FUNCTIONAL REPRESENTATION FOR THE TWO-DIMENSIONAL CONFORMAL GROUP*

R. Floreanini** and R. Jackiw
 Center for Theoretical Physics
 Laboratory for Nuclear Science
 and Department of Physics
 Massachusetts Institute of Technology
 Cambridge, Massachusetts 02139 U.S.A.

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** Permanent address: Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Trieste, Italy.

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ABSTRACT

A functional projective representation for the two-dimensional conformal group is explicitly constructed. The representation functionals are propagation kernels for self-dual fields.

In two dimensions, conformal transformations comprise an infinite dimensional Lie group, whose action on coordinates of Minkowski space $[x^\mu \equiv (t, x)]$, metric $g_{\mu\nu} = \text{diag}(1, -1)$ takes light-cone components $x^\pm \equiv \frac{1}{\sqrt{2}}(t \pm x)$ into arbitrary functions of themselves: $x^+ \rightarrow X^+(x^+)$, $x^- \rightarrow X^-(x^-)$. The infinitesimal transformation $\delta_f x^\mu \equiv X^\mu - x^\mu = -f^\mu$, satisfies the conformal Killing equation, $\partial_\mu f_\nu + \partial_\nu f_\mu = g_{\mu\nu} \partial_\lambda f^\lambda$, and a Lie algebra

$$[\delta_f, \delta_g] = -\delta_{(f,g)}, \quad (f, g)^\mu \equiv f^\alpha \partial_\alpha g^\mu - g^\alpha \partial_\alpha f^\mu \quad (1)$$

The finite transformation X^μ is related to the infinitesimal f^μ by $X^\pm(x^\pm) = \mathcal{F}_\pm^{-1}(\tau + \mathcal{F}_\pm(x^\pm))$, $\mathcal{F}_\pm(z) \equiv -\int^\tau \frac{dw}{\mathcal{F}_\pm(w)}$, where τ is a homotopy parameter.

Many two-dimensional field theories are invariant under this group of transformations,¹ and also conformal symmetry plays a central role in string theory.² Here we consider a massless spinless free field Φ which transforms as

$$\delta_f \Phi = f^\alpha \partial_\alpha \Phi \quad (2)$$

and construct a functional representation for the finite group elements in terms of functional matrix elements and propagation kernels. In the following, we restrict the discussion to one component of the group, transforming x^+ but not x^- .

In a canonical, fixed-time ($t = 0$) description, one deals with a pair of conjugate variables Φ and $\Pi = \dot{\Phi}$ which depend on x (but not on t) and the transformation (2) reads $\delta_f \Phi = \frac{1}{2} f(\Pi + \Phi')$, $\delta_f \Pi = \frac{1}{2} f'(\Pi + \Phi') + \frac{1}{2} f'(\Pi + \Phi')$, or in terms of the self-dual combination $\chi \equiv (\Pi + \Phi')/\sqrt{2}$

$$\delta_f \chi = (f\chi)' \quad (3)$$

Here, $f(x) = \sqrt{2} f^+(x/\sqrt{2})$ and the dash signifies derivation with respect to the argument. The quantity

$$Q_f = \frac{1}{2} \int dx f(x) \chi^2(x) \quad (4)$$

generates the infinitesimal transformation,

$$i[Q_f, \chi] = \delta_f \chi \quad (5)$$

as is verified with the help of

$$[X(x), X(y)] = i\delta'(x-y) \equiv k(x, y) . \quad (6)$$

The brackets denote either classical Poisson brackets (times i) or quantum mechanical commutators. It is recognized that the generator Q_J involves the $(++)$ component of a conserved, symmetric and traceless energy-momentum tensor.¹

In the classical theory, the generators satisfy the Lie algebra of the group (1), but in the quantum theory, there is a central extension owing to the necessary presence of a Schwinger term in the commutators of the energy-momentum tensor.^{3*}

$$[Q_J, Q_\delta] = iQ_{(J,\delta)} + \frac{i}{48\pi} \int dx (f'''g - g'''f) , \quad (f, g) \equiv fg' - gf' \quad (7)$$

The difference arises from the fact that the product X^2 is not well-defined in the quantum theory. The conventional way of handling this problem is to consider the Fock space of the quantum theory, and to normal order X^2 with respect to the Fock vacuum. However, it is possible to arrive at (7) by constructing a representation of the Lie group, observed to be projective, and no reference to the Fock vacuum need be made.

