CERN-TH.4424/86

REGULARIZED DETERMINANTS AND NON-PERTURBATIVE
DEFINITION OF CHIRAL ANOMALIES

D. Verstegen *)
CERN -- Geneva

ABSTRACT

We develop a non-perturbative regularization scheme for fermions interacting with external gauge fields. It is used to obtain the axial anomalies in theories with (classical) global and local axial-vector symmetries. The main result is that in the latter case, this regularization produces automatically the consistent anomaly, provided the cut-off operator is vector gauge covariant.

*) Member of the FNRS (Belgium).

1. - INTRODUCTION

An important feature of fermion field theories is the appearance of anomalies in the divergence of axial vector currents, despite the formal (or "classical") chiral invariance of the theory. This has been shown (for a global axial symmetry) by using perturbation theory¹⁾ but also with non-perturbative methods²⁾. The common feature of many of the regularization schemes that have been devised to establish the anomalous divergence equation is that they respect vector gauge invariance, while they explicitly break axial symmetry. For example, Fujikawa²⁾ showed that the axial variation of the fermionic integration measure (in the path integral formalism) involves $\text{Tr} \gamma_5$. ("Tr" is the functional trace, not just the Dirac trace). Regularizing this ill-defined quantity with the gauge covariant cut-off operator $\not{D}^2$, as in

$$(\text{Tr } \gamma_5)_{\text{reg.}} = \lim_{\Lambda \rightarrow \infty} \text{Tr } \gamma_5 e^{-\not{D}^2/\Lambda^2} \quad (1.1)$$

yields the anomaly, while the non gauge invariant expression $\text{Tr} \gamma_5 e^{-\not{D}^2/\Lambda^2}$ does not. A bothersome aspect of this calculation is that, although gauge invariant, $\text{Tr} \gamma_5 e^{-\not{D}^2/\Lambda^2}$ yields a vanishing anomaly. We will come back to this later.

The situation for a theory with γ_5 couplings (i.e., local axial symmetries) is worse: requiring vector gauge invariance does not yield a uniquely defined anomaly^{3),4)}. We consider here fermions in even-dimensional Euclidean space-time, coupled to external gauge fields V_μ^a and A_μ^a . The effective action W for these fields is given by:

$$Z[V, A] \equiv e^{-W[V, A]} \equiv \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \exp \left\{ - \int d^d x (\bar{\Psi} \not{D} \Psi + \mathcal{L}_{\text{gauge}}) \right\} \quad (1.2)$$

with

$$\not{D} \equiv \gamma_\mu D_\mu \equiv -i \left(\not{\partial} + \not{V} + \gamma_5 \not{A} \right) = -i \left(\not{\partial} + \not{V}^L \left(\frac{1+\gamma_5}{2} \right) + \not{V}^R \left(\frac{1-\gamma_5}{2} \right) \right),$$

$$V_\mu \equiv V_\mu^a T^a, \quad A_\mu \equiv A_\mu^a T^a, \quad V_\mu^L \equiv V_\mu - A_\mu, \quad V_\mu^R \equiv V_\mu + A_\mu \quad (1.3)$$

(T^a are the anti-Hermitian generators of the gauge group).

As is well known, gauge invariance of the effective action and current conservation are equivalent. Consider for example an infinitesimal vector gauge transformation in a vector theory:

$$\begin{aligned}
 Z[V] \rightarrow Z[V + [D, \omega]] &\simeq Z[V] + \int d^d x [D_\mu, \omega]^a \frac{\delta Z}{\delta V_\mu^a} \\
 &\simeq Z[V] - \int d^d x \omega^a [D_\mu, \frac{\delta Z}{\delta V_\mu^a}]
 \end{aligned}
 \tag{1.4}$$

from which follows that the current $j_\mu^a \equiv -(1/Z) (\delta Z / \delta V_\mu^a)$ is covariantly conserved. This can be made more rigorous by regularizing the theory in a gauge invariant manner.

The problem of gauge invariance of Z (or W) can be viewed in two ways:

- one performs (left- or right-handed) gauge transformations in the usual way on $\psi, \bar{\psi}, V$ and A ; the action is invariant but, like in the global U(1) case, the fermionic measure $\mathcal{D}\bar{\psi}\mathcal{D}\psi$ is not. Its chiral variation has been calculated in the Fujikawa spirit^{3),4)} and using the zeta-function regularization⁵⁾. It is now recognized that the choice of $\not{D}^2$ as cut-off operator leads to the same anomaly as the one calculated perturbatively using point-splitting⁶⁾ or Pauli-Villars regulator fields⁷⁾, and that on the other hand, the insertion of $e^{-\not{D}^2/\Lambda^2}$ or $e^{-\not{D}_V^2}$ in $\text{Tr} \gamma_5 T^a$ yields another expression. Our notation is:

$$\not{D}^+ = -i(\not{D} + \not{V} - \gamma_5 \not{A}) ; \not{D}_V = -i(\not{D} + \not{V}) \tag{1.5}$$

