TWO- AND FOUR-DIMENSIONAL ANOMALIES WITH AN INSTANTON BACKGROUND*

Stefano Forte

Center for Theoretical Physics
Laboratory for Nuclear Science
and Department of Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139 U.S.A.

CERN LIBRARIES, GENEVA

CM-P00066702

Submitted to: *Physics Letters*

Typeset in $\text{\TeX}$ by Roger L. Gilson

CTP #1342

March 1986

TWO- AND FOUR-DIMENSIONAL ANOMALIES WITH AN INSTANTON BACKGROUND

Stefano Forte

ABSTRACT

We illustrate the dimensional ladder which joins the d -dimensional gauge anomaly to the $d+2$ -dimensional chiral anomaly by solving explicitly the four- and two-dimensional problems of chiral fermions in a background instanton gauge potential.

It is well-known that the non-gauged chiral anomaly¹ is related to the Atiyah-Singer index theorem (see e.g., [2]), in that the integrated anomaly for a gauge theory coupled to massless Dirac fermions equals the index of the Dirac operator.

It has also been established that the gauge anomaly can be related to the index theorem in two dimensions higher.³ These results have been derived (in any even dimensional space) by general topological arguments,³ or by an adiabatic approximation for certain two-parameter families of gauge potentials.⁴ Here, we demonstrate explicitly the equality of the relevant topological invariants by choosing a specific four-dimensional gauge potential – an instanton or instanton-like configuration – which allows determining exactly the spectra of the Dirac operators involved.

We shall take the four-dimensional Euclidean spacetime manifold M^4 to be compact and boundaryless (this can be translated into a condition of slow fall-off at infinity in open space): $M^4 = S^2 \times S^2$. The four-dimensional index of the Dirac operator $i \not{D}[A]$ can be calculated by covering $S^2 \times S^2$ with two patches, $S^2 \times D^+$ and $S^2 \times D^-$, where $D^\pm$ are two disks (hemispheres) that cover the second S^2 , with $D^+ \cap D^- = S^1$.

The four-dimensional non-gauged chiral anomaly is equal to the Chern-Pontryagin density, $P = -\frac{1}{8\pi^2} \text{tr } F^2$, $F = dA + A^2$, which in turn equals the total derivative of the Chern-Simons 3-form: $P = d\Omega_0$; $\Omega_0 = -\frac{1}{8\pi^2} \text{tr } (AdA + \frac{2}{3}A^3)$. The four-dimensional index q is thus related to the variation of Ω_0 under a gauge transformation:

$$\begin{aligned} q &= \int_{S^2 \times S^2} P = \int_{S^2 \times D^+} d\Omega_0[A^+] + \int_{S^2 \times D^-} d\Omega_0[A^-] \\ &= \int_{S^2 \times S^1} \Omega_0[A^+] - \Omega_0[A^-] \\ &= \int_{S^2 \times S^1} \Delta^g \Omega_0[A^-] \end{aligned} \quad (1)$$

where the minus sign comes from the different orientation of the boundary S^1 with respect to $D^\pm$, and g is the transition function on $S^2 \times S^1$: $A^+ = (A^-)^g = g^{-1}(A^- + d)g$.

* This work is supported in part by funds provided by the U. S. Department of Energy (D.O.E.) under contract # DE-AC02-76ER03069.

If there exists a point on S^1 where g reduces to the identity of the gauge group G , then the functions $g: S^2 \times S^1 \rightarrow G$ fall into homotopy classes of $\pi_3(G)$, classified by the winding number n :

$$n(g) = \frac{1}{24\pi^2} \int_{S^1 \times S^2} \text{tr} (g^{-1} dg)^3 = \int_{S^2 \times S^1} \Delta^9 \Omega_0[A] . \quad (2)$$