In order to build a well-defined generator, $f(x)$ is promoted to a real, symmetric bilocal function $F(x, y)$ which in the local limit becomes $f(x)\delta(x-y)$. The regulated generator**

$$Q_F = \frac{1}{2} \int dx dy X(x) F(x, y) X(y) \quad (8)$$

is not beset by singularities: it satisfies the algebra without center***

$$[Q_F, Q_G] = iQ_{(F,G)} , \quad i(F, G) \equiv FKG - GFK , \quad (9)$$

but Q_F does not possess a local limit. As we shall see, a local limit can be taken if we add a c -number to Q_F , which is linear in F . So we define

$$:Q_F: \equiv Q_F - \frac{1}{4} \text{tr} \left\{ F^{1/2} \Omega F^{1/2} \right\} \quad (10)$$

and the formal definition (4) is replaced by $Q_J = \lim_{F \rightarrow J} :Q_F: \equiv \Omega(x, y)$ will be determined by the requirement that the representation of $e^{-i\tau Q_F}$ possesses a local limit. We emphasize that the colons do not signify normal ordering with respect to the

* There is also a conformally invariant theory in which the classical Lie algebra has a central extension. This is the Liouville model, which realizes conformal transformations non-linearly; see Ref. [4].

** Of course, Q_F no longer generates a symmetry, but it is a generator of a linear canonical transformation. The implementation of such transformations in quantum field theory without specifying Fock states was studied by Segal.⁵

*** Henceforth, we use a functional matrix notation.

Fock vacuum: they are defined by (10). Since the regulator is a c -number, it does not affect the infinitesimal transformation law (5), nor the finite transformation

$$e^{i\tau Q_J} X(x) e^{-i\tau Q_J} = \frac{f(X)}{f(x)} X(X) , \quad X = \mathcal{F}^{-1}(\tau + \mathcal{F}(x)) , \quad \mathcal{F}(x) \equiv \int_x^\infty \frac{dz}{f(z)} . \quad (11)$$

However, non-linear relations like (9) are changed.

$$[:Q_F: , :Q_G:] = i :Q_{(F,G)}: + \frac{i}{4} \text{tr} \{ (F, G) \Omega \} \quad (12)$$

To construct the representation, we consider eigenstates $|\varphi\rangle$ of Φ .

$$\Phi(x)|\varphi\rangle = \varphi(x)|\varphi\rangle , \quad \langle\varphi_1|\varphi_2\rangle = \delta(\varphi_1 - \varphi_2) \quad (13)$$

The scalar product involves a functional δ -function and functional integration is used to sum over φ . In this basis, X is represented by

$$\langle\varphi_1|X(x)|\varphi_2\rangle = \left(\frac{1}{i} \frac{\delta}{\delta\varphi_1(x)} + \varphi_1'(x) \right) \delta(\varphi_1 - \varphi_2) . \quad (14)$$

We seek the form of

$$U(\varphi_1, \varphi_2; \tau F) \equiv \langle\varphi_1|e^{-i\tau Q_F}|\varphi_2\rangle = U^*(\varphi_2, \varphi_1; -\tau F) , \quad (15)$$

which must be a Gaussian in φ_1 and φ_2 since Q_F is quadratic in X . U may be determined by solving the Schrödinger-like equation obtained by differentiating (15) with respect to τ , evaluating the matrix element of Q_F from (14) and using the boundary condition (13). The result is

$$U(\varphi_1, \varphi_2; \tau F) = e^{\Gamma(\tau F)} e^{i \int dx \varphi_1'(x) \varphi_2(x)} \\ \times \exp \frac{i}{2} \int dx dy [\varphi_1(x) - \varphi_2(x)] K_{\tau F}(x, y) [\varphi_1(y) - \varphi_2(y)] \quad (16)$$

where

$$K_{\tau F} = F^{-1/2} \left\{ k_F \text{ctn} \left(\frac{\tau}{2} k_F \right) \right\} F^{-1/2} \quad (17a)$$

$$\Gamma(\tau F) = -\frac{1}{2} \text{tr} \ln \left\{ F^{1/2} \frac{2\pi i \sin \left(\frac{\tau}{2} k_F \right)}{k_F} F^{1/2} \right\} \quad (17b)$$

$$k_F \equiv F^{1/2} k_F F^{1/2} . \quad (17c)$$

The functions of the kernels are defined either by power series, or by diagonalizing k_F .

