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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

CONSTRAINED HAMILTONIAN FORMALISM
FOR UNCONSTRAINED CLASSICAL MECHANICS

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1. INTRODUCTION

Recently, an alternative approach for constructing a Lagrangian from its equation of motion (usually known as the inverse problem of the calculus of variations) has been presented ¹⁾. It has been shown that any Lagrangian from which the regular n-dimensional second order system of differential equations

$$\ddot{x}^i = F^i(x^j, \dot{x}^j, t) \quad (I.1)$$

can be derived, may be written (up to a total time derivative) as a linear combination of its own equations of motion (I.1).

As it is stated in Ref.1 this universality is much wider than the usual T-V construction which is valid only when forces are derivable from a potential.

On the other hand it is sometimes impossible to find a (second order) Lagrangian depending on $x^j, \dot{x}^j$ and t only, from which arbitrary equations of the type (I.1) could be obtained ^{2),3)}.

These results are of fundamental importance not only in the understanding of classical systems, but for their quantizations. In fact, the whole set of equivalent Lagrangians (and Hamiltonians) of a given physical system can be treated on the same footing the peculiarities distinguishing one from another being completely contained in the coefficients of the linear combination. There are, however, two important issues to be addressed before any quantization scheme is attempted:

- a) the complete canonical treatment for acceleration-dependent Lagrangian systems; and b) an interpretation for the ambiguities residing in the infinite classes of classical equivalent Lagrangians. In the present article we discuss the above-mentioned problems at the classical level. The treatment of the quantization problem will be presented elsewhere ⁴⁾.

In section II the canonical formalism for Lagrangians $L(x, \dot{x}, t)$ is presented, following the method of Lanczos ⁵⁾. This method introduces a number of auxiliary variables and the resulting constraints are treated in section III using Dirac's formalism for constrained Hamiltonian systems ⁶⁾. Section IV is devoted to the discussion of the transformations that link equivalent Lagrangians. Some open problems and prospects at the quantum level are presented in section V.

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ABSTRACT

The Hamiltonian formalism for the multidimensional equation $\ddot{x} = F(x, \dot{x}, t)$ is presented. The link between the whole variety of Hamiltonians for a given F is established. It is shown that two equivalent Lagrangians are connected by a chain of successive transformations of the form $L \rightarrow KL + \dot{M}$ where both K and M are constants of the motion.

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II. a-LAGRANGIANS AND THEIR HAMILTONIANS

The so-called inverse problem of the calculus of variations for classical mechanics has been largely studied^{1)-3), 7)} and a variety of approaches exists. For the present purposes however, it is sufficient to recall that the whole set of s-equivalent Lagrangians⁸⁾ that originate the equations of motion

$$\ddot{x}^i - F^i(x^j, \dot{x}^j, t) = 0 \quad (II.1)$$

can be written as¹⁾

$$I_s = L(x^j, \dot{x}^j, t) = \mu_i(x^j, \dot{x}^j, t) [\ddot{x}^i - F(x^j, \dot{x}^j, t)] \quad (II.2)$$

provided

$$\frac{\partial \mu_i}{\partial \dot{x}^j} = \frac{\partial \mu_j}{\partial \dot{x}^i} \quad (II.3)$$

$$\frac{d^2}{dt^2} \mu_i + \frac{d}{dt} \left(\mu_j \frac{\partial F^j}{\partial \dot{x}^i} \right) - \mu_j \frac{\partial F^j}{\partial x^i} = 0 \quad (II.4)$$

$$\det \left[\frac{\partial}{\partial \dot{x}^j} \left(\frac{d}{dt} \mu_i + \mu_k \frac{\partial F^k}{\partial \dot{x}^i} \right) + \frac{\partial \mu_j}{\partial x^i} \right] \neq 0 \quad (II.5)$$

The symbol $\frac{d}{dt}$ stands for on-shell time derivative i.e.

$$\frac{d}{dt} = \frac{d}{dt} \Big|_{\ddot{x}^i - F^i = 0} = F^i \frac{\partial}{\partial \dot{x}^i} + \dot{x}^i \frac{\partial}{\partial x^i} + \frac{\partial}{\partial t} \quad (II.6)$$

The complete set of solutions of (II.3), (II.4) and (II.5) is given by

$$\mu_i = \left(C_1 \frac{\partial x_2}{\partial \dot{x}^i} + C_3 \frac{\partial C_4}{\partial \dot{x}^i} + \dots + C_{2n-1} \frac{\partial C_{2n}}{\partial \dot{x}^i} \right) \quad (II.7)$$

where the $C_i = C_i(x^j, \dot{x}^j, t)$ are $2n$ functionally independent constants of the motion of the equations (II.1).

