words, if one insists on having conformal invariance before and after truncation, then one has to require that $D(G)$ and $D(H)$ be the same. This is the consistency condition proposed in Ref. 3. It is believed that this implies that $T_n^H \equiv T_n^G$, since if one defines $L_n = T_n^G - T_n^H$, one can show that L_n generates a Virasoro algebra with vanishing central charge.⁷ All unitary representations of such an algebra are trivial and hence $L_n \equiv 0$ (Ref. 8). The consistency condition $D(G) = D(H)$ is trivially satisfied in Frenkel-Kac compactification, truncated to the Cartan subalgebra. Since the truncated theory is obviously conformally invariant, so is the Frenkel-Kac theory.

Motivated by these considerations we define a conformal subalgebra of a Kac-Moody algebra as a subalgebra with the property $D(H) = D(G)$. In Ref. 3 several examples were given for the case G simply laced, $k=1$. The

main purpose of this paper is to find all such conformal subalgebras of all Kac-Moody algebras. An important quantity is the ratio $D(H)/D(G)$, which we will call the conformal factor of the embedding. We will prove that it is always less than or equal to one.

Using (9) one can easily show that replacing $H \times H$ by its diagonal subgroup decreases the conformal factor, so that such embeddings do not have to be considered. Now suppose that one of the H_α , H_1 say, is embedded into two (or more) of the G_A , say G_1 and G_2 . By projecting onto each factor of $G_1 \times G_2$ we see that the embedding $H \subset G_1 \times G_2$ may be written $H_1 \subset H_1 \times H_1 \subset G_1 \times G_2$, where the first embedding is diagonal. Thus if the conformal factor of the embedding is not to decrease, each H_α must be embedded in one and only one G_A . Then, for a given α , the sums over A in (9) have only one term, and the overall expression may be rewritten as a sum over simple factors. We see that the conformal factor will not decrease provided it does not decrease on each simple factor of G . We may therefore assume that G is simple.

Consider $D(H)/D(G)$ as a function of the central charge k of G ,

$$\frac{D(H)}{D(G)} = \sum_{\alpha} \frac{2kI_2(G/H_{\alpha}) + I_2(G/H_{\alpha})C_G}{2kI_2(G/H_{\alpha}) + C_{H_{\alpha}}} \frac{\dim H_{\alpha}}{\dim G}.$$

Since $I_2(G/H_{\alpha})C_G > C_{H_{\alpha}}$ (by the index sum rule), we see that this is a monotonic decreasing function of k , maximized at $k=1$. Define $f(G/H)$ to be the conformal factor for $k=1$,

$$f(G/H) = \sum_{\alpha} \frac{2 + C_G}{\dim G} \frac{I_2(G/H_{\alpha}) \dim H_{\alpha}}{2I_2(G/H_{\alpha}) + C_{H_{\alpha}}}. \quad (10)$$

We will show that $f(G/H) \leq 1$, and hence from the considerations above we can conclude that Kac-Moody algebras with $k > 1$ do not have conformal subalgebras.

Most of the work will consist of determining the conformal factors for all maximal subalgebra of the simple Lie algebras. A simple chain rule will then extend the results to arbitrary subalgebras. Fortunately, the majority of cases is covered by the following theorem. Consider $H \subset G$, G simply laced, G/H isotropy irreducible.⁹ Then the conformal factor is equal to 1 if $(\dim G / \text{rank } G) \neq (\dim H / \text{rank } H)$, and less than 1 otherwise.

Proof. Choosing a Cartan-Weyl basis for G and H , with the Cartan subalgebra of H embedded in that of G , one can derive the following formula for the Casimir operator of H acting on the Cartan subalgebra of G . Let Y_i denote the generators of the Cartan subalgebra of G , then

$$C_2^H Y_i = \sum_{\mu, \alpha, j} |y_{\mu}^{\alpha}|^2 \alpha^i \alpha^j Y_j, \quad (11)$$

where μ denotes the roots of H , α the roots of G , and the coefficients y_{μ}^{α} relate the root generators of H and G ,

$$F_{\mu} = \sum_{\alpha} y_{\mu}^{\alpha} E_{\alpha}. \quad (12)$$

If H is simple and G/H isotropy irreducible, C_2^H has only two eigenvalues, $C_2^H(A) = C_A$, and $C_2^H(T)$ (T is the

