CERN-TH.4425/86

ON THE COMPACTIFICATION OF THE BOSONIC
STRING AT HIGHER LOOPS

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ABSTRACT

We analyze the procedure of string compactification on a lattice Λ in the framework of Polyakov's path integral for arbitrary genus of the string world sheet. It is demonstrated how compactification, factorizability into left and right movers and modular invariance are related. The constraints on Λ of being an even, self-dual lattice, as well as the relation among the radius of compactification and the string tension, known from the heterotic string occur naturally. We present the "correction factor" R for arbitrary genus of the string world sheet.

I. - INTRODUCTION

The present revival of interest in string theories is based to a large extent on the possibility of constructing string theories which contain gauge symmetries having a chance of being relevant phenomenologically. This development has been initiated by the discovery of the relation between the theory of representations of certain Kac-Moody algebras and string theory^{1),2)}. So far most of the interest centered around the heterotic string^{3),4)}. However, the heterotic string may not exhaust all variations of the vertex operator construction leading to a consistent theory⁵⁾⁻⁸⁾.

The central idea of the heterotic string is to generate non-Abelian gauge symmetries by compactifying some space-time dimensions of the left-moving part of the bosonic string on a weight lattice Λ of a simply laced Lie group. In order to obtain a consistent theory the lattice Λ has to be self dual. This self-duality constraint is intimately linked, via the so-called modular invariance, with the finiteness of the one-loop corrections to the Green's functions. For the closed string, the genus g of the string world sheet, i.e., its number of holes, serves as a loop counting parameter. At one loop the string world sheet is a torus. Modular transformations correspond to changes of the generators of the fundamental group Π_1 of the world sheet torus. Analyzing a particular Green's function it has been demonstrated in detail in 4) that:

- (i) keeping only the left-moving sector of an open string in certain dimensions, and
- (ii) compactifying the corresponding dimensions on an even, self-dual lattice, one obtains a result which exhibits modular invariance.

Writing down a consistent theory of a bosonic string involving only left movers in some dimensions without compactifying the corresponding space-time co-ordinates seems not to be an immediate procedure. In the operator formalism the difficulties appearing when trying to truncate a closed string so that it consists of left movers only, are related to the appearance of the zero mode, describing the movement of the string as a whole, i.e., of the superposition of left and right movers. At the level of the quantum theory the difficulties manifest themselves as the failure of even the vacuum amplitude to factorize into a right-moving and a left-moving part⁹⁾. However, as the analysis of 4) shows, these problems immediately disappear when the corresponding space-time co-ordinates are compactified on a self-dual lattice.

The considerations of 4) have been restricted to the one-loop level. One may therefore wonder whether the two crucial properties found for the one-loop corrections after compactification on a self-dual lattice, namely:

- (i) factorization of the closed string amplitudes into a left- and a right-moving part,
- (ii) modular invariance,

are spoiled by higher-loop corrections. We shall argue in the present paper that this is not the case.

In a recent paper¹⁰⁾, the vacuum amplitude for the closed oriented bosonic string has been investigated in the Polyakov formulation¹¹⁾ to arbitrary loop order without compactifying any dimensions. It has been found that the integrand does not factorize. The non-factorizing part has been identified. We shall demonstrate in the present paper that the vacuum amplitude factorizes for all those dimensions in which the string is compactified on a self-dual lattice for arbitrary genus g of the string world sheet. Moreover, we shall demonstrate that the compactification is consistent with the generalization of the modular invariance to arbitrary g , the invariance under the mapping class group.

The starting point of our analysis is the Polyakov path integral for closed oriented strings in $d = 26$. In the first part of this paper we shall analyze the vacuum amplitude for $g = 1$. The aim is to obtain a prescription which allows us to factorize the vacuum amplitude in the compactified dimensions into a left-moving and a right-moving part. Such a prescription is necessary to be able to define the amplitude of a single sector separately. By requiring factorization we obtain the constraint on Λ to be self dual, as well as the relation $\alpha' = R^2$ in a natural manner. Asking for modular invariance of the factorizing parts, we get the further constraint on Λ to be even.

In the second part we shall demonstrate that all those features found for $g = 1$ are essentially preserved by higher-loop corrections. Our procedure consists in constructing the correction factor R in terms of the Abelian differentials.

In the present work we shall only be concerned with those aspects which are related with the compactification of D dimensions. We have nothing to say about the problem of constructing a well-defined theory for the $d-D$ uncompactified dimensions.

The rest of the paper is organized as follows: in the next section we shall summarize some fundamental facts about Riemann surfaces, which we shall make use of. Section 3 presents the analysis for $g = 1$. The vacuum amplitude for arbitrary g is analyzed in Section 4. Some comments are added in Section 5.

2. - RIEMANN SURFACES AND THEIR RELATION TO POLYAKOV'S PATH INTEGRAL

In this section we shall give a brief outline of the relation between Riemann surfaces and the Polyakov path integral. Within the framework of Polyakov's path integral the string world sheet may be interpreted as a Riemann surface. We shall sketch some of the results on Riemann surfaces which are important for the evaluation of the Polyakov path integral. For more detailed information on Riemann surfaces the reader is referred to the mathematical literature, e.g., Refs. 12) to 14), 21) and 24).

