Hard bremsstrahlung amplitudes for e^+e^- collisions with polarized beams
at LEP/SLC energies

Ronald Kleiss

CERN - Geneva

Abstract

Cross sections are presented for the processes $e^+e^- \rightarrow e^+e^-\gamma$ and $e^+e^- \rightarrow f\bar{f}\gamma$ (where f is any light fermion) in the form of helicity amplitudes in which all complex phase information is kept. Owing to a special choice of the bremsstrahlung polarization and the use of spinor techniques the resultant expressions are extremely compact and lend themselves to fast numerical evaluation. An algorithm is given to obtain from the helicity amplitudes the cross section for arbitrary polarization of the incoming e^+e^- beams. Special attention is given to the case of bremsstrahlung at very small angles to the beams, where a complicated dependence on the electron mass has to be taken into account.

Ref. CERN-TH.4448/86

May 1986

1. Introduction

This paper deals with the calculation of bremsstrahlung amplitudes in e^+e^- collisions where the beams have nonzero polarization. The motivations for this study are twofold. In the first place, it has become clear that polarized beams can be a useful tool in the study of both standard-model interactions [1] and 'nonstandard' ones (see, for instance, [2]). For many processes the cross sections have been calculated for the case of either longitudinally or transversely polarized beams (a very complete list of the existing calculations for transverse polarization is given in [3]). A recent paper [4] even deals with the production and decay of generally polarized Z_0 bosons in e^+e^- collisions. As far as precision tests of the standard model are concerned, Greco et al [5] have calculated radiative corrections for $e^+e^- \rightarrow \mu^+\mu^-$ for transversely polarized beams, taking soft photon bremsstrahlung into account. Bohm and Hollik [6] have performed a similar calculation including the possibility of longitudinal polarization as well. On the other hand, the phenomenology of the standard-model interactions can only be studied at an acceptable level of accuracy if also the hard photon bremsstrahlung is well understood [7]. Therefore, the cross sections for hard photon emission in processes such as $e^+e^- \rightarrow e^+e^-$, $e^+e^- \rightarrow \mu^+\mu^-$ and $e^+e^- \rightarrow \text{hadrons}$ have to be known also for the case of polarized beams. So far, only Kuraev et al [8] have derived expressions valid for muon pair production at high energies with purely transverse beam polarization, and without the effect of a Z^0 boson. In this paper we shall present cross

section formulae that are valid at LEP energies, i.e. with Z^0 exchange taken into account, and where arbitrary polarization of the beams is considered.

The second motivation is of a more technical nature. In the last few years much progress has been made in the calculation of cross sections for bremsstrahlung and related tree-level processes. Original attention focused on the calculation of bremsstrahlung amplitudes in QED for massless fermions [9]. More recently, it has been shown how to take effects of small but finite fermion masses into account [10] and the more complicated case of QCD has also been studied, while at the same time a number of techniques have been developed which enable one to manipulate expressions involving massless spinors [11] [12] [13] [14] [15]. It is shown in this paper that the techniques developed for these calculations, which make use of fermion states with a definite helicity, can also be used to handle more generally polarized states. Any detailed phenomenology of the radiative corrections for polarized beams is outside of the scope of this paper: here, we restrict ourselves to finding compact and elegant expressions for the cross section, which we define in this paper to be the square of the matrix element, suitably summed and averaged over the spins.

The layout of the paper is as follows. In section 2 a brief review is given of the spinor techniques developed in refs. [11] [12] [13] which we shall use in this paper as well. Also, a particular representation for the bremsstrahlung polarization vector is discussed, which greatly simplifies the calculation. In section 3 we present the results for the helicity amplitudes of the processes $e^+e^- \rightarrow e^+e^-\gamma$ and $e^+e^- \rightarrow f\bar{f}\gamma$, where f stands for any light

charged fermion (i.e. $\mu, \tau, u, d, s, c, \dots$). In section 4 we discuss how of these helicity amplitudes, which only describe fermions with longitudinal polarization, complex linear combinations can be formed such that generally polarized states are described. In section 5 this is specialized to the case of e^+e^- collisions with polarized beams. Since our spinor techniques, and the helicity states formalism, are only justified for essentially massless particles, care has to be taken in situations where the bremsstrahlung photon is collinear with a charged fermion. In particular bremsstrahlung along the incoming beams has to be treated very carefully. This is discussed in section 6. Finally, a number of technicalities are collected in Appendices A through D.