- one transforms only the gauge fields; doing a transformation on the fermions is pointless because they are just integration variables. From now on, we stick to this viewpoint. In a (V,A) theory, a vector gauge invariant regularization leads to:

$$\frac{\delta W_{\text{reg}}}{\delta \omega_{L,R}^a} \equiv G_a^{L,R} \neq 0 \tag{1.6}$$

($\omega_{L,R}^a$ stands for the parameter associated with an infinitesimal left or right gauge transformation and thus $[D_\mu^{L,R}, J_a^\mu] = G_a^{L,R}$). Computing the commutator of two gauge transformations on W, Wess and Zumino⁸⁾ obtained consistency conditions for the anomaly; these conditions fix completely the whole anomaly once one of its parts (for example the one involving only V_μ) is known. This consistent anomaly has been obtained by Bardeen⁶⁾, Brown et al.⁷⁾ and by non-perturbative calculations with $\not{D}^2$ as

cut-off operators⁴⁾. Reuter uses the zeta-function technique^{*)} to define a regularized determinant¹⁰⁾. This approach is in principle better than those of Refs. 3)-5) because all expressions are well defined at every stage of the calculation. It nevertheless suffers from the same drawbacks as the previous ones: the choice of $\not{D}^2$ as regulator operator yields the consistent anomaly, while $\not{D}^\dagger \not{D}$ does not (and the reason for this is not clear).

The plan of the paper is as follows. In Section 2, we present a non-perturbative regularization in which all vector gauge invariant cut-off operators lead to the correct anomaly in the global U(1) case and to the consistent anomaly in the local non-Abelian case. It is inspired by the finite mode regularization proposed in Ref. 11) and developed in Ref. 12). The general arguments of Section 2 are illustrated with explicit calculations in Section 3 (Abelian case) and Section 4 (non-Abelian case). Section 5 contains our conclusions.

2. - REGULARIZED DETERMINANT

The idea of Andrianov, Bonora and Gamboa-Saravi^{11),12)} was to make the fermion functional integral a well-defined quantity by keeping only a finite number of eigenvalues λ_i of $\not{D}$ (the space is taken to be a finite box, so that the spectrum is discrete):

$$Z_{reg} \equiv \prod_{|\lambda_i| < \Lambda} \lambda_i = \det(1 - P_\Lambda + P_\Lambda \not{D} P_\Lambda) \quad (2.1)$$

where P_Λ projects on the finite subspace of eigenvectors of $\not{D}$ with $|\lambda_i| < \Lambda$, i.e.,

$$P_\Lambda = \theta(\Lambda^2 - \not{D}^2) \quad (2.2)$$

Our version of $\not{D}_{reg}$ is slightly different:

$$\not{D}_\Lambda \equiv \not{D} P + i\Lambda(1 - P) \quad (2.3)$$

*) The zeta-function technique has also been used by J. Lott and I.M. Singer⁹⁾, although in a more general and abstract framework.

where P is a smooth vector gauge covariant operator, close to $\mathbb{1}$ for small (compared to Λ) eigenvalues of the operator(s) it involves, and going rapidly to zero for large eigenvalues. Examples of P are:

$$e^{-A/\Lambda^2}, \quad e^{-A^2/\Lambda^4}, \quad \left(1 + \frac{A}{\Lambda^2}\right)^{-\ell}$$

(where ℓ is a large enough positive integer), with $A = \not{D}^2, \not{D}_V^2, \not{D}^* \not{D}, D^2, \dots$. We could also use $\not{D}_r = P \not{D} P + i\Lambda(\mathbb{1} - P)$ but this is an unnecessary complication.

The determinant of an infinite-dimensional operator which is not Hermitian and not even normal ($[\not{D}, \not{D}^*] \neq 0$) is an object whose existence is not guaranteed by standard theorems. To avoid potential problems, we define the regularized effective action as:

$$W_\Lambda = -\ln Z_\Lambda \equiv -\int_0^1 dt \quad \text{Tr} \left(\not{D}_r(t)^{-1} \frac{\partial \not{D}_r(t)}{\partial t} \right) \quad (2.4)$$

where $\not{D}_r(t)$ contains (everywhere) tV_μ and tA_μ instead of V_μ and A_μ ¹³⁾. We now sketch the proof that with the regularization introduced before this is a well-defined, finite quantity (for finite Λ). The functional trace of the operator L depending on $-i\partial^\mu$ is evaluated as follows (the dependence on V and A is implicit and for simplicity we take space to be infinite)

$$\text{Tr} L = \int d^d k \, d^d x \, d^d x' \, \langle k | x \rangle \, \hbar \langle x | L | x' \rangle \, \langle x' | k \rangle \quad (2.5)$$

(tr stands throughout this paper for the trace over Dirac and colour indices).