Let us start with Polyakov's representation of the vacuum amplitude¹¹⁾

$$Z = \sum_{g=0} \int \mathcal{D}g_{ab} \mathcal{D}X_{\mu} e^{-S}, \quad (2.1)$$

where the action S is

$$S = \frac{1}{4\pi\alpha'} \int d^2z \sqrt{g} g^{ab} \partial_a X^{\mu} \partial_b X^{\mu}. \quad (2.2)$$

The integral $\mathcal{D}g_{ab}$ has to be taken over the space of all Riemannian metrics on a surface of genus g . A precise definition of the formal expression (2.1) has been given in 15) and 16). Related work may be found in 17) to 20).

In order to evaluate (2.1), one has to fix the gauge of g_{ab} . Introducing complex co-ordinates on the string world sheet, the general form of the invariant distance ds^2 is given by

$$d^2s = e^{\phi(z)} |dz + \mu(z) d\bar{z}|^2. \quad (2.3)$$

Changing the dilatation factor $e^{\phi(z)}$ corresponds to a conformal transformation. On a Riemann surface all those metric tensors are identified, which differ only by a conformal transformation. If the genus of the Riemann surface satisfies $g > 1$ ($g=1$) the so-called Beltrami coefficient $\mu(z)$ depends on $3(g-1)$ (1) complex parameters, the Teichmüller parameters. These Teichmüller parameters take into account the different complex structures, which may be put onto a Riemann

surface, if viewed as a complex manifold. They are related to the presence of co-ordinate transformations, which are not homotopic to the identity transformation.

All the Riemann surfaces M with $g > 0$ are not simply connected, i.e., there are closed curves which cannot be continuously deformed into a point on M . Such curves may be transversed several times. Therefore, the mapping of M into the set of all paths is multiple valued. In order to obtain a single valued mapping, one has to introduce $\tilde{M}$, the universal covering space of M . This is done by first choosing a complete set of $2g$ closed curves on M , which serve as the generators of the fundamental group $\Pi_1(M)$. Such a set of curves is called a marking. There is a standard procedure of choosing $2g$ independent loops on M labelled by $a_1, \dots, a_g, b_1, \dots, b_g$, such that the only non-vanishing intersecting pairs are given by (a_i, b_i) $i = 1, \dots, g$ (see Fig. 1). Letting the intersection points of all the curves coincide, cutting M along these curves and unfolding M , one obtains a $4g$ sided polygon. $\tilde{M}$ is obtained by gluing together infinitely many of these polygons in a consistent manner. In this way, each closed path on M is mapped into an open curve on $\tilde{M}$.

The fundamental group of M is represented on $\tilde{M}$ by a group of transformations Γ , the so-called group of deck-transformations. Choosing on M a fundamental region Δ , which does not contain any two points which are related by Γ , each point on $\tilde{M}$ may be transformed into Δ by acting with Γ . In this way one obtains a projection p which may be interpreted as $p: \tilde{M} \rightarrow M$ and therefore $M = \tilde{M}/\Gamma$.

Let us describe the construction of $\tilde{M}$ in some more detail²¹⁾. Γ is constructed as follows. One chooses a set of $2g$ open curves on $\tilde{M}$, which have the common initial point z_0 . Let us denote them again by $a_1, \dots, a_g, b_1, \dots, b_g$. The endpoints of these curves define a set of mappings $\chi(a_i), \chi(b_i)$. These transformations generate Γ . By acting with all the elements of Γ on z_0 , one obtains a graph where each vertex corresponds to an element of Γ . At each vertex $2g$ lines of this graph meet, which correspond to the generators of Γ . These lines are images of the curves $a_1, \dots, a_g, b_1, \dots, b_g$ under Γ . In Fig. 2 we have represented a part of such a graph for a surface with $g = 2$, which contains Δ . Dashed curves are identified under p . Note that those sides of Δ , belonging to the same curve are identified on M .

One may easily extend the metric (2.3) from M to all of $\tilde{M}$ by requiring that

$$\mu(\gamma(z)) \bar{\gamma}'(z) / \gamma'(z) = \mu(z) \quad (2.4)$$

for $\gamma \in \Gamma$.

One of the central results about Riemann surfaces is the uniformization theorem which states that each Riemann surface with $g > 1$ is holomorphically equivalent to U/Γ , where U is the upper half complex plane and Γ is a subgroup of the group of Möbius transformations

$$z' = \frac{az + b}{cz + d} \quad ad - bc = 1 \quad (2.5)$$

with real coefficients. It is generated by $2g$ transformations (2.5) and the corresponding inverse transformations. The space of parameters of these $2g$ generating transformations is called the Fricke space. For $g = 1$, U has to be substituted by $\mathbb{C}$, the complex plane, and Γ by the group of translations.

Applying the uniformization theorem to $\tilde{M}$, one is immediately able to fix the group of deck-transformations on $\tilde{M}$, or equivalently the transition functions on M . The generators of Γ , $\chi(a_p)$ and $\chi(b_p)$ must obviously be transformations of the form (2.5). Due to the closure of the fundamental polygon, we must have in addition

$$\prod_{p=1}^g C_p = 1 \quad (2.6)$$

where C_p is the commutator

$$C_p = \chi(a_p) \chi(b_p) \chi(a_p)^{-1} \chi(b_p)^{-1} \quad (2.7)$$

Taking into account that one can always normalize Γ by giving three points on the real line, which are left invariant under some elements of Γ , it follows from (2.6) that the real dimension of the Fricke space is $6g-6$.

One may now try to change the co-ordinates on $\tilde{M}$, such that the Beltrami coefficient of (2.3) disappears. It is obvious that this can only be achieved by a non-holomorphic co-ordinate transformation, which changes the complex structure, respectively Γ . This indicates that instead of keeping Γ fixed and changing $\mu(z)$ in (2.3), we may set $\mu(z) = 0$ and vary Γ . This is a standard procedure for Riemann surfaces, which provides a map from the space of Teichmüller parameters into the Fricke space¹³⁾.

