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ABOUT CONVEXITY PROPERTIES OF THE BARYON MASS SPECTRUM

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A B S T R A C T

We construct analytic and numerical counter examples to conjectured convexity properties among baryon masses in potential models with flavour independence.

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INTRODUCTION

Flavour independence is a very important property of QCD, which has given rise to one of the most convincing achievements of potential models, namely that the same force ensures the binding of Charmonium and Upsilon systems. Fine and hyperfine interactions with explicit $1/m$ dependence are of course removed here by considering centres of gravity of multiplets.

The game was initiated by Eichten and Gottfried [1] who tried to anticipate the $b\bar{b}$ levels from the $c\bar{c}$ potentials. Quigg and Rosner proposed a logarithmic effective potential [2] to explain the approximate equality of Charmonium and Upsilon radial excitation energies. Later on, the test of flavour independence was extended down to strange [3] and even light [4] quarks, without regard for the obvious need for relativistic corrections, which have been partially accounted for in further papers [5].

In the baryon sector also, smooth flavour independent potentials allow for a unified description of baryon properties [6], and in particular for an accurate fit of all the observed ground states from the nucleon to the Λ^+ (csu) and the T^0 (ssc) [7].

Convexity properties for mesons :

In the two-body case, the assumption of flavour independence leads to the inequality between binding energies :

$$2 E(Mm) > E(MM) + E(mm) \quad (1)$$

which is well fulfilled in nature when data are available [8]. This

inequality can be proved as in Ref. [9] as a consequence of a very general convexity theorem [10], by observing that the hamiltonian depends linearly on the inverse of the reduced mass. Alternatively one can make use of the variational principle, as done by Nussinov [11]. Namely, with obvious notations :

$$E(MM) + E(mm) < \langle \psi(Mm) | H(MM) + H(mm) | \psi(Mm) \rangle \quad (2)$$

If one takes for H the reduced hamiltonian, one has $H(MM) + H(mm) = 2H(Mm)$ and the result follows. This, however, occurs only in the two-body case. Both methods can be generalized to derive inequalities involving the sum of the n first levels.

In a recent paper by E. Lieb [12] the inequality (1) was formulated in a rather more general way. Note also that the aim of Nussinov's article was much beyond the simple potential models and consisted in studying such inequalities in the framework of QCD, in line with previous work by Witten and Weingarten (quoted in [11]).

Convexity properties for baryons :

Given the inequality (1), it is very tempting to add a constituent m' to each state and write :

$$2E(Mmm') > E(MMm') + E(mmm') \quad (3)$$

With $M = s$ (strange quark), and $m = m' = q$ (ordinary quark), this reduces to :

$$F(M,m) = \frac{E(MMm) + E(mmm)}{2 E(Mmm)} < 1 \quad (4)$$

which is related to the Gell-Mann-Okubo mass formula (for Refs. see [7] or [12]):

$$\frac{N + E}{2} = \frac{\Lambda + 3E}{4} \quad (5)$$

and to the lower part of the "equal spacing rule for the decuplet" :

$$\Sigma^* - \Delta = E^* - \Sigma^* = \Omega^- - E^* \quad (6)$$

The upper part is reached with $M = q$ and $m = m = s$, for which the inequality (3) reduces to :

$$G(M, m) = \frac{E(Mmm) + E(MMm)}{2 E(MMm)} < 1 \quad (7)$$

As explained, e.g. in Ref. [7], the approximate equality observed for (5) and (6) in nature is reproduced in explicit quark model calculations thanks to a miraculous cancellation between the convexity (4) and (7) induced by the central potential and the effects of QCD hyperfine corrections [13].

With realistic potentials, e.g. Coulomb plus linear or r^α with $0 < \alpha < 2$, and $m' = M$ or m , the inequality is always satisfied, a result obtained numerically in the hyperspherical treatment of the 3-body problem [7], and confirmed by a variational method [14].