Care should be taken in using the correct Euler-Lagrange (E-L) operator E_i to derive equations (II.1) from Lagrangian (II.2) i.e.

$$E_i = - \frac{d^2}{dt^2} \frac{\partial}{\partial \dot{x}^i} + \frac{d}{dt} \frac{\partial}{\partial x^i} - \frac{\partial}{\partial x^i} \quad (II.8)$$

We now want to emphasize that (II.2) - together with (II.7) - solves completely the problem of finding a Lagrangian from the equations of motion. Any other Lagrangian $\tilde{L}$, that gives rise to the same equations is necessarily of the same form as L modulo a total time derivative of a function of the coordinates, the velocities and time:

$$\tilde{L} = \tilde{\mu}_i (\ddot{x}^i - F^i) + \dot{\lambda}(x^j, \dot{x}^j, t) \quad (II.9)$$

The Hamiltonian formalism for Lagrangian (II.9) will be constructed closely following Lanczos' procedure for Lagrangians containing higher than first derivatives⁵⁾. By adding extra variables, the variational problem will be converted into another one which does not involve derivatives higher than first. From now on, indices will be dropped and we will work formally in one dimension but all the results can be extended to an arbitrary number of degrees of freedom.

Let us define new variables v and a as

$$\dot{x} = v \quad (II.10)$$

$$\dot{v} = a \quad (II.11)$$

Then we can incorporate (II.10) and (II.11) into (II.9). Besides, to ensure that the above equations hold, we add them to the Lagrangian as subsidiary (constraint) conditions with appropriate Lagrange multipliers ω and α that should be considered as new variables as well.

Thus, our starting point will be the modified Lagrangian

$$L' = \mu(x, v, t) [a - F(x, v, t)] + \omega(\dot{x} - v) + \alpha(\dot{v} - a) + \lambda_x a + \lambda_v v + \lambda_t \quad (\text{II.12})$$

(λ_x denotes differentiation with respect to x etc.)

The equation of motion

$$\ddot{x} - F(x, \dot{x}, t) = 0, \quad (\text{II.13})$$

the defining equations (II.10), (II.11) and the equations for the Lagrange multipliers ω and α are obtained from L' by independent variations of x, v, a, ω and α , as can easily be verified.

To construct the associated Hamiltonian, the canonical momenta conjugated to x, v, a, ω and α must be calculated. From (II.12)

$$\begin{aligned} p_x &\equiv \frac{\partial L}{\partial \dot{x}} = \omega & (\text{II.14.a}) \\ p_v &\equiv \frac{\partial L}{\partial \dot{v}} = \alpha & (\text{II.14.b}) \\ p_a &\equiv \frac{\partial L}{\partial \dot{a}} = 0 & (\text{II.14.c}) \\ p_\omega &\equiv \frac{\partial L}{\partial \dot{\omega}} = 0 & (\text{II.14.d}) \\ p_\alpha &\equiv \frac{\partial L}{\partial \dot{\alpha}} = 0 & (\text{II.14.e}) \end{aligned}$$

The Hamiltonian is then

$$H = p_x v + p_v a - \mu(a - F) - \lambda_x v a - \lambda_v v - \lambda_t \quad (\text{II.15})$$

The equations (II.14) constitute a set of constraints in the sense that they do not allow ^{us} to solve the velocities in terms of the momenta and the coordinates. Consequently, to reach a consistent Hamiltonian picture use of Dirac's formalism for constrained systems ⁶⁾ has to be made.