Using

$$\begin{aligned} \langle k | x \rangle &= (2\pi)^{-d/2} e^{-ikx} ; \quad \langle x | L[-i\partial_x^\mu] | x' \rangle = L[-i\partial_x^\mu] \delta^d(x-x'); \\ -i\partial_x^\mu e^{ikx} &= e^{ikx} (-i\partial_x^\mu + k^\mu) \end{aligned} \quad (2.6)$$

we obtain:

$$\text{Tr} L = \int d^d x \, \frac{d^d k}{(2\pi)^d} \, \hbar L[-i\partial_x^\mu + k^\mu] \quad (2.7)$$

(with the implicit prescription that ∂_x^μ with nothing to its right yields 0). It is easy to show that for smooth V and A fields, the U.V behaviour of the k integral appearing in (2.7) is well behaved. Because of the shift $\not{D} \rightarrow \not{D} + \not{K}$, the choice $P = e^{-\not{D}(t)^2/\Lambda^2}$ (for example) gives rise to

$$e^{-(\not{D}(t)+K)^2/\Lambda^2} = e^{-(k^2 + \text{terms in } ktV, ktA, t^2VV\dots)/\Lambda^2} \approx e^{-k^2/\Lambda^2} \text{ for large } k^2.$$

The leading term (for large k^2) of $\not{D}_r(t)^{-1}$ is $(i\Lambda)^{-1}$ while that of $\partial \not{D}_r(t)/\partial t$ is proportional to e^{-k^2/Λ^2} : the integral converges thus for large k . (The small k region presents no problem.) To make W a well-defined functional, we assume further that the V_μ and A_μ fields vanish outside a finite region of space-time. Equation (2.4) differs by the irrelevant constant $\ln \not{D}_r(t=0)$ from the equation we would obtain after naïvely integrating the right-hand side.

Denoting δ an infinitesimal gauge transformation, we obtain

$$\begin{aligned} \delta W_R &= - \int_0^1 dt \operatorname{Tr} \left[\delta \left(\frac{\partial \not{D}_R(t)}{\partial t} \right) \cdot \not{D}_R(t)^{-1} + \frac{\partial \not{D}_R(t)}{\partial t} \cdot \delta (\not{D}_R(t)^{-1}) \right] \\ &= - \operatorname{Tr} \left[\delta (\not{D}_R(t=1)) \cdot \not{D}_R(t=1)^{-1} - \delta (\not{D}_R(t=0)) \cdot \not{D}_R(t=0)^{-1} \right] \\ &\quad + \int_0^1 dt \operatorname{Tr} \left[(\delta \not{D}_R(t)) \cdot \frac{\partial \not{D}_R(t)}{\partial t} + \frac{\partial \not{D}_R(t)}{\partial t} \cdot \not{D}_R(t)^{-1} \cdot (\delta \not{D}_R(t)) \cdot \not{D}_R(t)^{-1} \right] \end{aligned}$$

after integrating by parts the first term and

$$\delta W_R = - \operatorname{Tr} \left[\not{D}_R^{-1} \cdot \delta \not{D}_R \right] \quad (2.8)$$

after using the cyclicity of the trace and $\delta \not{D}_r(t=0) = 0$.

We would have found the same result had we applied δ to the naïve definition: $W = -\ln \det \not{D}_r$.

The regularization (2.3) was already proposed in Ref. 14), but the point of view of that paper (which dealt only with vector theories) was rather different. The reason why our method always yields the correct $U(1)$ anomaly is that both the effective action and the axial current are regularized in a vector gauge invariant way. The proof that it always leads to the consistent anomaly in a (V, A) theory proceeds in two steps: the calculation of that part of the anomaly containing only V_μ is very close that of the global $U(1)$ anomaly. The Wess-Zumino consistency conditions are automatically satisfied because we calculated the anomaly as the chiral variation of a well-defined effective action. This means that here the manipulations made to obtain the consistency conditions are meaningful and not just formal. The facts that the V_μ part comes out correctly and that the consistency conditions are automatic lead thus to the consistent anomaly for all vector gauge covariant cut-off operators. We think that this "universality" puts the non-perturbative proof of the anomalies on a more solid footing.

Although it seems formal this reasoning is perfectly sound. Before illustrating it with a few calculations, we state the differences between the approach of Andrianov and Bonora and the present one:

- they do not recognize properly the connection with the W-Z condition: they infer from their calculations that taking $\not{D}^2$ as cut-off operator leads to the consistent anomaly and "intend to discuss elsewhere the problem of choosing different bases" (i.e., different cut-off operators in our language);
- our formula (2.3) contains a factor $i\Lambda$ in front of $(1-P)$;
- using a smooth function like "exp" makes calculations simpler than with the step function θ ;
- we dropped the unnecessary requirement that the system should be in a finite box (this amounts to replacing sums over the momentum k by integrals).