For a while we were tempted to believe that the inequality (3), at least in the special cases (4) and (7), was a very general property, whose proof was, however, elusive. In Ref. [7], it was shown that one cannot make use of the linearity of the hamiltonian into the two reduced masses which show up when one introduces Jacobi variables and removes the centre of mass motion. The variational proof proposed in Ref. [11] was shown to fail [15] because of the neglect of the asymmetries of the (MMm) or (Mmm) wave functions. Then, E. Lieb [12] made two essential contributions :

(1) He showed that the inequality (3) is satisfied provided the 3-dimensional Fourier transform of $\exp(-\lambda V)$, where $V(r)$ is the two-body

potential, is positive for all λ (for 3-body terms, see [12]). There exist sufficient conditions for this to hold. One is :

$$V' > 0, V'' < 0 \text{ and } V''' > 0 \quad (8)$$

Condition (8) is satisfied for instance by the familiar potential $V = -a/r + br$.

Another condition is that all successive derivatives of V with respect to r^2 have alternate signs. This latter condition means that power law potentials $V = r^\alpha$ with $\alpha < 2$ are acceptable. Note that the Fourier transform of $\exp(-r^\alpha)$ oscillates for $\alpha > 2$ (the proof involves lengthly but straightforward integrations by parts), so that the above condition does not apply.

(2) He displayed a counter-example $G > 1$ involving an infinite mass and a square-well potential.

By continuity one expects also violation of the inequality (7) for finite potential sharpness and mass ratio. The aim of the present paper is to show that the violation occurs actually "rather soon", i.e. for reasonable potential shapes and constituent masses. More precisely, we shall investigate the case of two-body power-law potentials $V(r) = r^\alpha$ and study, for $m = m' = 1$ and $M > 1$, the regions of the (M, b) plane where a violation of (7) occurs, i.e. $G > 1$.

RESULTS FOR INFINITE M

Square well

As shown in Ref. [12], for a square-well potential corresponding to an infinite exponent α , $E(MMM) = 0$ in the limit of infinite M , $E(MMm) = e(\infty)$

where $e(\alpha)$ hereafter denotes the binding energy of a $m = 1$ particle in the central potential r^α , and $E(Mmm) > 2 e(\infty)$, so that one gets the violation of (7). On the other hand, the violation of (4) does not occur.

Analytic results for power-law potentials r^α

In this case the quantity (7) is denoted as $G(M,m,\alpha)$. We first recall that a single particle of mass m in a potential Kr^α has a binding energy $B(m,K)$ which is related to the $m = K = 1$ case through the scaling relation [2] :

$$B(m,K) = e(\alpha) K^{\frac{2}{\alpha+2}} m^{\frac{-\alpha}{\alpha+2}} \quad (9)$$

For $m = 1$ and M infinite we have as above :

$$E(MMM) = 0 \text{ and } E(MMm) = e(\alpha) 2^{\frac{2}{\alpha+2}} \quad (10)$$

whereas $E(Mmm)$ comes from :

$$H(Mmm) = \frac{\vec{p}_1^2 + \vec{p}_2^2}{2m} + r_1^\alpha + r_2^\alpha + |\vec{r}_1 - \vec{r}_2|^\alpha \quad (11)$$

Introducing $\vec{x} = \vec{r}_1 - \vec{r}_2$ and $\vec{y} = \vec{r}_1 + \vec{r}_2$, we have for $\alpha > 2$:

$$r_1^\alpha + r_2^\alpha > 2^{1-\alpha} (x^\alpha + y^\alpha) \quad (12)$$

an inequality proved in the Appendix. In this way we minorize (11) by a separable hamiltonian, and get, using (9) :

$$G(\infty,1,\alpha) > G_1(\alpha) = 2^{-\frac{2+2\alpha}{2+\alpha}} [1 + (1 + 2^{-1+\alpha})^{\frac{2}{2+\alpha}}] \quad (13)$$

so that the violation of (7) occurs at least for $\alpha > 5.6$, as seen in Fig. (1).