tangent vector in G/H). Taking the trace of C_2^H we find thus

$$r_H C_2^H(A) + (r - r_H) C_2^H(T) = 2I_2(G/H)(d_H - r_H)$$

(with $d = \dim G$, $d_H = \dim H$, $r = \text{rank } G$, $r_H = \text{rank } H$). To calculate the right-hand side we have used $\alpha^2 = 2$ and $\sum_{\alpha} |y_{\mu}^{\alpha}|^2 = I_2(G/H)$ (Ref. 11). We can eliminate $C_2^H(T)$ by using the index sum rule for the adjoint of G , the relation between C_2^H and the second index, and the formula for $C_2^G(A)$ valid for simply laced groups,

$$r = \frac{2d}{2 + C_1^G(A)}. \quad (14)$$

Then we obtain the equation

$$\frac{r_H d - d_H r}{d - d_H} \left[C_2^H(A) + 2I_2(G/H) \left[1 - \frac{d_H}{r} \right] \right] = 0. \quad (15)$$

If the first factor does not vanish, the equation implies that the embedding is conformal.

The first factor is proportional to the difference of the Coxeter numbers $(d/r - 1)$ and $(d_H/r_H - 1)$ of G and H . It is known—and can easily be verified—that the Coxeter number of H is smaller than or equal to the Coxeter number of G , for any G and H . The Coxeter numbers of G and H are equal only for $\text{SU}(N)/\text{Sp}(N)$, $\text{SO}(N+1)/\text{SO}(N)$ (N odd), E_6/F_4 , $\text{SO}(7)/G_2$, and $\text{SO}(8)/G_2$. All these embeddings have conformal factors less than one. To complete the proof of the theorem one has to extend the calculation to semisimple H , which is trivial. For each simple factor H_{α} of H one obtains an identity like (13). Instead of the factor $(r_H d - d_H r)$ which appears for simple H , one obtains now factors $(d - d_H)r_{\alpha} - (r - r_H)d_{\alpha}$ (r_{α} and d_{α} are rank and dimension of H_{α}). This factor is positive due to the inequality

$$\frac{d - d_H}{r - r_N} > \frac{d}{r} > \frac{d_{\alpha}}{r_{\alpha}},$$

where in the first step one uses the inequality for the Coxeter numbers of G and H , and in the second step the one for G and H_{α} . Therefore all these embeddings are conformal.

It is useful to extend this result (for G/H isotropy irreducible) to nonsimply laced G . This requires some explicit calculations, but there are only a few cases to be checked, which are all conformal, with the exception of $\text{SO}(2n+1)/\text{SO}(2n)$, $\text{Sp}(p+q)/\text{Sp}(p) \times \text{Sp}(q)$, $\text{SO}(7)/G_2$, $G_2/\text{SU}(3)$, and $F_4/\text{SO}(9)$, which have $f < 1$.

Next we consider the maximal embeddings which are not isotropy irreducible. For exceptional groups G several maximal subgroups have $\dim H < \text{rank } G$, which implies $f < 1$ [this follows from (10), plus the inequality $C_G \leq (2/r)(\dim G - r)$, which may be derived by considering the action of the Casimir operator of H on its Cartan subalgebra. It is an equality if and only if G is simply laced]. The four remaining cases, $E_7/\text{SU}(2) \times G_2$, $E_8/\text{SU}(5) \times \text{SU}(5)$, $E_8/\text{SO}(5)$, and $E_8/\text{SU}(2) \times \text{SU}(3)$ are all conformal.

For $G = \text{SU}(N)$, $\text{SO}(N)$, and $\text{Sp}(N)$ we distinguish two kinds of maximal subgroups: “generic” ones, listed below, and all the remaining ones. The generic embeddings can

be chosen in such a way that all remaining (nongeneric) maximal subgroups are simple and act irreducibly on the vector representation of G ; furthermore, the H representation embedded in the vector must be complex, real and pseudoreal for $G = \text{SU}(N)$, $\text{SO}(N)$, and $\text{Sp}(N)$, respectively. This is achieved by choosing the following generic subgroups:

$G = \text{SU}(N)$	$H = \text{S}(\text{U}(p) \times \text{U}(q))$	$(p + q = N)$
	$H = \text{SU}(p) \times \text{SU}(q)$	$(pq = N)$
	$H = \text{SO}(N)$	
	$H = \text{Sp}(N)$	$(N \text{ even})$
$G = \text{SO}(N)$	$H = \text{SO}(p) \times \text{SO}(q)$	$(p + q = N)$
	$H = \text{SO}(p) \times \text{SO}(q)$	$(pq = N)$
	$H = \text{Sp}(2p) \times \text{Sp}(2q)$	$(4pq = N)$
	$H = \text{SO}(N - 1)$	
	$H = \text{U}(N/2)$	$(N \text{ even})$
$G = \text{Sp}(2N)$	$H = \text{Sp}(2p) \times \text{Sp}(2q)$	$(p + q = N)$
	$H = \text{Sp}(2p) \times \text{SO}(q)$	$(pq = N)$
	$H = \text{U}(N)$	

Most of these are isotropy irreducible. The others have $f(G/H) \leq 1$, with equality for $\text{SO}(pq)/\text{SO}(p) \times \text{SO}(q)$, and $\text{SO}(4pq)/\text{Sp}(2q) \times \text{Sp}(2p)$.