Throughout the present paper, we shall always take $\mu = 0$ and let Γ change. This corresponds to fixing the gauge

$$d^2s = e^\phi |dz|^2. \quad (2.8)$$

Though S is independent of ϕ , Z may depend on ϕ via the measure of the path integral. It is well known, if $d = 26$ this dependence is trivial^{11),15,16)}. This makes the interpretation of the string world sheet as a Riemann surface into a perfect analogy. We shall always work with $d = 26$.

In this paper we shall mainly be concerned with the integral over $\mathbb{D}X^\mu$ in (2.1). Our argument does not require the knowledge of the precise definition of $\mathbb{D}g_{ab}$. Therefore, we shall take the formal expression (2.1) as our starting point in the following section.

3. - THE ONE-LOOP VACUUM AMPLITUDE

Due to the compactification of some of the space-time co-ordinates, the integrand of the vacuum amplitude gets modified by a correction factor R . We shall calculate this correction factor for surfaces of $g = 1$ in the present section and generalize the procedure in the next section to arbitrary g . We shall demonstrate that after compactification, it is possible to factorize the integrand of the vacuum amplitude into a left moving and a right moving part, in such a way that both parts exhibit modular invariance separately. Our result for R agrees with the one known from the operator formalism^{9),22)}.

3.1 Derivation of the correction factor R

Let us compactify D dimensions. To distinguish those dimensions which are left uncompactified from those which are compactified, we shall denote the latter by an index $I = 26-D+1, \dots, 26$. To compactify X^μ on a lattice Λ , we shall assume that Λ is generated by a set of vectors e_I^i , $i = 1, \dots, D$. We shall identify those points of the tangent space of $X_I(z, \bar{z})$, which differ by an amount of $R\pi\sqrt{2}e_I^i$

$$X_I \approx X_I + R\sqrt{2}\pi e_I^i, \quad i = 1, \dots, D. \quad (3.1)$$

In the present case, the Riemann surface M is topologically a torus. Its

universal covering space $\tilde{M}$ is the whole complex plane on which Γ acts as the group of translations generated by the two complex numbers ξ_a and ξ_b (see Fig. 3). We may therefore write

$$\begin{aligned}\chi(a)z &= z + \xi_a \\ \chi(b)z &= z + \xi_b\end{aligned}, \quad (3.2)$$

where $\chi(a)$ [$\chi(b)$] maps $b'(a')$ onto $b'(a'')$. The region over which the dz^2 integral of the action (2.2) extends is just the fundamental region Δ . Though it is a common practice to normalize Δ such that $\zeta_a = 1$, we shall not do this, in order to facilitate the comparison with the $g > 1$ case.

To evaluate the path integral (2.1), we have to prescribe boundary conditions for X_μ and X_I on $\partial\Delta$. As we are dealing with a closed string, periodic boundary conditions are appropriate for X_μ . In the compactified dimensions, however, the value of X_I on two opposite boundary curves may differ by a vector on the lattice Λ . This freedom is a consequence of the fact that the string may have a non-vanishing winding. In order to allow for non-vanishing winding numbers, we shall split $X_I(z, \bar{z})$ into two parts

$$X_I(z, \bar{z}) = \delta X_I(z, \bar{z}) + \hat{X}_I(z, \bar{z}). \quad (3.3)$$

$\delta X_I(z, \bar{z})$ satisfies periodic boundary conditions. Defining the action of $\chi(a)$ on $\hat{X}_I(z, \bar{z})$ by

$$\chi(a)\hat{X}_I(z, \bar{z}) = \hat{X}_I(\chi(a)z, \chi(a)\bar{z}), \quad (3.4)$$

and analogously for $\chi(b)$, we require that

$$\chi(a)\hat{X}_I(z, \bar{z}) = \hat{X}_I(z, \bar{z}) + \sqrt{2}\pi R w_I^a, \quad (3.5a)$$

$$\chi(b)\hat{X}_I(z, \bar{z}) = \hat{X}_I(z, \bar{z}) + \sqrt{2}\pi R w_I^b \quad (3.5b)$$

where $w_I^a, w_I^b \in \Lambda$. (3.5a,b) are solved by

$$\hat{X}_I(z, \bar{z}) = [w_I^a(z \bar{f}_b - \bar{z} f_b) - w_I^b(z \bar{f}_a - \bar{z} f_a)] / (\sqrt{2\pi} R) (f_a \bar{f}_b - \bar{f}_a f_b) \quad (3.6)$$

$\delta X_I(z, \bar{z})$ is decomposed in the standard manner into eigenfunctions of the Laplacian defined by the action. As will be demonstrated in the next section for general g , δX_I and $\hat{X}$ are orthogonal if inserted into the action. Hence

$$S = S_0 + \hat{S}(\underline{w}^a, \underline{w}^b) \quad , \quad (3.7)$$

where S_0 is just the action we would encounter without compactification.