Using the above mentioned concavity relation [10] allows one to decrease further the critical exponent below the value $\alpha = 5$. Indeed, the hamiltonian (11) can be considered as the $a = 1$ case of :

$$H(a) = \frac{\vec{p}_1^2 + \vec{p}_2^2}{2m} + a |\vec{r}_1 - \vec{r}_2|^\alpha + (2-a) (r_1^\alpha + r_2^\alpha) \quad (14)$$

so that, for any $a > 1$:

$$E(Mmm) > [(a-1) E(a=0) + E(a)] / a \quad (15)$$

since $H(a)$ is linear in a . The energy $E(a=0)$ is obvious and $E(a)$ can be bounded below by inserting the inequality (12) into the hamiltonian (14). We then obtain :

$$G(\infty, 1, \alpha) > G_2(\alpha) = \text{Max}_{a>1} g_2(a, \alpha) \quad (16)$$

where :

$$g_2(a, \alpha) = 1 + \frac{1}{a} \left[-1 + \frac{(2-a)^{\frac{2}{\alpha+2}} 2^{\frac{2-\alpha}{2+\alpha}} + (a 2^{\alpha/2} + (2-a) 2^{1-\alpha/2})^{\frac{2}{\alpha+2}}}{2^{\frac{\alpha+4}{\alpha+2}}} \right] \quad (17)$$

As seen in Fig. (1), one easily gets a violation of (7) for $\alpha = 5$, but not for $\alpha = 4$.

Numerical results for r^α

The two-body binding energy in eq. (10) is easily computed. For $E(Mmm)$, in the limiting case $M = \infty$, one can use either the coordinates r_1 and r_2 or X and Y and then expand the wave-function into hyperspherical harmonics (for references, see [7]). The convergence is very satisfactory, since the two Jacobi variables play a similar role and, in particular, have the same r.m.s. value. The results are shown in Fig. (1). For $\alpha > 2.6$ one gets a violation of (7). Meanwhile the violation of (4) is never observed.

RESULTS FOR FINITE MASS RATIOS

We now search, for a given α , above which value $M = M_0$ a violation of the inequality (7) appears. This is done first by crude majorations and then by numerical methods.

Upper bound for M_0

If α is not too large (i.e. just above the critical value $\alpha = 2.6$ found for $M = \infty$), we expect M_0 rather large and thus use :

$$E(MMM) + E(Mmm) > E(\infty, \infty, \infty) + E(\infty, m, m) > 2e(\alpha) 2^{\frac{2}{2+\alpha}} \quad (18)$$

We thus achieve a violation of (7) provided that :

$$\frac{E(M, M, m)}{E(\infty, \infty, m)} < G(\infty, 1, \alpha) \quad (19)$$

To check this we use a variational ansatz :

$$\psi = \exp\left[-\frac{\lambda}{2} (r_1^2 + r_2^2)\right] \phi(\vec{r}_3) \quad (20)$$

where $\phi(\vec{r})$ is the $M = \infty$ wave-function, i.e. the solution of :

$$\left(\frac{p^2}{2m} + 2r^\alpha - B(m, 2)\right) \phi(\vec{r}) = 0 \quad (21)$$

To compute the expectation value of r_{13}^α one writes it as :

$$[(\vec{r}_1 - \vec{r}_3)^2]^{\alpha/2} \quad (22)$$

When α is an even integer, one can expand and perform the angular integrations. This gives terms like $r_3^p r_1^q$. With our trial wave-function, $\langle r_1^q \rangle$ is obvious, and from the Schwarz inequality and the virial theorem :

$$\langle r_3^p \rangle < [e(\alpha) \frac{2}{2+\alpha}]^{\frac{p}{\alpha}}, \quad (23)$$

for $p < \alpha^*$.

* This inequality, and others, can be improved by using the remark of A.K. Common that the matrix elements are taken in the ground state, the wave function of which is a decreasing function of r (16).