3. - GLOBAL U(1) ANOMALY

Before turning to the next section on non-Abelian local anomalies, we use our method to rederive the classical global anomaly. This is meant as a warm-up exercise but is also of interest on its own.

We consider a vector theory with a semi-simple gauge group G . We use the trick of extending this to $G \otimes U(1)$ (with generators T^a and $\mathbf{1}$ respectively): $V_\mu = V_\mu^a T^a + V_\mu^0 \mathbf{1}$. This will allow us to obtain the global U(1) axial current through the infinitesimal U(1) axial transformation $\not{D}^0 \rightarrow \not{D}^0 + \gamma_5 [\not{D}, \omega]$, $\omega(x) = \omega_5^0(x) \mathbf{1}$. At the end of the calculation, we put $V_\mu^0 = 0$, which brings us back to the ordinary divergence equation: $\partial_\mu j_{\mu 5} = \text{anomaly}$.

We perform thus the transformation $\delta_{\text{axial}} \not{D} = \gamma_5 [\not{D}, \omega]$ on the regularized effective action:

$$W_\Lambda = - \ln Z_\Lambda = - \text{Tr} \ln [\not{D} P + i\Lambda(1-P)]$$

$$\text{with } P = P[\not{D}^2/\Lambda^2] \begin{cases} \simeq 1 & \text{for } \not{D}^2/\Lambda^2 \ll 1 \\ \rightarrow 0 & \text{rapidly for } \not{D}^2/\Lambda^2 \gg 1 \end{cases} \quad (3.1)$$

$$\delta W_2 = \text{Tr} \left[\left(\not{D} P + i \Lambda (1 - P) \right)^{-1} \left(\gamma_5 [\not{D}, \omega] P + (\not{D} - i \Lambda) \cdot \delta P \right) \right] \quad (3.2)$$

Our γ -matrix conventions are:

$$\begin{aligned} \{\gamma_\mu, \gamma_\nu\} &= 2 \delta_{\mu\nu} \quad ; \quad \gamma_\mu^+ = \gamma_\mu \\ \gamma_5 &= i \gamma_1 \dots \gamma_d \quad \text{if } d = 2 + 4p \\ \gamma_5 &= \gamma_1 \dots \gamma_d \quad \text{if } d = 4p \quad (p \text{ is integer}) \end{aligned} \quad (3.3)$$

Using the cyclicity of the trace and the identity $[\not{D}, P] = 0$, we obtain:

$$\frac{\delta W}{\delta \omega_5} = 2 \hbar \gamma_5 \cdot 1 \int \frac{d^d k}{(2\pi)^d} \left(\not{D} P + i \Lambda (1 - P) \right)^{-1} \left(\not{D} P + 2 (\not{D} - i \Lambda) \frac{\not{D}^2}{\Lambda^2} \cdot P' \right) \quad (3.4)$$

where $\not{D} = \not{D} + \not{K}$, P is now a functional of $\not{D}^2/\Lambda^2$ and P' is the derivative of P with respect to its argument $\not{D}^2/\Lambda^2$.

$$\begin{aligned} \not{D}^2 &= (D_\mu + k_\mu)(D_\mu + k_\mu) - \frac{1}{4} [\gamma_\mu, \gamma_\nu] (\partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu]) \\ &\equiv (D + k)^2 - \frac{1}{4} [\gamma_\mu, \gamma_\nu] F_{\mu\nu} \equiv (D + k)^2 + F \end{aligned} \quad (3.5)$$

$$\begin{aligned} \frac{\delta W}{\delta \omega_5} &= 2 \hbar \gamma_5 \cdot 1 \Lambda^d \int \frac{d^d k}{(2\pi)^d} \left[\left(\frac{F}{\Lambda^2} + K^2 \right) P^2 + (1 - P)^2 \right]^{-1} \cdot \\ &\cdot \left[\left(\frac{F}{\Lambda^2} + K^2 \right) (P^2 + P \cdot P' \cdot \left(\frac{F}{\Lambda^2} + K^2 \right)) - 2(1 - P) \cdot P' \cdot \left(\frac{F}{\Lambda^2} + K^2 \right) \right] \end{aligned} \quad (3.6)$$