For the remaining maximal subgroups we express the conformal factor in terms of the index $I_2(R)$ of the representation R of H embedded in the vector of G . If $G = \text{SU}(N)$ the result is

$$f = \frac{2I_2(R)\dim H}{2I_2(R) + C_H} \frac{1}{\dim R - 1}.$$

Clearly $f < 1$ if $\dim R > \dim H$. Therefore, only complex representations smaller than the adjoint have to be inspected. The obvious possibilities, the second-rank tensors of $\text{SU}(n)$, give rise to isotropy-irreducible embeddings. In fact, only three candidates remain [the rank-3 antisymmetric tensors of $\text{SU}(7)$ and $\text{SU}(8)$ and the spinor of $\text{SO}(14)$], which all have $f < 1$. The cases $G = \text{SO}(N)$ and $G = \text{Sp}(N)$ are just as simple to inspect, and yield no new conformal subalgebras.

This completes the search for all maximal conformal subalgebras. The result is listed in Tables I and II. The

extension to arbitrary subalgebras goes as follows. Consider $G \subset H_1 \subset H_2$. We may assume that $I_2(G/H_1) = 1$, since otherwise H_1 would have a Kac-Moody algebra with $k > 1$, which does not have conformal subalgebras. If $I_2(G/H_1) = 1$, it follows immediately that $f(G/H_2) = f(G/H_1)f(H_1/H_2)$, so that nonmaximal embeddings are conformal if and only if they can be obtained by a chain of maximal conformal embeddings, of which only the last one may have an index not equal to one. In other words, to obtain all conformal subalgebras of a given algebra one must build all possible trees of index 1 embeddings from Table I; at the end of every branch, a single embedding with index > 1 from Table II may be added. For example, the embedding of the adjoint representation of a D -dimensional semisimple group H in $\text{SO}(D)$ is always conformal.

Although it follows from general theorems⁸ that all conformal subalgebras have the same Sugawara tensor as the original algebra, this is not immediately obvious in explicit calculations. Let us assume that the Kac-Moody algebra of G is realized by the Frenkel-Kac construction.¹ In particular, the Cartan subalgebra generators of G are just the momentum operators P_i ($i = 1, \dots, r$). Writing the Cartan subalgebra generators K_ρ ($\rho = 1, \dots, r_H$) of H as $x_\rho^i P_i$ and the root generators as in (12) we derive the following expression for the Sugawara tensor of H (simple):¹³

$$T_m^H = \frac{1}{2k + C_H} \left[\sum_{ij} \left[\sum_\rho x_\rho^i x_\rho^j + \frac{1}{2} \sum_{\mu, \alpha} |y_\mu^\alpha|^2 \alpha^i \alpha^j \right] \times : \sum_n P_i^m - n P_i^n : \right],$$

where $k = I_2(G/H)$. For a conformal subalgebra T_m^H must be equal to $\frac{1}{2} : \sum_n P_i^m - n P_i^n :$, the uncompactified Virasoro generator. Thus the matrix

$$M^{ij} = \sum_\rho x_\rho^i x_\rho^j + \frac{1}{2} \sum_{\mu, \alpha} |y_\mu^\alpha|^2 \alpha^i \alpha^j \quad (16)$$

must be $\frac{1}{2}(2k + C_H)\delta^{ij}$. Because of the way we have normalized our generators, we have $x_\rho^i x_\sigma^j = I_2(G/H)\delta_{\rho\sigma} = k\delta_{\rho\sigma}$. Moreover, from (11) we see that x_ρ^i must be an eigenvector of the second term in (16) with eigenvalue

TABLE I. Maximal conformal subalgebras with index 1. The indications between parentheses have the following meaning: (S)=symmetric space; (I)=isotropy irreducible, not symmetric; (NI)=not isotropy irreducible.