III. DIRAC'S METHOD FOR $\ddot{x} - F(x, \dot{x}, t) = 0$

Due to the fact that equations (II.14) do not contain velocities at all, there are five primary constraints

$$\phi_1 = p_x - \omega \approx 0 \quad (\text{III.1})$$

$$\phi_2 = p_v - \alpha \approx 0 \quad (\text{III.2})$$

$$\phi_3 = p_\omega \approx 0 \quad (\text{III.3})$$

$$\phi_4 = p_\alpha \approx 0 \quad (\text{III.4})$$

$$\phi_5 = p_a \approx 0 \quad (\text{III.5})$$

In order to get the correct equations of motion with the usual Poisson brackets (P.B.) one must allow arbitrary linear combinations of these constraints to be added to the Hamiltonian (II.15):

$$\begin{aligned} H' &= H + \lambda^i \phi_i \\ &= p_x v + p_v a - \mu(a - F) - \partial \lambda + \lambda^i \phi_i \end{aligned} \quad (\text{III.6})$$

Here the λ^i are Lagrange multipliers and the $\mathcal{D}$ denotes total time derivative of any function of $x, \dot{x}$, and t , where $\dot{x}$ and $\ddot{x}$ have been replaced by v and a respectively.

The Poisson brackets are

$$[x^A, p_B] = \delta_B^A \quad (\text{III.7})$$

with the identification $x^1 = x, x^2 = v, x^3 = a, x^4 = \omega, x^5 = \alpha_j$ etc.

Consistency requires that the constraints must be preserved in time, so their time derivatives should also be weakly zero,

$$\dot{\phi}_i = [\phi_i, H'] + \frac{\partial \phi_i}{\partial t} \approx 0 \quad (\text{III.8})$$

$$\dot{\phi}_1 = [\mu(a-F) + \lambda_{1v}a + \lambda_{1x}v + \lambda_{1t}]_x - \lambda_3 \approx 0 \quad (\text{III.9})$$

$$\dot{\phi}_2 = -p_x + [\mu(a-F) + \lambda_{1v}a + \lambda_{1x}v + \lambda_{1t}]_v - \lambda_4 \approx 0 \quad (\text{III.10})$$

$$\dot{\phi}_3 = \lambda_1 \approx 0 \quad (\text{III.11})$$

$$\dot{\phi}_4 = \lambda_2 \approx 0 \quad (\text{III.12})$$

$$\dot{\phi}_5 = -p_v + \mu + \lambda_{1v} \approx 0 \quad (\text{III.13})$$

Equations (III.9) to (III.12) determine $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, whereas equation (III.13) is a new (secondary) constraint.

Let us impose that this new constraint,

$$\phi_6 = -p_v + \mu + \lambda_{1v} \approx 0 \quad (\text{III.14})$$

be maintained in time:

$$\begin{aligned} \dot{\phi}_6 &= \partial \mu + \partial(\lambda_{1v}) - [p_v, H'] \\ &= \partial \mu + \partial(\lambda_{1v}) + p_x - [\mu(a-F)]_v - (\partial \lambda)_{1v} \approx 0 \end{aligned} \quad (\text{III.15})$$

Equation (III.15) is a new constraint that by virtue of the algebraic identities

$$\partial(\lambda_{1v}) - (\partial \lambda)_{1v} = -\lambda_x \quad (\text{III.16})$$

and

$$\partial \mu - [\mu(a-F)]_v = (\mu F)_{1v} + \mu_{1x}v + \mu_{1t} \quad (\text{III.17})$$

can be written as

$$\phi_7 = p_x - \lambda_x + (\mu F)_{1v} + \mu_{1x}v + \mu_{1t} \approx 0 \quad (\text{III.18})$$

Imposing once again that the recently born constraint must be preserved in time we get an equation for $\dot{\lambda}^6$, and hence no new constraints appear in the theory.

The equations of motion are derived from the total Hamiltonian (where all the constraints are now incorporated),

$$H_T = p_x v + p_v a - \mu(a-F) \partial \lambda + \lambda^i \phi_i \quad i=1, \dots, 7 \quad (\text{III.19})$$

in the usual way:

$$\dot{x} = [x, H_T] = v + \lambda^1 + \lambda^7 \quad (\text{III.20.a})$$

$$\dot{v} = [v, H_T] = a + \lambda^2 - \lambda^6 \quad (\text{III.20.b})$$

$$\dot{\omega} = [\omega, H_T] = \lambda^3 \quad (\text{III.20.c})$$

$$\dot{\alpha} = [\alpha, H_T] = \lambda^4 \quad (\text{III.20.d})$$

$$\dot{a} = [a, H_T] = \lambda^5 \quad (\text{III.20.e})$$

$$\begin{aligned} \dot{p}_x &= [p_x, H_T] = [\mu(a-F) \partial \lambda]_x - \lambda^6 (\mu + \lambda_{1v})_x - \\ &\quad - \lambda^7 [(\mu F)_{1v} + \mu_{1x}v + \mu_{1t} - \lambda_x]_x \end{aligned} \quad (\text{III.20.f})$$