In the path integral we must sum over all the configurations with different winding numbers. $\hat{S}$ just determines the relative weight of their contributions. In addition, we may allow for a phase depending on $\underline{w}^a$ and $\underline{w}^b$. Therefore, compactifying D dimensions has the consequence of modifying the integrand of (2.1) by the factor

$$R = \sum_{\underline{w}^a, \underline{w}^b \in \Lambda} e^{-\hat{S}(\underline{w}^a, \underline{w}^b)} J(\underline{w}^a, \underline{w}^b) \quad , \quad (3.8)$$

$$|J(\underline{w}^a, \underline{w}^b)| = 1 \quad .$$

Later it will turn out that the presence of a non-trivial phase factor J is a necessary condition for factorization. Inserting (3.6) into the action (2.2), and taking into account that the area of the fundamental region Δ is given by

$$|f_b|^2 \text{Im} \tau \quad (3.9)$$

$$\tau = f_b / f_a \quad , \quad (3.10)$$

we get

$$R = \sum_{\underline{w}^a, \underline{w}^b \in \Lambda} e^{-\frac{\tilde{\eta}}{\gamma \operatorname{Im} \tau} |\underline{w}^b - \tau \underline{w}^a|^2} \mathcal{J}(\underline{w}^a, \underline{w}^b) \quad , \quad (3.11)$$

where

$$\gamma := 2d'/R^2 \quad . \quad (3.12)$$

τ is the Teichmüller parameter of the torus.

3.2 Modular transformations

Before proceeding to the factorization of R , let us briefly discuss modular transformations. These transformations describe changes of the marking of the torus. They are generated by the so-called Dehn twists. Modular invariance of the integrand is of crucial importance to be able to define a finite amplitude.

To perform a Dehn twist, one cuts the surface along a closed curve to obtain two disconnected arms, rotates one arm by 2π , and glues both arms together again. In the present case, we have two Dehn twists τ_a and τ_b , which correspond to rotations around the curves a and b . τ_a and τ_b induce the following mappings of the generators of the fundamental group $\Pi_1(M)$

$$\begin{aligned} \tau_a : (a, b) &\rightarrow (a, ba) \\ \tau_b : (a, b) &\rightarrow (ab, a) \end{aligned} \quad . \quad (3.13)$$

τ_a and τ_b together with their inverses generate the modular group of the torus. They induce the following transformations of the periods ξ_a and ξ_b

$$\begin{aligned} \tau_a : (\xi_a, \xi_b) &\rightarrow (\xi_a, \xi_b + \xi_a) \\ \tau_b : (\xi_a, \xi_b) &\rightarrow (\xi_a + \xi_b, \xi_b) \end{aligned} \quad , \quad (3.14)$$

and consequently

$$\tau_a: \tau \rightarrow \tau + 1, \quad \tau_b: \tau \rightarrow \frac{\tau}{1+\tau} \quad (3.15)$$

It is a common procedure to replace the generator τ_b by a new generator which we shall label κ . κ is defined by

$$\kappa := \tau_a^{-1} \tau_b^{-1} \tau_a^{-1} \quad (3.16)$$

As $\Pi_1(M) = \mathbb{Z}^2$ for $g = 1$ is an Abelian group, κ acts on the generators of $\Pi_1(M)$ respectively τ like

$$\kappa: \begin{cases} (a, b) \rightarrow (aba^{-1}, a^{-1}) \cong (b, a^{-1}) \\ \tau \rightarrow -1/\tau \end{cases} \quad (3.17)$$

As the sum in (3.11) is invariant under the replacements

$$\underline{w}^b \rightarrow \underline{w}^b - \underline{w}^a; \quad (\underline{w}^a, \underline{w}^b) \rightarrow (\underline{w}^b, -\underline{w}^a)$$

R is invariant if $J = 1$ for arbitrary Λ . As we shall see in the next section, the condition $J = 1$ is not necessary for R to be modular invariant, if Λ satisfies further specifications.

3.3 Factorization

As explained in the Introduction, the option of treating left and right movers independently is of supreme importance in constructing the heterotic string. Suppose such a procedure is known, then it should be possible to factorize the integrand of the vacuum amplitude completely into two independent parts, being the complex conjugate of each other. Conversely once it is understood how to factorize Z , one can employ the same prescription to define the amplitude of a left moving part, which has no right moving counterpart.

In Ref. 17), it has been shown that the integrand $I(\tau, \bar{\tau})$ of the vacuum amplitude for $g = 1$ of the closed, oriented bosonic string in $d = 26$, without compactification almost factorizes

$$I(\tau, \bar{\tau}) = (\text{Im } \tau)^{-12} |\bar{F}(\tau)|^2 \quad (3.18)$$

$$\bar{F}(\tau) = q^{-1} \left(\prod_{n=1}^{\infty} (1 - q^n) \right)^{-24}, \quad q = e^{2\pi i \tau}$$

Integrating $I(\tau, \bar{\tau})$ with respect to the invariant measure $d\tau d\bar{\tau} (\text{Im } \tau)^{-2}$, one gets Z .