G	H	G	H
$\text{SU}(N)$	$\text{S}(\text{U}(p) \times \text{U}(q))$	$(p + q = N)$ (S)	E_7 $\text{SU}(3) \times \text{SU}(6)$ (I)
$\text{SO}(N)$	$\text{SO}(p) \times \text{SO}(q)$	$(p + q = N)$ (S)	E_7 $\text{G}_2 \times \text{Sp}(6)$ (I)
$\text{SO}(2N)$	$\text{U}(N)$	(S)	E_8 $\text{SO}(16)$ (S)
E_6	$\text{SO}(10) \times \text{U}(1)$	(S)	E_8 $\text{E}_7 \times \text{SU}(2)$ (S)
E_6	$\text{SU}(6) \times \text{SU}(2)$	(S)	E_8 $\text{SU}(9)$ (I)
E_6	$\text{Sp}(8)$	(S)	E_8 $\text{E}_6 \times \text{SU}(3)$ (I)
E_6	$\text{SU}(3)^3$	(I)	E_8 $\text{G}_2 \times \text{F}_4$ (I)
E_7	$\text{E}_6 \times \text{U}(1)$	(S)	E_8 $\text{SU}(5) \times \text{SU}(5)$ (NI)
E_7	$\text{SO}(12) \times \text{SU}(2)$	(S)	F_4 $\text{SU}(2) \times \text{Sp}(6)$ (S)
E_7	$\text{SU}(8)$	(S)	

TABLE II. (a) Maximal conformal subalgebras with index $\neq 1$; infinite series. Notation as in Table I. These embeddings are conformal for any sensible value of the parameters, but the isotropy irreducibility may be different for some small groups. In particular $(*)$ is isotropy irreducible for $q=1$, $p>1$, and has one index equal to 1. (b) Maximal conformal subalgebras with index $\neq 1$; exceptional cases. In some cases one factor of H has index 1; conformal embeddings in these factors are, of course, allowed. For completeness we have included also groups G which are too large to emerge in compactification of the 26-dimensional bosonic string.

(a)						
G	H		Index	Restrictions		
$SU(N)$	$SO(N)$	(S)	2			
$Sp(2N)$	$U(N)$	(S)	2			
$SO(\dim R)$	H	(I)	$\frac{1}{2}C_H$	H simple, R =adjoint		
$SU(\dim R)$	$SU(n)$	(I)	$n+2$	R =symmetric tensor		
$SU(\dim R)$	$SU(n)$	(I)	$n-2$	R =antisymmetric tensor		
$SO(\dim R)$	$SO(n)$	(I)	$n+2$	R =symmetric tensor		
$SO(\dim R)$	$Sp(n)$	(I)	$\frac{1}{2}(n-2)$	R =antisymmetric tensor		
$SU(pq)$	$SU(p) \times SU(q)$	(I)	(q,p)			
$SO(4pq)$	$Sp(2p) \times Sp(2q)$	(NI)	(q,p)	$(*)$		
$SO(pq)$	$SO(p) \times SO(q)$	(NI)	(q,p)			
$Sp(2q)$	$SU(2) \times SO(q)$	(I)	$(q,4)$			

(b)						
G	H	Index		G	H	Index
G_2	$SU(2)$	(I)	28	$SU(16)$	$SO(10)$	(I) 4
G_2	$SU(2) \times SU(2)$	(S)	$(3,1)$	$SU(27)$	E_6	(I) 6
F_4	$SU(3) \times SU(3)$	(I)	$(2,1)$	$SO(16)$	$SO(9)$	(I) 2
F_4	$SU(2) \times G_2$	(I)	$(8,1)$	$SO(26)$	F_4	(I) 3
E_6	$SU(3) \times G_2$	(I)	$(2,1)$	$SO(42)$	$Sp(8)$	(I) 7
E_6	G_2	(I)	3	$SO(70)$	$SU(8)$	(I) 10
E_6	$SU(3)$	(I)	9	$SO(128)$	$SO(16)$	(I) 16
E_7	$SU(2) \times F_4$	(I)	$(3,1)$	$Sp(14)$	$Sp(6)$	(I) 5
E_7	$SU(2) \times G_2$	(NI)	$(7,2)$	$Sp(20)$	$SU(6)$	(I) 6
E_7	$SU(3)$	I	21	$Sp(32)$	$SO(12)$	(I) 8
E_8	$SO(5)$	(NI)	12	$Sp(56)$	E_7	(I) 12
E_8	$SU(2) \times SU(3)$	(NI)	$(16,6)$			

$\frac{1}{2}C_H$. Thus, M^{ij} has the correct eigenvalues on the Cartan subalgebra of H . It is by no means obvious why M^{ij} has the correct eigenvalue on the rest of the Cartan subalgebra of G . In particular, if T' is any irreducible component of the tangent space of G/H , and if T' contains an element of the special Cartan subalgebra of G ,¹⁴ then the quadratic Casimir of H on each such component T' must have eigenvalue $k + \frac{1}{2}C_H$. Isotropy irreducibility guarantees that the quadratic Casimir is constant on the tangent space of G/H , but it is perhaps somewhat surprising that for most isotropy-irreducible embeddings it also has the right value.