The composition law for U may be verified. Only trigonometric identities are needed to prove $\int D\varphi_2 U(\varphi_1, \varphi_2; \tau_1 F) U(\varphi_2, \varphi_3; \tau_2 F) = U(\varphi_1, \varphi_3; (\tau_1 + \tau_2) F)$. However, the more general rule

$$\int D\varphi_2 U(\varphi_1, \varphi_2; \tau F) U(\varphi_2, \varphi_3; \tau G) = U(\varphi_1, \varphi_3; \tau F \circ G) \quad (18)$$

can be established only in a τ power series, since $F \circ G$ is given by such a series: $F \circ G = F + G + \frac{1}{2}(F, G) + \dots$. We have verified (18) through $O(\tau^2)$.

Let us now examine the local limit $F \rightarrow f$. The spectrum of k_f is the real line and the normalized and complete eigenfunctions are

$$\int dy k_f(x, y) \psi_\lambda^f(y) = \lambda \psi_\lambda^f(x) \quad (19)$$

$$\psi_\lambda^f(x) = \frac{1}{\sqrt{2\pi f(x)}} \exp \left\{ -i\lambda \int^x \frac{dz}{f(z)} \right\}.$$

Consequently,

$$K_{\tau f}(x, y) = \frac{1}{f(x)} \int \frac{d\lambda}{2\pi} \lambda \operatorname{ctn} \left(\frac{\tau}{2} \lambda \right) \exp \left\{ -i\lambda \int_y^x \frac{dz}{f(z)} \right\} \frac{1}{f(y)}. \quad (20)$$

The λ integration requires a prescription for handling the singularities: continue to the imaginary axis $\tau \rightarrow -i\tau'$, evaluate the integral and continue back $\tau' \rightarrow i\tau + 0^+$. This gives

$$K_{\tau f}(x, y) = -i\pi \frac{1}{\tau f(x)} P \operatorname{csc}^2 \left\{ \pi \int_y^x \frac{dz}{\tau f(z)} \right\} \frac{1}{\tau f(y)} \quad (21)$$

(P means principal value.) For the prefactor, we proceed as follows. After continuation, we have

$$\begin{aligned} \Gamma(-i\tau' F) &= -\frac{1}{2} \operatorname{tr} \ln \left\{ F^{1/2} \frac{2\pi \sinh \left(\frac{\tau'}{2} k_F \right)}{k_F} F^{1/2} \right\} \\ &= -\frac{1}{2} \operatorname{tr} \ln \left\{ F^{1/2} \frac{2\pi \sinh \left(\frac{\tau'}{2} \omega_F \right)}{\omega_F} F^{1/2} \right\} \\ &= \frac{1}{2} \operatorname{tr} \ln \left(\frac{1}{\pi} \omega \right) - \frac{1}{2} \operatorname{tr} \ln \left\{ 1 - e^{-\tau' \omega_F} \right\} - \frac{\tau'}{4} \operatorname{tr} \omega_F \end{aligned} \quad (22)$$

where $\omega_F(x, y) \equiv |k_F|(x, y)$ and $\omega(x, y) \equiv |k|(x, y) = \int \frac{d\lambda}{2\pi} e^{-i\lambda(x-y)} |\lambda| = -P \frac{1}{\pi(x-y)^2}$. The first term is an infinite τ' -independent constant, $\ln Z \equiv \frac{1}{2} \operatorname{tr} \ln \left(\frac{1}{\pi} \omega \right) \sim \frac{1}{2} \int dx \int \frac{d\lambda}{2\pi} \ln \frac{1}{\pi} |\lambda|$, which may be removed by redefining the functional integration measure $D\varphi \rightarrow Z^{-1} D\varphi$. The second attains a finite local limit.

$$\gamma(-i\tau' f) = -\frac{1}{2} \int \frac{dx}{f(x)} \int \frac{d\lambda}{2\pi} \ln(1 - e^{-\tau' |\lambda|}) = \frac{\pi}{12} \int \frac{dx}{\tau' f(x)} \quad (23)$$

The third has no local limit. However, we observe that

$$\begin{aligned} \omega_f(x, y) &= \frac{1}{f^{1/2}(x)} \int \frac{d\lambda}{2\pi} |\lambda| \exp \left\{ -i\lambda \int_y^x \frac{dz}{f(z)} \right\} \frac{1}{f^{1/2}(y)} \\ &= -\frac{1}{f^{1/2}(x)} P \frac{1}{\pi \left(\int_y^x \frac{dz}{f(z)} \right)^2} \frac{1}{f^{1/2}(y)} \\ &= \frac{1}{f^{1/2}(x)} \frac{(x-y)^2}{\left(\int_y^x \frac{dz}{f(z)} \right)^2} \frac{1}{f^{1/2}(y)} \omega(x, y) \end{aligned} \quad (24)$$