As the result (3.18) indicates, the presence of the factor $\text{Im } \tau$ prevents factorization. It is now one of the nice consequences of compactifying D coordinates, to cancel precisely this factor for each of the compactified dimensions, i.e., to turn the exponent of the factor in (3.18) from 12 into $(24-D)/2$. To see this, one has to perform a Jacobi transformation with respect to the sum over $\underline{w}^b$ in (3.11) to obtain

$$R \cdot (\gamma \text{Im } \tau)^{D/2} \det^{-1/2} (c q) \quad (3.19)$$

$$\sum_{\underline{w}^a \in \Lambda, \underline{w}^b \in \tilde{\Lambda}} \exp \left\{ \frac{i\pi}{2\gamma} \left(\tau [\underline{w}^a + \gamma \underline{w}^b]^2 - \bar{\tau} [\underline{w}^a - \gamma \underline{w}^b] \right) \right\}$$

g^{ij} is the metric on Λ

$$g^{ij} = \sum_I e_I^i e_I^j, \quad (3.20)$$

and the dual lattice $\tilde{\Lambda}$ is spanned by $\tilde{e}_I^i$, where

$$\tilde{e}_I^i = \sum_{j=1}^D (c q^{-1})^{ij} e_I^j. \quad (3.21)$$

We have set $J = 1$ to obtain (3.12). The factorization of $R \cdot I$ would now be immediate, were it not that in general

$$\sum_{\underline{w}^a, \underline{w}^b} \neq \sum_{\substack{\underline{w}^a + \gamma \underline{w}^b \\ \underline{w}^a - \gamma \underline{w}^b}} \quad (3.22)$$

even if $\gamma = 1$ and Λ self-dual. It is at this stage that the presence of a non-trivial J becomes important. To illustrate how J acts, let us take

$$J(\underline{w}^a, \underline{w}^b) = \exp \left(i \sum_{i=1}^D \theta^i n_b^i \right) \quad (3.23)$$

where the $n_b^i \in \mathbb{Z}$ come from

$$w_I^b = \sum_i e_I^i n_b^i \quad (3.24)$$

The effect of $J(\underline{w}^a, \underline{w}^b)$ on (3.11) is to shift the elements of $\tilde{\Lambda}$ by an amount $\theta^i/2\pi$

$$w_I^b \rightarrow w_I^b + \frac{1}{2\pi} \sum_{i=1}^D \theta^i \tilde{e}_I^i \quad (3.25)$$

If we now choose $\theta^i = 0$ if n_a^i is even, and $\theta^i = \pi$ if n_a^i is odd, i.e.

$$J(\underline{w}^a, \underline{w}^b) = \exp \left(+i\pi \sum_{i=1}^D n_a^i n_b^i \right), \quad (3.26)$$

then R can be rewritten as

$$R = (2 \operatorname{Im} \tau)^{D/2} \det^{-1/2} (g) \cdot \sum'_{\underline{w}^0 \in \tilde{\Lambda}/\Lambda} \sum_{\underline{w}^a, \underline{w}^b \in \Lambda} \exp \{ i\pi (\tau [\underline{w}^a + \underline{w}^a] - \bar{\tau} [\underline{w}^b + \underline{w}^0]) \} \quad (3.27)$$

provided that $\Lambda \subset \tilde{\Lambda}$ and $\gamma = 2$. In (3.27) the symbol Σ' indicates that the sum extends only over the generators of $\tilde{\Lambda}/\Lambda$. (3.27) is just the correction factor of Ref. 22). Note that $\gamma = 2$ is equivalent to the relation $\alpha' = R^2$ between the slope parameter α' and the radius of compactification R.

From (3.27) it is obvious that we can obtain a completely factorized expression for $R \cdot I$ if $\tilde{\Lambda} = \Lambda$. As $\Lambda = \tilde{\Lambda}$ implies $\det g = 1$, we finally have

$$R = (2 \operatorname{Im} \tau)^{D/2} |\hat{R}(\tau)|^2 \quad (3.28)$$

where

$$\hat{R}(\tau) = \sum_{\underline{w} \in \Lambda} e^{+i\pi \tau \underline{w}^2} \quad (3.29)$$

is independent of $\bar{\tau}$. Note that (3.29) is just a Θ -function defined on Λ ,

compare, e.g., Ref. 23). Consequently

$$R \cdot I(\tau, \bar{\tau}) = 2^{D/2} (Im \tau)^{-(24-D)/2} |\hat{R}(\tau) F(\tau)|^2 \quad (3.30)$$

Though with (3.30) we have demonstrated that it is possible to factorize $R \cdot I$, we still have to check modular invariance as $J(\underline{w}^a, \underline{w}^b) \neq 1$. Let us first analyze the behaviour under κ . As $J(\underline{w}^b, -\underline{w}^a) = J(\underline{w}^a, \underline{w}^b)$, $R \cdot I$ must be invariant under κ . To see this explicitly, note that

$$\hat{R}(-1/\tau) = e^{-i\pi D/4} \tau^{D/2} \hat{R}(\tau) \quad (3.31)$$

From (3.18) we know that $F(\tau)$ transforms like

$$F(-1/\tau) = F(\tau) \tau^{-12} \quad (3.32)$$

Consequently $F \cdot \hat{R}$ is invariant under κ . So all that remains, is to check invariance under τ_a . From (3.18) it follows that $F(\tau-1) = F(\tau)$. Therefore, $\hat{R}$ should be invariant under τ_a . Equation (3.19) then transforms this into the requirement that Λ has to be an even lattice.

So far we have demonstrated that the integrand $R \cdot I$ factorizes for all the compactified dimensions. The next step is then to amputate $R \cdot I$. This can be achieved by dividing out the contributions of, e.g., the right movers associated with the compactified dimensions

$$\hat{R}(\tau) F(\tau)^{D/24} \quad (3.33)$$

However, such a procedure is consistent only if simultaneously further right moving degrees of freedom are added, like the heterotic construction, to compensate for the lack of modular invariance of the expression left after removing (3.33). Though this expression is invariant under $\tau \rightarrow -1/\tau$, it picks up a phase of $2\pi D/24$ under $\tau \rightarrow \tau+1$.

4. - THE MULTILOOP AMPLITUDE

In the present section we shall generalize the arguments of the foregoing section to string world sheets of arbitrary genus. The key role herein is played by the Abelian differentials, which are well-known objects in the theory of Riemann surfaces.