We have rescaled $k \rightarrow k\Lambda$ and have written K_μ for $k_\mu + D_\mu/\Lambda$. F is the only object containing γ matrices; it is easy to show that in the $\Lambda \rightarrow \infty$ limit, one needs to keep only the term of order $F^{d/2}$ in this expression and one may replace K^2 by k^2 . The relevant term is singled out using the operator $(1/n!)(\delta^n/\delta F^n)|_{F=0}$ (with $n \equiv d/2$). This is equivalent to inserting $(1/n!)(\delta^n/\partial x^n)$ (with $x = k^2$) inside the above integral with $F = 0$:

$$\frac{\delta W}{\delta \omega_s^0} = 2 t \gamma_s \cdot \frac{F^n}{n!} \frac{2 \pi^n}{(n-1)! (2\pi)^d} \cdot \frac{1}{2} \int dx x^{n-1} \frac{\partial^n A(x)}{\partial x^n} \quad (3.7)$$

with

$$A(x) = (x P^2 + (1-P)^2)^{-1} \cdot (x P^2 + 2 x^2 \cdot P \cdot P' - 2 x (1-P) P') \quad (3.8)$$

(P has the same dependence here on x as it had before on $\not{D}^2/\Lambda^2$). Integrating by parts, we find¹⁵⁾:

$$\partial_\mu j_{\mu 5}^0 = - \frac{\delta W}{\delta \omega_s^0} \Big|_{V_\mu^0=0} = c_d \cdot \varepsilon_{\alpha_1 \beta_1 \dots \alpha_n \beta_n} t_{\text{colour}} F_{\alpha_1 \beta_1} \dots F_{\alpha_n \beta_n} \quad (3.9)$$

with $c_d = \frac{-2}{n! (4\pi)^n} \begin{cases} -i \\ 1 \end{cases}$ if $\begin{cases} d=2+4p \\ d=4p \end{cases}$

if we identify

$$j_{\mu 5}^0 = (\gamma_5 \gamma_\mu)_{ij} \frac{\delta W_\pi}{\delta \chi_{ij}^0} \Big|_{V_\mu^0=0} \quad (3.10)$$

and provided $P(0) = 1$, where $P(x)$ is smooth around 0 and goes rapidly to 0 for $x \rightarrow \infty$. The current $j_{\mu 5}^0$ is regularized because it comes from a regularized effective action.

We turn next to the choice of $D^2 = -(\partial_\mu + V_\mu)(\partial_\mu + V_\mu)$ as regulator operator^{*)}. The calculations are more complicated because D^2 does not commute with $\not{D}$ and we have only done them in two and four dimensions.

$$\delta_A D_\mu = -\gamma_5 [D_\mu, \omega] \Rightarrow \delta_A D^2 = -\gamma_5 [D^2, \omega] \Rightarrow \delta_A P = -\gamma_5 [P, \omega] \quad (3.11)$$

Denoting $\not{D}_r \equiv \not{D}P + i\Lambda(1-P)$, we get:

$$\begin{aligned} \frac{\delta W_\pi}{\delta \omega_s} &= - \text{Tr} \gamma_5 \left(-P \not{D}_r^{-1} \not{D} - \not{D} P \not{D}_r^{-1} - \not{D}_r^{-1} (\not{D} - i\Lambda) P + P \not{D}_r^{-1} (\not{D} - i\Lambda) \right) \\ &= 2 \text{Tr} (\gamma_5 \not{D} P \not{D}_r^{-1}) \end{aligned} \quad (3.12)$$

For the sake of simplicity, we choose $P(D^2/\Lambda^2) = e^{-D^2/\Lambda^2}$, but the following calculation can easily be generalized. Proceeding as before, we find that P

*) It has also been suggested by E. Brézin and D. Friedan that any gauge covariant cut-off operator should yield the correct global U(1) anomaly if calculated with the Andrianov-Bonora method (private communication through O. Napolé).

gives rise to $e^{-(k+D/\Lambda)^2}$ and $\not{D}_r$ to

$$\left(k + \frac{\not{D}}{\Lambda}\right) e^{-(k + \frac{D}{\Lambda})^2} + i \Lambda \left(1 - e^{-(k + \frac{D}{\Lambda})^2}\right)$$

$$\simeq \left[\not{k} e^{-k^2} + i(1 - e^{-k^2})\right] \Lambda + \text{terms of order } \Lambda^0, \Lambda^{-1}, \dots$$

We expand next P and $\not{D}_r^{-1}$ in powers of Λ^{-1} . All terms are of the form

$$f(k^2) \cdot \Lambda^{d-l} \cdot \{l \text{ powers of } D_\mu \text{ coming from } \not{D}, k \cdot D \text{ or } D^2, \text{ with insertions of } \not{K}\}$$

It is easy to show that for trace reasons (i) terms containing D^2 vanish and (ii) there may be at most one $k \cdot D$ factor. We may thus write

$$e^{-(k + \frac{D}{\Lambda})^2} \simeq e^{-k^2} \left(1 - 2 \frac{k \cdot D}{\Lambda}\right) \quad (3.13)$$

Denoting $x = k^2$ and $y = e^{-x}$, we need to invert

$$\left[\not{k} y + i(1-y)\right] \Lambda + y \left[\not{D} - 2 \not{D} \frac{k \cdot D}{\Lambda} - 2(\not{k} - i) k \cdot D\right]$$