If $\pi_1(G) = 0$, then $\pi_3(G) = \pi_1(\mathcal{G})$, where $\mathcal{G}$ is the space of functions $\hat{g}: S^2 \rightarrow G$. This is the same homotopy group that is relevant to the two-dimensional gauge anomaly. That anomaly arises because the functional determinant of a theory of massless Weyl fermions coupled to an external gauge field acquires a phase upon gauge transformation:

$$\text{Det} \left(i \hat{D}[A^\theta] \right) = e^{2\pi i \omega_1(A; \hat{g}(\theta))} \text{Det} \left(i \hat{D}[A] \right) ; \quad (3)$$

$\hat{D} \equiv \not{\partial} + A_+ = \not{\partial}_- + \not{D}_+$ is the Weyl operator. If one defines a loop of gauge functions $\hat{g}$ starting from the identity, $\hat{g} = \hat{g}(x, \theta)$, $x \in S^2$, $\theta \in S^1$, $\hat{g}(x, 0) = 1$, the phase factor provides a mapping from a loop embedded in $\mathcal{G}$ to the S^1 of unimodular complex numbers. These maps fall into homotopy classes of $\pi_1(\mathcal{G})$, and are classified by the winding number

$$w = \int_{S^1} d\theta \omega_1(A; \hat{g}(\theta)) .$$

The content of the established connection between the four- and two-dimensional anomalies is that $w = n$; hence, $w = q$, the four-dimensional index. We shall demonstrate this explicitly by calculating the winding number of the gauge variation of a two-dimensional fermionic determinant, with a specific choice of A .

To define A , we begin with an $SU(2)$ instanton-like gauge potential on 4-space:

$$A_\mu^n = \frac{x^2}{1+x^2} \tilde{g}_\mu^{-1} \partial^\mu \tilde{g}_n \quad (4)$$

$x \in \mathbf{R}^4$, $\tilde{g}_n = (\tilde{g}_1)^n$, $\tilde{g}_1$ is the standard map of the first homotopy class; in the fundamental representation of $SU(2)$ it is given by

$$\tilde{g}_1 = \frac{x^4 - i\vec{x} \cdot \vec{\sigma}}{\sqrt{x^2}} . \quad (5)$$

When $n = \pm 1$ this is the BPST instanton,⁵ for general n it is no longer a solution to the classical Yang-Mills equation, although it is a field configuration with instanton number n .

We can define a bundle on S^4 by stereographically projecting (4) from the south pole.² This provides a connection on the upper patch covering S^4 ; on the lower patch, which can be taken to be infinitesimal (around the south pole of S^4) the connection is

$$A' = \tilde{g}_n(A + d)\tilde{g}_n^{-1} . \quad (6)$$

2

Having constructed an instanton bundle on S^4 , we can build a bundle on $S^2 \times S^2$ through the following steps: (a) cover S^4 with two disk-like patches, $\bar{D}^\pm$, with boundary S^3 , $\delta \bar{D}^\pm = S^3$; (b) introduce a differentiable map $\beta: S^1 \times S^2 \rightarrow S^3$; (c) remove the origins $0^\pm$ from $\bar{D}^\pm$ and factor $\bar{D}^\pm - \{0\} = S^3 \times I$, with I a semi-open interval; (d) pull-back the connections on $\bar{D}^\pm$ to $\bar{M}^\pm = S^2 \times (S^1 \times I \cup \{0^\pm\})$. This defines a connection on the two manifolds $\bar{M}^\pm$, whose boundary is $S^2 \times S^1$, which cover $S^2 \times S^2$.

The transition function is now $g_n = \tilde{g}_n \circ \beta: S^2 \times S^1 \rightarrow SU(2) \sim S^3$. These maps fall into homotopy classes of $\pi_3(SU(2)) = \mathbf{Z}$, provided β preserves the base points, i.e., g_n is equal to the identity on the base points (the north poles) of S^1 and S^2 .⁶ An explicit choice of β yields

$$g_n(\psi, \varphi, \theta) = \exp -n\theta r^i T^i \exp n\theta T^3 , \quad (7)$$

with $0 \leq \theta \leq 2\pi$, $\theta \in S^1$; r^i are stereographic coordinates on S^2 ,⁷ $r^1 = \sin \psi \cos \varphi$, $r^2 = \sin \psi \sin \varphi$, $r^3 = \cos \psi$; $T^i = \frac{\sigma_i}{2}$ are generators of $SU(2)$. The connection on the upper patch $S^2 \times D^+$ is $A = t g_n^{-1}(\varphi, \psi; \theta) d g_n(\varphi, \psi; \theta)$; the radial coordinate $0 \leq t \leq 1$ and the polar angle θ parameterize D^+ . The index of the corresponding Dirac operator is, of course, n .