4.1 Abelian differentials

In Section 2, we have discussed the procedure of representing a Riemann surface M in terms of the covering surface $\tilde{M}$ and a group of deck-transformations Γ . By performing a conjugation in Γ , we can always choose those transformations which map the corresponding arcs of the polygon representing Δ into each other to be the generators $\chi(a_p)$ and $\chi(b_p)$ of Γ (see Fig. 4). Letting the transformations χ act on a function of z and $\bar{z}$ by the obvious generalization of (3.4), the boundary conditions on $\hat{X}(z, \bar{z})$ read

$$\begin{aligned}\chi(a_p) \hat{X}_I(z, \bar{z}) &= \hat{X}_I(z, \bar{z}) + \sqrt{2} \pi R w_{I,p}^a \\ \chi(b_p) \hat{X}_I(z, \bar{z}) &= \hat{X}_I(z, \bar{z}) + \sqrt{2} \pi R w_{I,p}^b\end{aligned}\quad \rho=1, \dots, g \quad (4.1)$$

$w_{I,p}^a$ ($w_{I,p}^b$) represents a set of g vectors laying on Δ .

To solve (4.1), we need a set of functions $f_p(z)$ which transform like additive automorphic functions²⁴⁾, i.e.,

$$\chi f_p(z) = f_p(z) + \text{const.} \quad (4.2)$$

Such functions are easily constructed as

$$f_p(z) = \int_{z_0}^z dz' g_p(z') \quad (4.3)$$

provided the integral is independent of the path of integration. This is equivalent to requiring that $\omega_p := g_p(z)dz$ is a holomorphic one-form on M . In general such forms are called Abelian differentials. It is well known^{12), 24)} that the Abelian differentials span a g dimensional vector space. Let us denote a basis by ω_p , $p = 1, \dots, g$. The constants in (4.2) can then be represented by the so-called period matrices Π_a and Π_b

$$(\Pi_a)_{pq} := \int_{a_q} \omega_p \quad (\Pi_b)_{pq} := \int_{b_q} \omega_p \quad (4.4)$$

We have

$$\left. \begin{array}{l} \chi(a_q) \\ \chi(b_q) \end{array} \right\} f_p(z) = f_p(z) + \begin{cases} (\tilde{\pi}_a)_{pq} \\ (\tilde{\pi}_b)_{pq} \end{cases} \quad (4.5)$$

It is possible to define a scalar product $\langle \omega_1, \omega_2 \rangle$ of two one-forms ω_1 and ω_2 in such a way that in a preferred basis in which we shall work throughout

$$\langle \omega_p, \omega_q \rangle = -2i (\tilde{\pi}_b)_{pq} \quad (4.6)$$

If ω is an arbitrary one-form

$$\omega = u dz + v d\bar{z} \quad (4.7)$$

the Hodge operator^{*} acts like

$$\omega^* = -i u dz + i v d\bar{z} \quad (4.8)$$

and the exterior product of two one-forms ω_1 and ω_2 is

$$\omega_1 \wedge \omega_2 = (u_1 v_2 - v_1 u_2) dz \wedge d\bar{z} \quad (4.9)$$

Then $\langle , \rangle$ is defined by

$$\langle \omega_1, \omega_2 \rangle := \int_{\Delta} \omega_1 \wedge \bar{\omega}_2^* \quad (4.10)$$

where the bar above ω_2 denotes complex conjugation.

The period matrices have the properties

$$\tilde{\pi}_b = \tilde{\pi}_b^t \quad (4.11)$$

$$\tilde{\pi}_a \tilde{\pi}_b = \tilde{\pi}_b \tilde{\pi}_a^t \quad (4.12)$$

$$\operatorname{Re}(\tilde{\pi}_a) = 1 \quad (4.13)$$

$$\operatorname{Re}(\tilde{\pi}_b) = 0 \quad (4.14)$$

which we shall employ freely in the following. (t denotes transposition.)

On the torus

$$\omega = N dz \quad (4.15)$$

is an Abelian differential. In the basis which we have chosen, N has to be

$$N = 2 \bar{f}_b (f_a \bar{f}_b - \bar{f}_a f_b)^{-1} . \quad (4.16)$$

4.2 The correction factor R

With the help of the tools presented in Section 4.1, a function $\hat{X}(z, \bar{z})$, which solves (4.1), is immediately constructed

$$\begin{aligned} \hat{X}_I(z, \bar{z}) = \frac{\pi R}{\sqrt{2}} \left\{ [W_I^{a,t} - W_I^{b,t} \hat{\pi}_b^{-1} \hat{\pi}_a] f(z) \right. \\ \left. + [W_I^{a,t} - W_I^{b,t} \bar{\pi}_b^{-1} \bar{\pi}_a] \bar{f}(\bar{z}) \right\} . \end{aligned} \quad (4.17)$$

To verify that (4.17) indeed solves (4.1) one has to make use of (4.13) and (4.14).

In order to perform the next step, one has to recognize that the action (2.2) can be rewritten in terms of the scalar product (4.10) like

$$S = \frac{2\pi}{g} \langle \partial_z X^\nu dz, \partial_{\bar{z}} X^\nu d\bar{z} \rangle \quad (4.18)$$

where

$$\partial_z := \frac{1}{2} (\partial_1 - i \partial_2) . \quad (4.19)$$

We shall furthermore need the result that

$$\langle \partial_z \delta X^I dz, \partial_{\bar{z}} \hat{X}^I d\bar{z} \rangle = 0 . \quad (4.20)$$

(4.20) is immediately demonstrated by starting from the fact that

$$\int_{\partial \Delta} \delta X^I \partial_{\bar{z}} \hat{X}^I d\bar{z} = 0 \quad (4.21)$$

and applying Stoke's theorem.