Since the quantity multiplying ω in the last formula of (24) becomes $f(x)$ for $x = y$, we see that*

$$\operatorname{tr} \omega_F = \operatorname{tr}(F^{1/2} \omega_F^{1/2}) + \left\{ \text{terms that vanish in the local limit} \right\}. \quad (25)$$

Thus, if we chose Ω in (10) to coincide with ω , $e^{-i\tau F} Q_F$ has the representation

$$Z^{-1/2} \langle \varphi_1 | e^{-i\tau F} Q_F | \varphi_2 \rangle Z^{-1/2} = Z^{-1} e^{\frac{i\tau}{2} \operatorname{tr}(F\omega)} U(\varphi_1, \varphi_2; \tau F) \quad (26)$$

which tends in the local limit to

$$\begin{aligned} : U : (\varphi_1, \varphi_2; \tau f) &= \exp -i \left\{ \frac{\pi}{12} \int \frac{dx}{\tau f(x)} - \int dx \varphi_1'(x) \varphi_2(x) \right\} \times \\ &\exp \frac{\pi}{2} \int dx dy [\varphi_1(x) - \varphi_2(x)] \frac{1}{\tau f(x)} P \operatorname{csc}^2 \left(\pi \int_y^x \frac{dz}{\tau f(z)} \right) \frac{1}{\tau f(y)} [\varphi_1(y) - \varphi_2(y)] \end{aligned} \quad (27)$$

But the composition law changes. From (26) it follows that a projective rule with a 2-cocycle is satisfied.

$$\int D\varphi_2 : U : (\varphi_1, \varphi_2; \tau F) : U : (\varphi_2, \varphi_3; \tau G) = e^{i\omega_2(F, G)} : U : (\varphi_1, \varphi_3; \tau F \circ G) \quad (28)$$

The 2-cocycle in the local limit is given by**

$$\omega_2(f, g) = \lim_{F \rightarrow f} \frac{\tau}{4} \operatorname{tr} \{ (F + G - F \circ G) \omega \}. \quad (29)$$

* Here is a shortcoming in our argument. We do not have sufficient control over ω_F to establish (25) without the singular manipulations in (24). However, we have confidence in our result, since we see *a posteriori* that it agrees with the Fock space approaches to the problem.

** Formulas for this 2-cocycle also appear in Refs. [6]. Presumably, they are cohomological to ours.

Before the limit is taken, the cocycle is trivial: it is a coboundary which arises because the phase of U has been redefined, and ω_2 can similarly be removed. However, after the local limit, the nontrivial expression remains which to $O(\tau^2)$ produces the center in (7). This is seen as follows. The $O(\tau^2)$ contribution to (29) is $-\frac{\tau^2}{8} \text{tr} \{ \tilde{F}, \tilde{G} \} \omega$. In terms of a Fourier representation $\tilde{F}(x, y) = \int \frac{dp}{2\pi} \frac{dq}{2\pi} e^{-i\frac{p}{2}(x+y)} e^{-iq(x-y)} \tilde{F}(p, q)$, $\tilde{F}(p, q) = \tilde{F}(p, -q)$, $\tilde{F}(p, q) \xrightarrow{p \rightarrow -j} \tilde{f}(p) = \int dx e^{ipx} f(x)$, and similarly for $\tilde{G}$, this reads

$$-\frac{\tau^2}{8} \text{tr} \{ \tilde{F}, \tilde{G} \} \omega = \frac{i\tau^2}{8} \text{tr} \{ (F k G - G k F) \omega \} \\ = \frac{i\tau^2}{8} \left\{ \int \frac{dp}{2\pi} \frac{dq}{2\pi} \epsilon \left(q + \frac{p}{2} \right) \left[\frac{d}{dq} \left(\frac{q^3}{3} - \frac{p^2 q}{4} \right) \right] \tilde{f}(p, q) \tilde{g}(-p, q) - (F \leftrightarrow G) \right\}.$$

Next, an integration by parts in q is performed. No surface terms survive owing to the rapid decrease of $\tilde{F}$ and $\tilde{G}$ for large q . Then, in the local limit, the q derivative annihilates $\tilde{F}$ and $\tilde{G}$, and we are left with $-\frac{i\tau^2}{96\pi} \int \frac{dp}{2\pi} p^3 \left\{ \tilde{f}(p) \tilde{g}(-p) - \tilde{g}(p) \tilde{f}(-p) \right\} = -\frac{\tau^2}{2} \frac{1}{48\pi} \int dx (f''' g - g''' f)$, which gives the center in (7).