This will give rise to powers of $xy^2 + (1-y)^2 \equiv f$ in the denominator. Without the factor "i" in $\not{D}_{\text{reg}} = \not{D}P + i\Lambda(1-P)$, we would find instead $xy^2 - (1-y)^2$, which vanishes for a finite value of k^2 . After some straightforward algebra, we obtain in two dimensions:

$$\frac{\delta W_\Lambda}{\delta \omega_5} = 2 \int \frac{d^2 k}{(2\pi)^2} f^{-2} \left[y^2 (1-y)^2 - 2xy^2(1-y) \right] \text{tr } \not{\gamma}_5 \not{D}^2$$

$$= 2 \int_0^\infty \frac{\pi dx}{(2\pi)^2} \frac{\partial}{\partial x} \left(f^{-1} x y^2 \right) \text{tr } \not{\gamma}_5 \not{D} \left(D^2 - \frac{i}{4} [\not{\gamma}_\mu, \not{\gamma}_\nu] F_{\mu\nu} \right) \quad (3.14)$$

which agrees with the result of the previous section. Similarly, the calculation in four dimensions gives the right result too. For the sake of simplicity we have evaluated numerically the integral over $x = k^2$. (This can be done with great accuracy if one treats analytically the regions of small and large x .) We have also used an analytical integration routine and have obtained the same answer.

4. - LOCAL NON-ABELIAN ANOMALIES

We consider the theory defined in Eqs. (1.2) and (1.3) and regularized as in Eqs. (2.3) and (2.4). Left and right chiral transformations take the form

$$\delta_L = \frac{1}{2} (\delta_V - \delta_A) \quad ; \quad \delta_R = \frac{1}{2} (\delta_V + \delta_A)$$

$$\delta_V \mathcal{D} = [\mathcal{D}, \omega] \quad ; \quad \delta_A \mathcal{D} = \gamma_5 [\mathcal{D}, \omega] \quad , \quad \omega = \omega^a T^a \quad (4.1)$$

It is easy to check that, as expected, vector gauge covariant cut-off operators like $\mathcal{D}^2$, $\mathcal{D}^\dagger \mathcal{D}$, D^2 , $\mathcal{D}_V^2$, ..., lead to $\delta_V W_{\text{reg}} = 0$. On the other hand $\delta_A W_r = 0$. In this section, we concentrate on operators which were found unsatisfactory in previous approaches.

$(\delta W / \delta \omega_{\text{axial}}^a)$ contains two types of terms: those involving even powers of A_μ ("abnormal parity" terms) which come with an $\varepsilon_{\mu\nu\lambda\theta}$ in front, and those involving odd powers of A_μ ("normal parity" terms). The former cannot be eliminated by adding local counterterms in V_μ and A_μ to the action (without spoiling vector gauge invariance). As a general feature, the latter are unphysical^{13),16)}; they depend on the details of the regularization scheme and not only on the choice of the cut-off operator. All previous authors found that they can be eliminated by adding counterterms to the action. For these reasons, we calculate only the physical, abnormal parity part of the anomaly (except in one instance, in order to illustrate the previous considerations).

We take first $P = e^{-\mathcal{D}^\dagger \mathcal{D} / \Lambda^2}$

$$\delta_A \mathcal{D}^\dagger = -\gamma_5 [\mathcal{D}^\dagger, \omega] \quad ; \quad \delta_A P = -\gamma_5 [P, \omega]$$

$$\delta_A W_A = 2 \text{Tr} \gamma_5 \omega \mathcal{D} P [\mathcal{D} P + i \Lambda (\mathcal{D} - P)]^{-1} \quad (4.2)$$

Before proceeding, let us remark that the part of $\delta_A W$ involving only V_μ is easy to calculate: it is given by:

$$2 \text{Tr} \gamma_5 \omega \mathcal{D}_V P_V [\mathcal{D}_V P_V + i \Lambda (\mathcal{D} - P_V)]^{-1}$$

where $P_V = e^{-\mathcal{D}_V^2 / \Lambda^2}$,

$$\text{i.e.} \quad 2 \text{Tr} \gamma_5 \omega \mathcal{D}_V^2 P^2 [\mathcal{D}_V^2 P_V^2 + \Lambda^2 (\mathcal{D} - P_V)^2]^{-1} \quad (4.3)$$

After that the calculation of the functional trace is like in the global U(1) case (the only difference is the presence of $\omega = \omega^a T^a$ in the trace). Similar considerations apply to the other cases considered in this section. We return now to the complete computation.