Inserting now (4.17) into the generalization of (3.8) and making use of (4.6), (4.17) and (4.20), one obtains the following correction factor

$$R = \sum_{\underline{w}_p^a, \underline{w}_q^b \in \Lambda} e^{-\hat{S}} J(\underline{w}_p^a, \underline{w}_q^b) \quad (4.22)$$

where

$$\hat{S} = \frac{1}{\gamma} [\underline{w}^{a,t} \hat{\Pi} - \underline{w}^{b,t}] [Im \hat{\Pi}]^{-1} [\hat{\Pi}^t \underline{w}^a - \underline{w}^b], \quad (4.23)$$

and

$$\hat{\Pi} = \hat{\Pi}_a^{-1} \hat{\Pi}_b. \quad (4.24)$$

4.3 Mapping class transformations

Let us now turn to the generalization of the modular transformations to arbitrary g . The generalized modular group is called the mapping class group in the mathematical literature [e.g., Ref. 25)].

This group is generated by the Dehn twists which we have already introduced in Section 3.2. The problem of finding a finite set of generators for the mapping class group in terms of Dehn twists has been solved by Lickorish²⁶⁾. The finite set is represented by the Dehn twists around the curves c_r , $r = 1, \dots, g-1$ shown in Fig. 5, in addition to the Dehn twists around the canonical set of generators of $\Pi_1(M)$. We have, therefore, the following set of generators of the mapping class group

$$\begin{aligned} \tau_{a_p} : b_p &\rightarrow b_p a_p \\ \tau_{b_p} : a_p &\rightarrow a_p b_p \\ \tau_{c_p} : a_p &\rightarrow a_p b_p^{-1} b_{p+1} \quad a_{p+1} \rightarrow b_{p+1}^{-1} b_p^{+1} a_{p+1}, \end{aligned} \quad (4.25)$$

together with their inverse transformations. In (4.25), we have only displayed those generators of $\Pi_1(M)$ which are not left unchanged by the corresponding Dehn twist.

The transformations (4.25) act in general on the Teichmüller parameters of M . In the foregoing section, we saw that for $g = 1$ the Teichmüller parameter τ is just the ratio of two fundamental periods, see (3.10). The equivalence of fundamental periods and Teichmüller parameters happens to be a coincidence which only appears if $g = 1$. It ceases to hold for surfaces of higher genus. In the general case, the period matrix is a holomorphic function of the Teichmüller parameters.

Fortunately the Teichmüller parameters enter R only via the period matrices. Therefore, we can analyze the transformation properties of R solely in terms of the behaviour of Π_a and Π_b under (4.25). In other words, we have to consider the action of Siegel's modular group, which acts on $a_1, \dots, a_g, b_1, \dots, b_g$ considered as the generators of the first homology group of M . The transformation rules of the matrices Π_a and Π_b are immediately derived from (4.5).

$$\begin{aligned} \tau_{a\tau}: \quad (\tilde{\Pi}_a)_{pq} &\rightarrow (\tilde{\Pi}_a)_{pq} \\ (\tilde{\Pi}_b)_{pq} &\rightarrow (\tilde{\Pi}_b)_{pq} + (\tilde{\Pi}_a)_{p\tau} \delta_{q\tau} \end{aligned} \quad (4.26a)$$

$$\begin{aligned} \tau_{b\tau}: \quad (\tilde{\Pi}_a)_{pq} &\rightarrow (\tilde{\Pi}_a)_{pq} + (\tilde{\Pi}_b)_{p\tau} \delta_{q\tau} \\ (\tilde{\Pi}_b)_{pq} &\rightarrow (\tilde{\Pi}_b)_{pq} \end{aligned} \quad (4.26b)$$

$$\begin{aligned} \tau_{c\tau}: \quad (\tilde{\Pi}_a)_{pq} &\rightarrow (\tilde{\Pi}_a)_{pq} + [-(\tilde{\Pi}_b)_{p\tau} + (\tilde{\Pi}_b)_{p, \tau+1}] \cdot \\ &\quad \cdot [\delta_{q, \tau} - \delta_{q, \tau+1}] \\ (\tilde{\Pi}_b)_{pq} &\rightarrow (\tilde{\Pi}_b)_{pq} \end{aligned} \quad (4.26c)$$

We can now in analogy with (3.15) introduce the operator

$$\mathcal{X} := \prod_{\rho=1}^g \tau_{a\rho}^{-1} \tau_{b\rho} \tau_{a\rho}^{-1} \quad (4.27)$$

and drop τ_{b_r} in favour of κ from the list of generators. κ acts on the period matrices like

$$\mathcal{R}: \tilde{\Pi}_a \rightarrow \tilde{\Pi}_b, \quad \tilde{\Pi}_b \rightarrow -\tilde{\Pi}_a \quad (4.28)$$

i.e.,

$$\mathcal{R}: \tilde{\Pi} \rightarrow -\tilde{\Pi}^{-1} \quad (4.29)$$

The transformation properties of R , as given by (4.22) under mapping class transformations, can now be studied in almost the same fashion as previously for $g = 1$. It is immediately obvious that if $J = 1$, R is invariant.