$$\dot{p}_v = [p_v, H_T] = -p_x + [\mu(a-f) \partial \lambda]_{,v} - \lambda^6 (\mu + \lambda_v)_{,v} - \lambda^7 [(\mu f)_{,v} + \mu_{,v} v + \mu_{,v} - \lambda_{,v}]_{,v} \quad (\text{III.20.g})$$

$$\dot{p}_\omega = [p_\omega, H_T] = \lambda^1 \quad (\text{III.20.h})$$

$$\dot{p}_\alpha = [p_\alpha, H_T] = \lambda^2 \quad (\text{III.20.i})$$

$$\dot{p}_a = [p_a, H_T] = -p_v + \mu + \lambda_{,v} \quad (\text{III.20.j})$$

Out of all the constraints ϕ_i , $i = 1 \dots 7$, only ϕ_5 is first class (it has vanishing P.B. with all the rest), it occurs because a has no dynamical equations and therefore p_a is an arbitrary parameter of the theory. The six remaining constraints are second class (have non-vanishing P.B. among themselves) and arise from the introduction of the auxiliary variables v, a, ω, α , and their conjugate momenta. These variables are redundant in the sense that they do not correspond to genuine independent degrees of freedom and therefore should be eventually eliminated from the theory. This can be done in a systematic manner by defining the Dirac brackets ⁹⁾. We do this in two stages:

a) eliminate ϕ_1, ϕ_2, ϕ_3 and ϕ_4 - defining the one-star bracket $[,]^*$ - and b) eliminate ϕ_6, ϕ_7 - via the two-star brackets $[,]^{**}$

a) Define

$$C_{ij} = [\phi_i, \phi_j] \quad \phi_i = 1, \dots, 4 \quad (\text{III.21})$$

and the brackets

$$[\phi_i, \phi_j]^* = [\phi_i, \phi_j] - [\phi_i, \phi_k] C_{kj}^{-1} [\phi_k, \phi_j] \quad (\text{III.22})$$

where C^{-1} is the inverse matrix of C defined in (III.21)

$$C^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad (\text{III.23})$$

The only non-vanishing brackets of any pair of variables of the set x, v, ω, α, a and their conjugate momenta are:

$$[x, \omega]^* = [x, p_x]^* = [v, \alpha]^* = [v, p_v]^* = [a, p_a]^* = 1 \quad (\text{III.24})$$

Since the star bracket of any dynamical function with the constraints

$\phi_1 \dots \phi_4$ vanishes identically, we can now set these constraints strongly to zero, solve the corresponding strong equations for redundant variables $\omega, \alpha, p_\omega, p_\alpha$ and substitute them whenever they occur. This amounts to simply dropping $\phi_1 \dots \phi_4$ from H_T :

$$H^* = H_T |_{\phi_1, \dots, \phi_4=0} = p_x v + p_v a - \mu(a-f) - \partial \lambda + \lambda^1 \phi_i \quad i = 5, 6, 7 \quad (\text{III.25})$$

The only remaining variables are now x, v, a and their conjugate momenta.

In order to

b) eliminate ϕ_6 and ϕ_7 we define the 2x2 matrix

$$D_{ij} = [\eta_i, \eta_j] \quad i, j = 1, 2 \quad (\text{III.26})$$

and the (two-star) brackets

$$[\eta_i, \eta_j]^{**} = [\eta_i, \eta_j]^* - [\eta_i, \eta_k] D_{kj}^{-1} [\eta_k, \eta_j] \quad (\text{III.27})$$

where, by definition $DD^{-1} = I$, and $\eta_i = \phi_{5+i}$.

$$D = \begin{pmatrix} 0 & \Phi \\ -\Phi & 0 \end{pmatrix} \quad (\text{III.28})$$

with

$$\Phi = [(\mu F)_{,v} + \mu_{,x} v + \mu_{,t}]_{,v} + \mu_{,x} \quad (\text{III.29})$$

The quantity Φ is different from zero as required by equation (II.5) so that

$$D^{-1} = \begin{pmatrix} 0 & -\Phi^{-1} \\ \Phi^{-1} & 0 \end{pmatrix} \quad (\text{III.30})$$

is well defined.