Note the following further properties of the representation functional (27). : U : is invariant against the replacement of $\varphi_1(x)$ by $\varphi_1(X(x))$. This is a consequence of the identity $\frac{dX}{f(X)} = \frac{dx}{f(x)}$ and the coordinate invariance of the symplectic structure $\int dx \varphi_1'(x) \varphi_2(x) = -\int \varphi_1 d\varphi_2$. The occurrence of this symplectic form is natural for self-dual scalar fields. A Lagrangian for a self-dual field χ is $L_\chi = \frac{1}{2} \int dx dy \chi(x) \epsilon(x-y) \dot{\chi}(y) - \frac{1}{2} \int dx \chi^2(x)$, and leads to the Hamiltonian $H_\chi = \frac{1}{2} \int dx \chi^2(x) = Q_\chi$. Canonical quantization of L_χ produces precisely the commutator (6). So we recognize that : U : $(\varphi_1, \varphi_2; \tau)$ is the propagation kernel for time evolution of a self-dual field. Moreover, Q_f and : U : $(\varphi_1, \varphi_2; \tau f)$ may be viewed as the generator and propagation kernel for a self-dual field $f\chi$, whose propagation is described with time intervals that vary in space as $\Delta t = \frac{1}{f(x)}$.

Thus far, we have avoided discussing the Fock states of our massless free field. Let us remark in conclusion that the representation functional may be used to exhibit the effect of a conformal transformation on them. The Hamiltonian $H = \frac{1}{2} \int dx (\Pi^2 + \Phi^2)$ possesses the ground state functional $\Psi_0(\varphi) = e^{-\frac{1}{2} \int dx dy \varphi(x) \omega(x, y) \varphi(y)}$. The occurrence of precisely the kernel ω in Ψ_0 shows, of course, that our group theoretically motivated subtraction in Q_F coincides with the conventional subtraction. From (25), we see that ω_f may also be written as $\omega_f(x, y) = f^{1/2}(x) \left\{ \frac{1}{f(x)} \omega \left(x, \int^x \frac{dx}{f(x)} \int^y \frac{dy}{f(y)} \right) \frac{1}{f(y)} \right\} f^{1/2}(y)$. A Gaussian with the kernel as in the curly brackets is an eigenstate of the unsubtracted Q_f , Eq. (4), with eigenvalue $\frac{1}{2} \text{tr} \omega_f$. Hence, we may also view our regularization of Q_f as normal ordering with respect to the "vacuum" of Q_f .

An arbitrary Gaussian

$$\Psi(\varphi) = e^{-\frac{1}{2} \int dx dy \varphi(x) \Omega(x, y) \varphi(y)} \quad (30)$$

changes under the transformation f into $\Psi_f(\varphi)$ which is given by

$$\Psi_f(\varphi) = \int D\varphi_1 : U : (\varphi, \varphi_1; \tau f) \Psi(\varphi_1) = e^{\sigma(\tau f)} e^{-\frac{1}{2} \int dx dy \varphi(x) \Omega_{\tau f}(x, y) \varphi(y)} \quad (31)$$

where

$$\Omega_{\tau f} = \Omega - (\Omega + k)(\Omega - iK_{\tau f})^{-1}(\Omega - k) \quad (32a)$$

and

$$\sigma(\tau f) = \gamma(\tau f) - \frac{1}{2} \text{tr} \ln(\Omega - iK_{\tau f}). \quad (32b)$$

For the vacuum, we have $(\omega + k)(\omega - k) = 0$: hence, $\Omega_{\tau f} = \omega$ whenever $(\omega \pm k)$ commutes with $(\omega - iK_{\tau f})$. This always occurs for the restricted $SO(2, 1)$ conformal transformations ($f = 1, x, x^2$) for which there is no center in the Lie algebra, and the vacuum is invariant (up to a phase). Of course, the general conformal transformation cannot transform the vacuum into itself owing to the center.

By repeated action of (32), one may verify that $\Omega_{\tau f}$ gives a representation for the Lie algebra without center, while the center resides in the composition law for the σ 's.

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