4.4 Factorization

Let us finally show that the property of factorization, demonstrated in Section 3.3 for $g = 1$, is not spoiled at higher loops. Putting

$$\gamma(\underline{w}_p^a, \underline{w}_q^b) = \exp \left\{ +i\tilde{\Pi} \sum_{j=1}^D \sum_{\rho=1}^g n_{j,\rho}^b n_{j,\rho}^a \right\} \quad (4.30)$$

into (4.22), R can be factorized as

$$R = (2)^{\frac{1}{2}g \cdot D} \det^{\frac{1}{2}D} (Im \tilde{\Pi}) |\hat{R}(\tilde{\Pi})|^2 \quad (4.31)$$

with

$$\hat{R}(\tilde{\Pi}) = \sum_{\underline{w}_\rho \in \Lambda} \exp \{ +i\tilde{\Pi} \underline{w}^t \tilde{\Pi} \underline{w} \} \quad (4.32)$$

Note that $\hat{R}(\Pi)$ is a generalization of the Riemann Θ -function which is usually defined for Riemann surfaces of arbitrary g in terms of $\Pi^{(12)}$.

The generalization of (3.18) reads

$$I(\epsilon, \bar{\epsilon}) = \det^{-d/2} (Im \tilde{\Pi}) |\bar{F}(\epsilon)|^2 \quad (4.33)$$

where $F(\tau)$ is an analytic function of τ which is unknown in general. Consequently

$$R \cdot I(\epsilon, \bar{\epsilon}) = (2)^{\frac{1}{2}g \cdot D} (\det Im \tilde{\Pi})^{-\frac{1}{2}(d \cdot D)} |\hat{R}(\tilde{\Pi}) \bar{F}(\epsilon)|^2 \quad (4.34)$$

where τ stands generically for the $3(g-1)$ Teichmüller parameters.

All that remains to be done is to check that $\hat{R}(\Pi)F(\tau)$ is indeed invariant under mapping class transformations. Under κ we have

$$\hat{R}(-\hat{\Pi}^{-1}) = e^{-i\frac{1}{4}\bar{g}\cdot D} \det^{\frac{1}{2}D}(\hat{\Pi}) \hat{R}(\hat{\Pi}). \quad (4.35)$$

Exploiting the fact that (4.33) is invariant under κ we have

$$\bar{F}(\tau) \xrightarrow{\kappa} \bar{F}'(\tau) = \pm e^{i\hat{\Pi}g\cdot D/4} (\det \hat{\Pi})^{-1/2} \bar{F}(\tau). \quad (4.36)$$

The phase factor in (4.36) has been determined by noticing that $\kappa = \kappa^{-1}$. The overall sign cannot be fixed solely from (4.33), its determination requires further knowledge about $F(\tau)$. The product $\hat{R}(\Pi)F(\tau)$ is indeed invariant under κ , provided $F(\tau)$ transforms with the plus sign. Invariance of $|\hat{R}(\Pi)F(\tau)|^2$ under τ_{a_p} is established in the same way as for $g = 1$. It does not lead to any further restriction on Γ , beyond being even. To show invariance under τ_{c_p} , it is advantageous to exchange first Π_a and Π_b by κ , and then use the same method as for τ_{a_p} .

5. - FINAL REMARKS

Let us briefly comment on our results. In the first part of this paper, we have shown that it is possible to factorize the integrand of the vacuum amplitude of the bosonic oriented string, represented in terms of Polyakov's path integral for $g = 1$ into the product of two analytic functions representing left and right movers. It has been demonstrated that this factorization is possible provided Λ is of the same type as required for the construction of the heterotic string. We would like to emphasize that we have made no systematic search for the most general type of Λ , which allows for factorization.

In the second part, we have demonstrated that the pattern found for $g = 1$ is not upset if $g > 1$. We have restricted our consideration in the present paper to the vacuum amplitude, because on the one hand it is technically simple enough to allow for the necessary manipulations. On the other hand, we believe that it displays all those aspects related to factorization and mapping class invariance, which will be encountered for other non-trivial Green functions. However, this belief has to be substantiated before one can conclude that our prescription leads to a well-defined theory.

A further point which should be emphasized concerns the process of amputation. As we have indicated at the end of Section 3.3, a consistent amputation of, e.g., the right movers associated with the compactified dimensions necessitates the inclusion of further right moving degrees of freedom in the uncompactified dimensions. To proof consistency after amputation, one has obviously to consider these additional degrees of freedom as well. However, this should not result in any further requirements on Λ .

Conclusions similar to ours, concerning the consistency of higher loop contributions to the heterotic string have previously be drawn in Ref. 27). In Ref. 27) the fermionic representation of the internal degrees of freedom of the heterotic string has been employed. This method may be viewed as an alternative way of defining the path integral measure which does not make use of the factorization employed in the present paper. For some progress using the fermionic representation, see Refs. 28) and 29).

ACKNOWLEDGEMENTS

The author would like to thank M.A. Namazie, A. Neveu, A. Schellekens and J. Thierry-Mieg for instructive discussions.

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FIGURE CAPTIONS

Fig. 1 : Canonical basis of fundamental curves for a Riemann surface with $g = 2$.

Fig. 2 : Fundamental region of a Riemann surface with $g = 2$ on $\tilde{M}$.

Fig. 3 : Fundamental region of the torus.

Fig. 4 : Action of the generators $\chi(a_p)$ and $\chi(b_p)$ of Γ .

Fig. 5 : Generating curves of the mapping class group of M .

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5