Upon setting $\phi_6 = \phi_7 = 0$, the only subsisting independent dynamical variables are x and v . Hence, the only independent, non-trivial bracket left, is:

$$[x, v]^{**} = - \frac{1}{[(\mu F)_{,v} + \mu_{,x} v + \mu_{,t}]_{,v} + \mu_{,x}} \quad (\text{III.31})$$

The constraints η_1 and η_2 are now set strongly equal to zero so

$$p_v = \mu + \lambda_{,v} \quad (\text{III.32})$$

and

$$p_x = \lambda_{,x} - [(\mu F)_{,v} + \mu_{,x} v + \mu_{,t}] \quad (\text{III.33})$$

The Hamiltonian is then

$$H^{**} = -[(\mu F)_{,v} + \mu_{,x} v + \mu_{,t}]_{,v} + \mu_{,x} - \lambda_{,t} \quad (\text{III.34})$$

Since μ , λ and F are functions of x , v and t , no further reduction can be carried out.

It should be stressed that although the dynamical variables are x and v , they are not in general the canonical conjugates of each other with respect to the two-star brackets. In fact, the canonical conjugate of x can be checked to be the quantity

$$\pi(x, v, t) = -\left\{(\mu F)_{,v} + \mu_{,x} v + \mu_{,t} + \int \mu_{,x} dv + P(x, t)\right\} \quad (\text{III.35})$$

where $P(x, t)$ is an arbitrary function of x and t .

In terms of it, the Hamiltonian (III.34) takes the form

$$H^{**}(x, \pi, t) = (\pi + P + \int \mu_{,x} dv)_{,v} + \mu_{,x} - \lambda_{,t} \quad (\text{III.36})$$

where v is an implicit function of x , π and t .

(see Eq. (III.31))

It is apparent from the form of the P.B. $[x, v]^{**}$ that different choices of the functions μ might in general give rise to different realizations of the canonical algebra and therefore, to different quantum theories, originating from the same classical equations (II.1).

In the multi-dimensional case equation (III.31) reads

$$[x^i, v^j]^{**} = -w^{ij}, \quad (\text{III.37})$$

where W is the inverse matrix of

$$M_{ij} = \frac{\partial}{\partial x^i} \left(\frac{\partial}{\partial x^j} \mu_i + \mu_k \frac{\partial F^k}{\partial x^i} \right) + \frac{\partial \mu_j}{\partial x^i} \quad (\text{III.38})$$

M is a (symmetric) non-singular matrix as ensured by equation (II.5). On the other hand, if π_j is defined through

$$[x^i, \pi_j] = \delta^i_j$$

then equation (III.37) implies that

$$\frac{\partial \pi_k}{\partial \dot{x}^k} = -M_{jk}$$

These facts confirm that now there are no constraints left in the theory since all variables can be expressed as functions of coordinate and momenta ($\dots$).

Moreover, by virtue of the symmetry of M

$$\frac{\partial \pi_k}{\partial \dot{x}^k} = \frac{\partial \pi_k}{\partial \dot{x}^k}$$

Thus, relation (III.35) can be written in the general case

$$\pi_i = - \left[\frac{\partial}{\partial \dot{x}^i} (M_k F^k) + \frac{\partial M_i}{\partial \dot{x}^k} v^k + \frac{\partial H_i}{\partial t} + \int \frac{\partial M_k}{\partial \dot{x}^i} d\dot{x}^k + P_i(x^i, t) \right]$$

(III.42)

IV. LINK BETWEEN EQUIVALENT LAGRANGIANS (AND HAMILTONIANS)

As we already know, the most general solution of equations (II.3) to (II.5) is given by

$$\mu_j(x^i, \dot{x}^i, t) = C_1 \frac{\partial C_2}{\partial \dot{x}^1} + C_3 \frac{\partial C_4}{\partial \dot{x}^2} + \dots + C_{2n-1} \frac{\partial C_{2n}}{\partial \dot{x}^n} \quad (IV.1)$$

where the C_ℓ are $2n$ suitably chosen functionally independent constants of motion of equations (II.1).