For $\alpha = 8$, one gets that way, using $B = B(m,2)$

$$\begin{aligned} \frac{E(M,M,m)}{E(\infty,\infty,m)} &< \frac{3\lambda}{MB} + \frac{17}{16} \frac{945}{B} \lambda^{-4} + \frac{315}{2} \frac{B^{-3/4}}{10^{1/4}} \lambda^{-3} \\ &+ \frac{189}{2} \frac{B^{-1/2}}{10^{1/2}} \lambda^{-2} + 18 \frac{B^{-1/4}}{10^{3/4}} \lambda^{-1} + 1 \end{aligned}$$

If we take $\lambda = CM^{1/2}$, we see that the finite mass corrections are bounded by $cM^{-1/2}$ and can be made arbitrarily small by taking M big enough.

The two-body binding energy $B(m,2)$ entering into the above expression can be bounded below by optimizing x and y satisfying (for $\alpha > 2$) :

$$r^\alpha > x r^2 + y \quad \forall r \quad (25)$$

This is actually rather poor and can be improved by using more accurate bounds (for references, see [17]).

Minimizing with respect to λ the r.h.s. of eq. (24), using the simple lower bound (25) of $B(m,2)$, and the single bound $G_1(\alpha)$ for $G(\infty,1,\alpha)$ in (19) gives the very conservative bound $M_0 \gtrsim 2000$ for $\alpha = 8$. If we instead use the numerical values of $B(m,2)$ and $G(\infty,1,\alpha)$, we obtain $M_0 \gtrsim 200$ for $\alpha = 8$ and $M_0 \gtrsim 700$ for $\alpha = 4$.

With some algebra one can improve the above calculation by removing the centre of mass motion and using instead of (20) :

$$\psi = \exp - \frac{\lambda}{2} (\vec{r}_1 - \vec{r}_2)^2 \quad \phi(\vec{r}_3 - \frac{\vec{r}_1 + \vec{r}_2}{2}) \quad (26)$$

If one uses again the numerical values of $B(m,2)$ and $G(\infty,1,\alpha)$, the result is $M_0 \gtrsim 50$ for $\alpha = 8$ and $M_0 \gtrsim 200$ for $\alpha = 4$.

Numerical results

With equal masses (MMM), or nearly equal masses for a configuration (Mmm) with one heavy quark and two light quarks, the hyperspherical expansion converges very well, and, for simple potentials like r^α , one obtains essentially exact binding energies with a maximal "grand" orbital momentum $M = 8$.

For the (MMm) case with $M/m \gg 1$, however, the method suffers from a lack of flexibility to account for the dissymmetry :

$$\langle r_{12} \rangle \ll \langle r_{13} \rangle = \langle r_{23} \rangle \quad (27)$$

of the wave-function. An empirical variational parametrization of the wave-function generalizing eq. (26) is more appropriate [18]. This was done thanks to an algorithm supplied to us by S. Zouzou. Typical results are

$$\begin{aligned} &\text{for } r^8, G > 1 \text{ if } M/m \gtrsim 3 \\ &\text{for } r^3, G > 1 \text{ if } M/m \gtrsim 200. \end{aligned}$$

CONCLUSION

The counterexamples we have obtained appear to reduce the gap between the potentials for which the convexity property holds, for instance $\varepsilon(\alpha)r^\alpha$, $\alpha < 2$, and those for which it is violated - numerically $\alpha > 2.6$. However they are irrelevant for the physics of baryons because we believe, from lattice QCD, that the quark-quark potential is concave (19). We notice on the other hand that the conditions under which the convexity property holds are not yet completely satisfactory. It is not physically plausible that an information on the 3rd derivative of the potential is really needed, but it might take a long time before this presumably superfluous requirement is removed.

APPENDIX

It is easy to see that :

$$|\vec{r}_1 - \vec{r}_2|^\alpha + |\vec{r}_1 + \vec{r}_2|^\alpha < |r_1 - r_2|^\alpha + |r_1 + r_2|^\alpha, \text{ for } \alpha > 2,$$

where $r_i = |\vec{r}_i|$, by maximizing with respect to the angle $(\vec{r}_1, \vec{r}_2)$.