We expand in powers of Λ^{-1} and drop the terms that contribute only to the normal parity part of the anomaly. P gives rise to:

$$e^{-k^2} \left(1 - 2 \frac{k \cdot D}{\Lambda} - \frac{D^+ D}{\Lambda^2} + \frac{k \cdot D D^+ D}{\Lambda^3} + \frac{D^+ D k \cdot D}{\Lambda^3} + \frac{1}{2} \frac{(D^+ D)^2}{\Lambda^4} + \dots \right) \quad (4.4)$$

After a good deal of algebra, we find:

$$\begin{aligned} \frac{\delta W_A}{\delta \omega^a} = & 2 \hbar \gamma_5 T^a \int \frac{d^4 k}{(2\pi)^4} \left\{ \left[f^{-2} y^2 (1-y)(x+y-1) \right. \right. \\ & + f^{-3} y^3 \left((1-y - \frac{3x}{2})(1-y)(xy+y-1) - \frac{x^2 y}{2} \right) + \frac{xy^6}{2f^4} (1-y)(2x+y-1) \Big] D(D^+ D + D^+ D) D \\ & \left. + f^{-4} y^4 (1-y)^3 (1-y-2x) D^4 \right\} \end{aligned} \quad (4.5)$$

[$x = k^2$; $y = e^{-x}$ and $f = xy^2 + (1-y)^2$ as before]. The qualification "modulo normal parity terms" is implicit throughout this section. Performing the k integrals we obtain:

$$\frac{\delta W_A}{\delta \omega^a} = \frac{1}{48\pi^2} \hbar \gamma_5 T^a (D^4 + D D^+ D D + D D D^+ D) \quad (4.6)$$

Some more algebra yields the well-known result:

$$\begin{aligned} \frac{\delta W_A}{\delta \omega^a} = & \frac{1}{4\pi^2} \varepsilon_{\mu\nu\lambda\theta} \hbar_{\text{colour}} T^a \left[\frac{1}{4} V_{\mu\nu} V_{\lambda\theta} + \frac{1}{12} A_{\mu\nu} A_{\lambda\theta} \right. \\ & \left. + \frac{2}{3} A_\mu A_\nu A_\lambda A_\theta - \frac{2}{3} (A_\mu A_\nu V_{\lambda\theta} + A_\mu V_{\nu\lambda} A_\theta + V_{\mu\nu} A_\lambda A_\theta) \right] \end{aligned} \quad (4.7)$$

with

$$V_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu] + [A_\mu, A_\nu], \quad (4.8a)$$

$$A_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [V_\mu, A_\nu] + [A_\mu, V_\nu] \quad (4.8b)$$

We now comment on the normal parity terms of order Λ^2 appearing in $\delta W / \delta \omega^a$. Taking into account the term $2(k \cdot D)^2 / \Lambda^2$ dropped in (4.4), we find:

$$\begin{aligned} - 2 \Lambda^2 \hbar_{\text{colour}} T^a \int \frac{d^4 k}{(2\pi)^4} f^{-2} y^2 \left((y-1+x)^2 + x^2 \right) [\partial_\mu + V_\mu, A_\mu] \\ \equiv 2a \Lambda^2 [\partial_\mu + V_\mu, A_\mu] \end{aligned} \quad (4.9)$$

Such a term can be eliminated by replacing $W[V, A]$ by

$$\tilde{W}[V, A] = W[V, A] - a \Lambda^2 \int_{\text{column}} \frac{1}{\Lambda} A_\mu A_\mu d^4x \quad (4.10)$$

In contradistinction to the other integrals of this paper, a [Eq. (4.9)] is non-universal: another choice than $\exp(-)$ for P (but with the same behaviour at 0 and ∞) would yield another value for a .

We consider next the case $P = e^{-D^2/\Lambda^2}$. The axial variations of P and W_r are as in the previous case. The simplifications indicated in Section 3 (for $A_\mu = 0$) remain valid here, but care must be exercised when interchanging $\not{X}$ with $k \cdot D$ or $\not{D}$:

$$\begin{aligned} \not{X} k \cdot D &= k \cdot D^+ \not{X}, \text{ where } D_\mu^+ = -i(\partial_\mu + V_\mu + A_\mu \gamma_5) \\ \not{X} \not{D} &= -\not{D}^+ \not{X} + 2 k \cdot D \\ \not{D} \not{X} &= -\not{X} \not{D}^+ + 2 k \cdot D^+ \end{aligned} \quad (4.11)$$

The result is:

$$\begin{aligned} \frac{\delta W_r}{\delta \omega^a} &= 2 \frac{1}{\Lambda} \gamma_5 T^a \int \frac{d^4k}{(2\pi)^4} (y/f)^4 (2x+y-1) [(1-y)^3 \not{D}^4 \\ &\quad + \frac{xy^2(1-y)}{2} \not{D}(\not{D} \not{D}^+ + \not{D}^+ \not{D}) \not{D}] \end{aligned} \quad (4.12)$$

which yields again (4.6) and (4.7).