We will now show that any other choice for μ_j , say,

$$\tilde{\mu}_j(x^i, \dot{x}^i, t) = \tilde{C}_1 \frac{\partial \tilde{C}_2}{\partial \dot{x}^1} + \tilde{C}_3 \frac{\partial \tilde{C}_4}{\partial \dot{x}^2} + \dots + \tilde{C}_{2n-1} \frac{\partial \tilde{C}_{2n}}{\partial \dot{x}^n} \quad (IV.2)$$

can be obtained from (IV.1) by iteration of the following transformation process:

$$\mu \rightarrow K\mu + M_{1W} \quad (IV.3)$$

where K and M are (matrix functions of) appropriately selected constants of motion. To see this it suffices to observe that multiplication of μ by any function K generates extra terms proportional to $(\dot{K}/dt)$ or $(\dot{K}^2/dt^2)$ in the equation (II.4) for μ . Thus if K is a constant of the motion (i.e. $\dot{K} = 0$ on shell), $K\mu$ is a solution of the equation for μ as well. Furthermore, if we add a term of the form M_W (with $\frac{dM}{dt} = 0$) to any solution of (II.4) one obtains the extra term $\frac{d}{dt} \left(\frac{dM}{dt} v \right) - \left(\frac{dM}{dt} \right) \dot{x}$ which vanishes identically.

As can easily be verified, μ_j can be turned into $\tilde{\mu}_j$ if it is acted upon $(2n+1)$ transformations of the type (IV.3) defined by

$$K_s = C_{2s-3} [C_{2s-1}]^{-1} \quad \text{with } C_{-1} \equiv 1 \quad (IV.4)$$

$$M_s = -C_{2s}$$

for $s = 1, 2, \dots, n$ and

$$K_s = \tilde{C}_{2(s-n)-3} [\tilde{C}_{2(s-n)-1}]^{-1}$$

with $\tilde{C}_{-1}$ arbitrary, $\tilde{C}_{2n+1} \equiv 1$

(IV.5)

$$M_s = \tilde{C}_{2(s-n)}, \quad \text{with } \tilde{C}_{2n+2} \equiv 0$$

for $s = n+1, n+2, \dots, 2n+1$

consequently, any two Lagrangians $L, \tilde{L}$ which are constructed in the form (II.2) from ν and $\tilde{\nu}$ respectively are related by successive transformations of the form

$$L \rightarrow KL + M_{,\nu}(\ddot{x} - F) \quad (IV.6)$$

Correspondingly, as it can be deduced from (III.34) and using the constancy of K and M , the associated Hamiltonians can be found by iteration of

$$H \rightarrow KH - M_{,\nu} \dot{x} \quad (IV.7)$$

V. QUANTIZATION: OPEN PROBLEMS AND PROSPECTS.

In the preceding section we have outlined the canonical framework for the whole set of Lagrangians generating the same classical orbits.

This can be done directly from the equations of motion of the system.

This could be the starting point for a quantization of such systems. One is immediately faced, however, with the usual problems and ambiguities connected with the inversion of the limiting process $\hbar \rightarrow 0$: given the classical limit ($\hbar = 0$) of some quantum theory, how can we unambiguously reproduce the original theory ($\hbar \neq 0$)?

In the present case, over and above the usual ordering problem¹⁰⁾ for the operators in the quantum Hamiltonian, etc. we have a new source of ambiguity, namely, among the whole set of choices of the functions μ , which one should be chosen (if there is more than one) when we try to realize the algebra (III.31)? Are different choices of μ equivalent? and, if so, in what sense? In order to shed some light on these issues, which will be more extensively discussed elsewhere⁴⁾, let us consider two Lagrangians connected by a single transformation of the type (IV.6). This relation can be written in the more familiar form

$$\tilde{L} = KL + \frac{d}{dt} M, \quad (V.1)$$

where use has been made of the property that M is a constant of the motion. Consider the particular case $K = 1$. Now, equation (V.1) reduces to the standard result that two Lagrangians differing by a total time derivative give the same E-L equations. This is true irrespective of the nature of M : the (appropriate) E-L operator vanishes identically on any total time derivative. In the particular case where the function whose total derivative is added to L depends on x and t only, this transformation corresponds to a gauge transformation in the quantum theory. To illustrate this, consider the case of one degree of freedom with simple Lagrangian