Then it is sufficient to show that :

$$r_1^\alpha + r_2^\alpha > 2^{1-\alpha} (|r_1 - r_2|^\alpha + |r_1 + r_2|^\alpha).$$

We can assume without loss of generality :

$$\begin{aligned} r_1 &= \frac{1}{2}(y + 1) \\ r_2 &= \frac{1}{2}(y - 1) \end{aligned} \quad y > 1$$

and study the sign of : $f(y) = (y+1)^\alpha + (y-1)^\alpha - 2(1+y^\alpha)$

Since : $f(1) = 2^\alpha - 4 > 0$ if $\alpha > 2$

and $f'(y) > 0$ if $\alpha > 2$ due to the concavity of $f(x) = x^{\alpha-1}$

we get $f(y) > 0$ if $\alpha > 2$ and the inequality (12) is established.

REFERENCES

- [1] E. Eichten and K. Gottfried, Phys. Lett., 66B, 286 (1977)
- [2] C. Quigg and J.L. Rosner, Phys. Lett., 71B, 153 (1977)
and Phys. Rep., 56, 167 (1979), see also M. Machacek and
Y. Tomozawa, Ann. Phys. (N.Y.) 110, 407 (1978)
- [3] A. Martin, Phys. Lett., 100B, 511 (1981)
- [4] R.K. Bhaduri et al., Nuovo Cimento, 65A, 376 (1981)
- [5] S. Godfrey and N. Isgur, Phys. Rev., D32, 189 (1985)
J.L. Basdevant and S. Boukraa, Z. Physik, C28, 413 (1985) and
30, 103 (1986) and Annales de Physique, 10, 475 (1985)
- [6] See, e.g., N. Isgur, in "New Flavours and Hadron Spectroscopy",
Proc. XVI, Rencontre de Moriond, J. Tran Thanh Van Ed. (Editions
Frontières, France, 1981)
- [7] J.M. Richard and P. Taxil, Ann. Phys. (N.Y.) 150, 267 (1983)
and Phys. Lett. 123B, 4531 (1983), see also P. Taxil, in
"Flavour mixing and CP violation", Proc. V, Moriond Workshop,
Ed. J. Tran Thanh Van (Editions Frontières, France)
- [8] Particle Data Group, "Review of Particle Properties", Phys. Lett.
111B, 1 (1982)
- [9] R. Bertlmann and A. Martin, Nucl. Phys. B168, 111 (1980)
- [10] See, e.g., W. Thirring, "Quantenmechanik von Atomen und Molekulan",
Lehrbuch der Mathematischen Physik, Vol. 3 (Springer, Berlin, 1979)
- [11] S. Nussinov, Phys. Rev. Lett., 52, 966 (1984)
- [12] E. Lieb, Phys. Rev. Lett., 54, 1987 (1985)
- [13] A. de Rujula, H. Georgi and S.L. Glashow, Phys. Rev. D12, 147 (1975)
- [14] N. Papanicolaou and P. Spathis, J. Phys. G : Nucl. Phys. 11, 149
and 679 (1985)
- [15] J.M. Richard and P. Taxil, Phys. Rev. Lett. 54, 847 (1985)

- [16] A.K. Common, Nucl. Phys. B 184, 323 (1981)
- [17] Essays in Mathematical Physics in Memoriam of V. Glaser, A. Martin and M. Jacob eds., Phys. Rep. 134, 273 (1986)
- [18] S. Zouzou, Thèse de Troisième Cycle, Université P. et M. Curie Paris, 1985
- [19] C. Bachas, Phys. Rev. D 33, 2723 (1986)

FIGURE CAPTIONS

Fig. 1 - Result of a numerical calculation of $G(M = \infty, m = 1, \alpha)$ (full curve), compared to the bounds $G_1(\alpha)$ (dashed curve) and $G_2(\alpha)$ (dotted dashed curve).