This becomes even more surprising for the many non-isotropy-irreducible embeddings that can be constructed from the table. In most cases, H is semisimple, and the condition is then that the weighted Casimir operator

$$\tilde{C}_2^H = \sum_{\alpha} \frac{1}{2I_2(G/H_{\alpha}) + C_{\alpha}} C_{\alpha}^{H_{\alpha}}$$

[with $H = \Pi H_{\alpha}$, $C_{\alpha} = C_2^H(\text{Ad}j)$] must be equal to $\frac{1}{2}$ on those irreducible pieces of the tangent space of G/H

which contain an element of the special Cartan subalgebra of G . In fact we find that $\tilde{C}_2^H = \frac{1}{2}$ on the entire tangent space, although we have not checked this systematically. If one could show that $\tilde{C}_2^H(T) \leq \frac{1}{2}$ for every irreducible factor of the tangent space on any coset manifold G/H , it would follow that $f(G/H) \leq 1$, with equality if and only if $\tilde{C}_2^H = \frac{1}{2}$ on the entire tangent space. In other words, conformal subalgebras would then be precisely those that saturate the upper bound on $\tilde{C}_2^H$.

This last property underscores the special nature of the conformal subalgebra, and it would not be surprising if they had other applications than the one that originally motivated our interest in them, namely, the construction of bosonic strings with nonsimply laced gauge groups.³

After the completion of this paper we were informed by Bais that he and Bouwknegt have also obtained a list of conformal subalgebras.¹⁵

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- ¹I. B. Frenkel and V. G. Kac, *Inv. Math.* **62**, 23 (1980); G. Segal, *Commun. Math. Phys.* **80**, 301 (1981); P. Goddard and D. Olive, in *Proceedings of the MSRI Workshop on Vertex Operators*, edited by J. Lepowsky (Springer, Berlin, to be published).
- ²H. Sugawara, *Phys. Rev.* **170**, 1659 (1968).
- ³F. A. Bais, F. Englert, A. Taormina, and P. Zizzi, Brussels University report (unpublished).
- ⁴We will only consider untwisted Kac-Moody algebras G and H , with an embedding $H \subset G$ as indicated.
- ⁵W. McKay and J. Patera, *Tables of Dimensions, Indices and Branching Rules* (Dekker, New York, 1981).
- ⁶Conventions for simple Lie algebras G : $[\lambda^i, \lambda^j] = if^{ijk}\lambda^k$; $\text{Tr}\lambda^i(R)\lambda^j(R) = I_2(R)\delta^{ij}$ for any representation R . I_2 is the second index, normalized as in Ref. 5. Quadratic Casimir eigenvalues for irreducible representations: $C_2(R) = (\dim G / \dim R)I_2(R)$. If R is the adjoint we denote the quadratic Casimir eigenvalue as $C_2(A)$ or C_G .
- ⁷P. Goddard, A. Kent, and D. I. Olive, *Phys. Lett.* **152B**, 88 (1985).
- ⁸J. F. Gomes, Imperial College Report No. TP/85-86/10, 1986 (unpublished); D. Friedan, Z. Qiu, and S. Shenker, in *Proceedings of the MSRI Workshop on Vertex Operators* (Ref. 1).
- ⁹ G/H is isotropy irreducible if the tangent vector is a real-irreducible representation of H . This includes all symmetric spaces, listed, for example, in Ref. 10. The nonsymmetric ones were classified in Ref. 11 (a list appears also in Ref. 12).
- ¹⁰S. Helgason, *Differential Geometry and Symmetric Coset Spaces* (Academic, New York, 1961).
- ¹¹J. Wolf, *Acta Math.* **120**, 59 (1968); **152**, 141(E) (1984).
- ¹²K. Pilch and A. Schellekens, *Nucl. Phys.* **B256**, 109 (1985).
- ¹³In general, additional terms with nonzero lattice momenta are present, which cancel for conformal subalgebras.
- ¹⁴By the special Cartan subalgebra of G we mean a Cartan subalgebra of G that contains a Cartan subalgebra of H as a subspace.
- ¹⁵F. A. Bais and P. Bouwknegt, Amsterdam report (unpublished).