We now define a two-parameter (t, θ) family of two-dimensional gauge potentials on S^2 by taking the ψ, φ components of A , and considering D^+ as a parameter space:

$$A^{t, \theta} = t \hat{g}_n^{-1} d \hat{g}_n . \quad (8)$$

When $t = 1$ (i.e., on the rim of the disk D^+) the potentials form a loop of pure gauges; when $0 < t < 1$ they are not pure gauges. We shall show that the fermionic determinant has winding number n along the rim of D^+ . To this purpose, we study the eigenvalue problem for the Weyl operator.

On S^2 the Weyl operator has the form⁷

$$\hat{D}[A] = (S^{ab})_{ij} \delta_{mn} L_{ab} + \delta_{ij} \delta_{mn} - 2(S^{ab})_{ik} r_a A_b^k T_{mn}^p (P_+)_k{}^j \quad (9)$$

where i, j are spin indices, m, n are isospin indices, and

$$S^{ab} = \frac{1}{2} \epsilon^{abc} \sigma^c ; \quad L_{ab} = -i(r_a \partial_b - r_b \partial_a) \quad (10)$$

The γ -matrices are $\Gamma_\mu = i\gamma_\mu \gamma_5$, $\mu = 1, 2$; $\Gamma_3 = \gamma_5$; where γ^μ, γ^5 are flat-space γ -matrices. A representation is $\gamma^1 = -\sigma_2$, $\gamma^2 = \sigma_1$, $\gamma_5 = i\gamma^1 \gamma^2 = -\sigma_3$; $\Gamma^1 = -\sigma^1$. The chiral projectors are $P_\pm = \frac{1}{2}(1 \mp \vec{r} \cdot \vec{\sigma})$.

The eigenvalue equation for $i \hat{D}$ with the gauge potentials (8) is $\hat{D}[A^{t, \theta}] \hat{\psi} = \lambda \hat{\psi}$ is suitably written by redefining the spinor field $\hat{\psi}$:

$$\begin{aligned} \psi_{jn} &= U_{jk} \left(\exp \frac{i}{2} n \theta \sigma_3 \right)_{np} \hat{\psi}_{kp} ; \quad U_{jk} = i(\sigma_1 \sigma_3)_{jk} \\ U^\dagger S^{ab} U &= -(S^{ab})^{tr} . \end{aligned} \quad (11)$$

3

One gets

$$\left\{ i \hat{D} [A_0]_{ijnm} + [(t-1)2S_{ik}^{ab}r_a S_{nm}^{bc}r_c - 2tS_{ik}^{ab}r_a [(\cos n\theta - i \sin n\theta p^p)S_{nm}^{bc}r_c] \frac{1}{2}(\delta_{kj} - r^d \sigma_{kj}^d) \right\} \psi_{jn} = \lambda \psi_{jn} \quad (12)$$

$$i \hat{D} [A_0]_{ijnm} = (S^{ab})_{ij} \delta_{mn} L_{ab} + \delta_{mn} \delta_{ij} + [2(S^{ab})_{ik} r_a (S^{bc})_{nm} r_c] \frac{1}{2}(\delta_{kj} - r^d \sigma_{kj}^d) \quad (13)$$

Note that due to the transformation (11) there is now no need to distinguish between spin and isospin indices, and ψ can be treated as a 2×2 matrix.⁸