Finally, we take $P = e^{-\not{D}_V^2/\Lambda^2}$. We use $\delta_A \not{D}_V = -i[\not{X}, \omega]$ and the identity

$$\delta e^X = \int_0^1 dt e^{(1-t)X} (\delta X) e^{tX} \quad (4.13)$$

to obtain:

$$\begin{aligned} \delta_A W_r &= T_r \gamma_5 \omega \left\{ -P \not{D}_r^{-1} \not{D} - \not{D} P \not{D}_r^{-1} \right. \\ &\quad \left. + \frac{i}{\Lambda^2} \gamma_5 \int_0^1 dt [\Gamma \not{D}_V \not{X} - \not{X} \Gamma \not{D}_V + \not{D}_V \Gamma \not{X} - \not{X} \not{D}_V \Gamma] \right\} \end{aligned} \quad (4.14)$$

where

$$\Gamma \equiv e^{-t \not{D}_V^2/\Lambda^2} \not{D}_r^{-1} (\not{D} - i\Lambda) e^{-(1-t) \not{D}_V^2/\Lambda^2} \quad (4.15)$$

The purely vector part comes out correctly because it is equal to that computed for $P = e^{-\not{D}\not{D}/\Lambda^2}$. The remaining algebra being more involved, we calculated only the terms involving A^4 , in $\delta W/\delta\omega^a_{\text{axial}}$:

$$\begin{aligned} & -\frac{1}{8\pi^2} \text{tr} \delta_5 T^a A^4 \int_0^1 x (y/f)^4 (1-y) (xy^2 - (1-y)^2) (2x+y-1) dx \\ & = -\frac{1}{48\pi^2} \text{tr} \delta_5 T^a A^4 \end{aligned} \quad (4.16)$$

which is the right answer.

5. - CONCLUSIONS

Our method for regularizing fermion gauge theories consists of a modification and reinterpretation of an earlier proposal by Andrianov, Bonora, and Gamboa-Saravi^{11),12)}. It provides well-defined effective actions and currents. We have checked that because of this, it leads to the correct global U(1) anomaly for all vector gauge covariant cut-off operators.

We have also argued that the same condition on cut-off operators yields automatically the consistent anomaly for currents associated with local non-Abelian axial-vector symmetries. Our general argument is supported by several explicit computations. This represents theoretical progress over other regularization schemes in which some vector gauge covariant regulator operators lead to the consistent anomaly (for unclear reasons) while others do not.

ACKNOWLEDGEMENTS

I am grateful to M. Forger, O. Lechtenfeld, O. Nappi, R. Petronzio, S. Samuel and R. Stora for fruitful discussions and/or for reading the manuscript.

REFERENCES

- 1) S. Adler, Phys. Rev. 177 (1969) 2426;
J. Bell and R. Jackiw, Nuovo Cimento 60A (1969) 47.
- 2) K. Fujikawa, Phys. Rev. Lett. 42 (1979) 1195; Phys. Rev. D21 (1980) 2848;
G.A. Christos, Z. Phys. C18 (1983) 155.
- 3) D.W. McKay and B.-L. Young, Phys. Rev. D28 (1983) 1039;
K. Fujikawa, Phys. Rev. D29 (1984) 285.
- 4) M.B. Einhorn and D.R.T. Jones, Phys. Rev. D29 (1984) 331;
S.K. Hu, B.-L. Young and D.W. McKay, Phys. Rev. D30 (1984) 836.
- 5) A.P. Balachandran, G. Marmo, V.P. Nair and C.G. Trahern, Phys. Rev. D25 (1982) 2713.
- 6) W. Bardeen, Phys. Rev. 184 (1969) 1848.
- 7) R.W. Brown, C.C. Shih and B.-L. Young, Phys. Rev. 186 (1969) 1491.
- 8) J. Wess and B. Zumino, Phys. Lett. 37B (1971) 95.
- 9) J. Lott, Commun. Math. Phys. 97 (1985) 371;
I.M. Singer, Cartan Symposium Lyon (1984).
- 10) M. Reuter, Phys. Rev. D31 (1985) 1374; see also:
R.E. Gamboa-Saravi, M.A. Muschietti, F.A. Shaposnik and J.E. Solomin, Ann. Phys. 157 (1984) 360.
- 11) A. Andrianov, L. Bonora and R. Gamboa-Saravi, Phys. Rev. D26 (1982) 2821.
- 12) A. Andrianov and L. Bonora, Nucl. Phys. B233 (1984) 232, 247.
- 13) H. Leutwyler, BUTP preprint 84/33 (1984).
- 14) D. Versteegen, Phys. Rev. D29 (1984) 2405.
- 15) P.H. Frampton and T.W. Kephart, Phys. Rev. D28 (1983) 1010;
P.K. Townsend and G. Sierra, Nucl. Phys. B222 (1983) 493; second reference of Ref. 12).
- 16) L. Alvarez-Gaumé and E. Witten, Nucl. Phys. B234 (1983) 269.