$$L = \frac{1}{2}\dot{x}^2 - V(x), \quad (V.2)$$

and its equivalent

$$\tilde{L} = L + \dot{A}(x,t). \quad (V.3)$$

The corresponding canonical momenta are

$$p \equiv \frac{\partial L}{\partial \dot{x}} = \dot{x} \quad (V.4)$$

$$\tilde{p} \equiv \frac{\partial \tilde{L}}{\partial \dot{x}} = \dot{x} + \Lambda_{,x} = p + \Lambda_{,x} \quad (V.5)$$

and, therefore, the Hamiltonians are, respectively

$$H = \frac{p^2}{2} + V \quad (V.6)$$

and

$$\tilde{H} = \frac{1}{2}(\tilde{p} - \Lambda_{,x})^2 + V - \Lambda_{,t}. \quad (V.7)$$

Following the usual correspondence rule of substituting the canonical momenta conjugate to a variable z by $-i\hbar \frac{\partial}{\partial z}$ and H by $i\hbar \frac{\partial}{\partial t}$, one obtains the Schrödinger representations associated to (V.6) and (V.7) as follows:

$$\hat{H}\psi = \left[-\frac{\hbar^2}{2} \frac{\partial^2}{\partial x^2} + V(x) \right] \psi(x,t) = i \frac{\partial}{\partial t} \psi(x,t) \quad (V.8)$$

and

$$\hat{H}\psi = \left[\frac{1}{2} \left(i\hbar \frac{\partial}{\partial x} + \Lambda_x \right)^2 + V(x) - \Lambda_x \right] \tilde{\psi}(x,t) = i \frac{\partial}{\partial t} \tilde{\psi}(x,t) \quad (V.9)$$

At first glance, equations (V.8) and (V.9) may seem to describe completely different systems due to the arbitrariness of the function Λ . Their equivalence, however, can be manifestly exhibited if

$$\tilde{\psi} = e^{i\Lambda(x,t)} \psi \quad (V.10)$$

is replaced in equation (V.9). Thus the addition of a total time derivative into the Lagrangian can be compensated by the local gauge transformation (V.10). This result, nevertheless, breaks down if Λ is a function of $\dot{x}$ as well¹¹⁾, since in that case $\dot{x}$ is a much more complicated function of x and p_x and consequently $\tilde{H}$ is not simply a quadratic expression in $(\tilde{p}, \tilde{\Lambda}_x)$. As a result of this, the relation between the different pictures will not be of the form (V.10). It should be mentioned though that in particularly simple cases (e.g. Λ linear in $\dot{x}$), the complications are manageable and one can still prove the equivalence of the quantum theories derived from the classical Lagrangians related in this manner. This is the case of the supersymmetric point particle¹²⁾, where under a supersymmetry transformation, the Lagrangian changes by a total time derivative of a function linear in the coordinates and velocities in the right combination⁴⁾.

In the general case - when K is an arbitrary constant of the motion - the problem becomes extremely difficult and delicate. We only mention here, for the sake of completeness, that in general the transformation that links $\tilde{\psi}$ with ψ will not be a simple change of basis in the Hilbert space, so that the classical equivalence will not be automatically extendible to the quantum theory.

VI. CONCLUDING REMARKS

We have carried out the Hamiltonian analysis for the whole class of Lagrangians that give rise to the same classical orbits. This is done via acceleration-dependent Lagrangians taking advantage of the fact that any Lagrangian can be written generically as a linear combination of its own equations of motion. After all the redundant variables of the theory are removed using Dirac's method, it is shown how the freedom in the choice of Lagrangians is mapped into the Poisson Bracket algebra.

The quantum realizations are expected to be radically different depending on the choice of the Lagrangian: in some cases (as $L = T - V$, for instance), the right hand side of the algebra of P.B. consists of constants (c-numbers in the quantum theory) whereas for another s-equivalent Lagrangian they are functions of the dynamical variables (operators in the quantized version). The problem of the s-equivalent Lagrangians is reduced to the class of transformations involving multiplication by arbitrary constants of motion and addition of total time derivatives of them.

This scheme makes apparent that the main obstacle for the quantization of an arbitrary classical system will be to understand how the classical equivalence is reflected at the quantum level.

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Here one should vary with respect to $\ddot{x}$ as well, and the equation
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