The spectrum of eq. (12) is determined by calculating the matrix elements of $i \hat{D} [A^{t,\theta}]$ with respect to a basis of eigenfunctions of the total angular momentum operator J^2 , $J^2 = \frac{1}{2} J^{ab} J^{ab}$, $J_{ij}^{ab} = L^{ab} + S_{ij}^{ab} - S_{nm}^{ab}$ (acting on the spinors ψ as defined in (11)). These can be obtained by combining normalized spherical harmonics $Y_m^\ell(\psi, \varphi)$ with eigenfunctions of the total spin and isospin. One finds

$$j = 0 \quad f_0^\pm = \frac{1}{2}(\delta_{jk} \pm \vec{r} \cdot \vec{\sigma}_{jk}) \delta_{kn} Y_0^0$$

$$j = \ell; \ell \geq 1 \quad \begin{cases} f_\ell^{1\pm} = \frac{1}{2}(\delta_{jk} \pm \vec{r} \cdot \vec{\sigma}_{jk}) \delta_{kn} Y_\ell^\ell \\ -\ell \leq m \leq \ell \quad f_\ell^{2\pm} = \frac{1}{2}(\delta_{jk} \pm \vec{r} \cdot \vec{\sigma}_{jk}) \frac{1}{\sqrt{\ell(\ell+1)}} S_{kn}^{ab} L_{ab} Y_m^\ell \end{cases} \quad (14)$$

This basis is complete and orthonormal with respect to the scalar product

$$\langle \xi | \chi \rangle = \text{tr} \int d\Omega \xi^\dagger \chi.$$

The Weyl operator is block-diagonal in each fixed- ℓ subspace of (14). The spectrum, obtained by diagonalization, is

$$\lambda_0^2 = (1-t) + t e^{in\theta}$$

$$\ell \geq 1 \quad (\lambda_\ell^\pm)^2 = \frac{1}{2} \left[2\ell(\ell+1) + \lambda_0^2 \pm \sqrt{\lambda_0^2(\lambda_0^2 + 4\ell(\ell+1))} \right] \quad (15)$$

There is full degeneracy in m , i.e., each eigenvalue is $(2\ell+1)$ times degenerate.

The spectral flow of the eigenvalues (15) considered as functions of t, θ displays several interesting features. The lowest-lying eigenvalues, λ_0^2 , have n nontrivial zeroes located on a ring with $t = t_0 = \frac{1}{2}$ and equally spaced in θ , occurring when $\theta = \theta_0 = \frac{(2k+1)\pi}{n}$, $0 \leq k < n$. Each zero contributes one unit of winding number; along the rim

of D^+ , where $t = 1$, $\lambda_0^2 = e^{in\theta}$ has winding number n . The other eigenvalues do not contribute to w ; in fact $(\lambda_\ell^+)^2 (\lambda_\ell^-)^2 = [\ell(\ell+1)]^2$, so that the phase dependence of $\lambda_\ell^+ \lambda_\ell^-$ cancels that of λ_ℓ^- .

Also, on the zeroes of λ_0^2 , i.e., on the topologically nontrivial zero modes of $\hat{D} [A^{t,\theta}]$, both $A^{t,\theta}$ and $\hat{D}$ reduce to interesting forms. $A^{t_0, \theta_0} = \frac{1}{2} e^{-i\vec{r} \cdot \vec{\sigma}} \rho_a e^{i\vec{r} \cdot \vec{\sigma}} = -i S^{ab} \tau_b$ (up to a constant gauge transformation) with ρ_a indicating derivative operators on the sphere,⁷ is the stereographic projection on S^2 of a two-dimensional meron.⁹ Furthermore, $\hat{D} [A^{t_0, \theta_0}] = i \hat{D} [A_0]$,¹³ on right-handed fermions this operator acts as the "square-root" of the total angular momentum: $(i \hat{D} [A_0])^2 = J^2$ (on left-handed fermions $\hat{D} [A^{t,\theta}]$ is just the free Dirac operator).

Now, the fermionic determinant can be defined as the regulated product of the eigenvalues of $i \hat{D}$. In a Pauli-Villars scheme

$$\det i \hat{D} = \prod_k \frac{i M \lambda_k}{\lambda_k + i M}. \quad (16)$$

If $\lambda_k(\theta) = \lambda_k(0) e^{i\varphi_k(\theta)} = |\lambda_k| e^{i(\varphi_k(\theta) + \varphi(0))}$, the θ -dependence of the imaginary part of the effective action Γ is given by

$$\begin{aligned} \text{Im} [\Gamma(\theta) - \Gamma(0)] &= \text{Im} \left[i n \det i \hat{D}(\theta) - i n \det i \hat{D}(0) \right] \\ &= \lim_{M \rightarrow \infty} \sum_k - \left(\tan^{-1} \frac{M |\lambda_k| \cos \varphi_k(\theta)}{|\lambda_k|^2 + M |\lambda_k| \sin \varphi_k(\theta)} \right. \\ &\quad \left. - \tan^{-1} \frac{M |\lambda_k| \cos \varphi(0)}{|\lambda_k|^2 + M |\lambda_k| \sin \varphi(0)} \right) \\ &= \sum_k (\varphi_k(\theta) - \varphi_k(0)). \end{aligned} \quad (17)$$

Hence, the total winding number is given by the sum of the winding numbers of the single eigenvalues.

This completes our construction: the winding number of the fermionic determinant family $\det i \hat{D} [A^{t,\theta}]$, with the gauge potential (8), along the rim of D^+ coincides with the index of the four-dimensional Dirac operator coupled to the potential A , if the same gauge function g_n is used, i.e., index and winding number calculated from g_n coincide.

This treatment can be extended to the three-dimensional parity anomaly,¹⁰ thereby providing a clarification of some controversial points about the structure of the eigenvalue flow that generates that anomaly.¹¹ These results, as well as details of the above calculations, will be presented in a forthcoming paper.

ACKNOWLEDGEMENTS

I thank Professor R. Jackiw for suggesting this problem, and for his assistance and guidance during this work; R. Percacci for his help and for explaining to me the methods of ref. [6]; L. Alvarez-Gaumé and P. Ginsparg for their interest and suggestions, and P. Sodano and M. Bos for discussions. I acknowledge financial support from Fondazione Angelo Della Riccia.

REFERENCES

1. For a general review on anomalies and topology, see R. Jackiw, in S. Treiman, R. Jackiw, B. Zumino and E. Witten, "Current Algebra and Anomalies", World Scientific, Singapore, 1985.
2. T. Eguchi, P. B. Gilkey and A. J. Hanson, *Phys. Rep.* **66**, 213 (1980).
3. M. Atiyah and I. Singer, *Proc. Nat. Acad. Sci. U.S.A.*, **81**, 2597 (1984).
4. L. Alvarez-Gaumé and P. Ginsparg, *Nucl. Phys.* **B243**, 449 (1984).
5. A. A. Belavin, A. M. Polyakov, A. S. Schwartz and Y. S. Tyupkin, *Phys. Lett.* **59B**, 85 (1975).
6. N. K. Pak and R. Percacci, *Nucl. Phys.* **B188**, 355 (1981).
7. M. M. Anousirian, *Phys. Lett.* **70B**, 301 (1977); N. K. Nielsen and B. Schroer, *Nucl. Phys.* **B127**, 493 (1977).
8. R. Jackiw and C. Rebbi, *Phys. Rev.* **D14**, 517 (1976); R. Jackiw and C. Rebbi, *Phys. Rev.* **D16**, 1052 (1977).
9. V. de Alfaro, S. Fubini and G. Furlan, *Phys. Lett.* **65B**, 163 (1976).
10. A. N. Redlich, *Phys. Rev. Lett.* **52**, 18 (1984); A. N. Redlich, *Phys. Rev.* **D29**, 2366 (1984).
11. Y. I. Kogan and A. Y. Morozov, *Sov. J. Nucl. Phys.* **41**, 693 (1985).