2. Spinor techniques

It is clear that if we want to construct general polarization states as complex combinations of helicity states, the complex phases of the helicity amplitudes are very important, and we will need computational techniques that are adequate to handle these phases. The original treatment of bremsstrahlung amplitudes in [9] is not very well suited for this purpose. Instead, we shall use so-called SPINOR TECHNIQUES. We shall begin this section by briefly reviewing those spinor techniques which are relevant to the present calculation. A more complete treatment can for instance be found in [13]. We start by choosing a pair of four-vectors k_0^μ and k_1^μ that satisfy

$$k_0 \cdot k_0 = 0, \quad k_1 \cdot k_1 = -1, \quad k_0 \cdot k_1 = 0.$$

For any lightlike four-momentum p^μ the two spinors¹ corresponding to states with definite helicities $+$ and $-$ can be defined as

$$\begin{aligned} u_+(p) &= \not{p} u_-(k_0) / \sqrt{2p \cdot k_0} , \\ u_-(p) &= \not{p} \not{K}_1 u_-(k_0) / \sqrt{2p \cdot k_0} , \end{aligned}$$

where $u_-(k_0)$ is a negative-helicity spinor with momentum k_0 :

$$\begin{aligned} u_-(k_0) \bar{u}_-(k_0) &= \omega_- \not{K}_0 , \\ \omega_\pm &= (1 \pm \gamma^5) / 2 . \end{aligned}$$

The spinors defined² in this way satisfy the Dirac equation for massless helicity states:

$$u_\pm(p) \bar{u}_\pm(p) = \omega_\pm \not{p} .$$

¹ Since the particles are massless the distinction between particles and antiparticles is irrelevant to the calculation, and we shall use the same symbol to denote both spinors and antispinors.

² Of course this is only possible if the vectors are such that $p \cdot k_0$ does not vanish. Some care is therefore necessary in the choice of k_0 . In practice, this is never a serious problem, and we shall always use the choice made here.

It is clear that this way of constructing the spinors is nothing but the use of a particular convention for the complex phases of the spinors. One useful property of this convention is that the following relation holds :

$$\bar{u}_{\pm}(p_1) \gamma^{\mu} u_{\pm}(p_2) \gamma_{\mu} = 2(u_{\pm}(p_2) \bar{u}_{\pm}(p_1) + u_{\mp}(p_1) \bar{u}_{\mp}(p_2)),$$

which can be used to eliminate repeated indices. We furthermore define a SPINOR PRODUCT as follows:

$$\begin{aligned} s_{+}(p_1, p_2) &= \bar{u}_{+}(p_1) u_{-}(p_2) = -s_{+}(p_2, p_1) , \\ s_{-}(p_1, p_2) &= \bar{u}_{-}(p_1) u_{+}(p_2) = -s_{+}(p_1, p_2)^{*} . \end{aligned}$$

With the above definition of the spinors we can employ standard trace techniques to obtain the expression for $s_{\pm}(p_1, p_2)$ in terms of the four-vectors k_0 , k_1 , p_1 and p_2 . If we adopt the following convention for k_0 and k_1 :

$$\begin{aligned} (k_0^0, k_0^1, k_0^2, k_0^3) &= (1 , 1 , 0 , 0) , \\ (k_1^0, k_1^1, k_1^2, k_1^3) &= (0 , 0 , 1 , 0) , \end{aligned}$$

we obtain for the spinor products:

$$s_{+}(p_1, p_2) = (p_1^2 + ip_1^3) \sqrt{(p_2^0 - p_2^1)} / \sqrt{(p_1^0 - p_1^1)} - (p_1 \cdot p_2) ,$$

which can easily be evaluated by complex computer arithmetic. The VECTOR PRODUCT of p_1 and p_2 can be obtained from the relation

$$s_{+}(p_1, p_2) s_{-}(p_2, p_1) = 2 p_1 \cdot p_2 .$$

This finishes our presentation of the spinor techniques.

We are now in principle able to compute the Feynman diagrams for our bremsstrahlung processes in terms of spinor products, as soon as we have an expression for the photon polarization which can also be treated with these methods. We want to have a representation which has a clear complex phase structure, and leads to a simple result. The form used by the CALKUL collaboration in [9] gives very simple results for the amplitudes, apart from a complex phase difference between bremsstrahlung from different fermion lines. The reason for this is as follows. The polarization vector is defined using the photon momentum and two other auxiliary momenta. It turns out that these can most conveniently be chosen to be the momenta of the two fermions at the endpoints of the fermion line which radiates the photon. Therefore, diagrams with radiation from different fermion lines (belonging to different gauge-invariant sets) are characterized by different auxiliary momenta. This leads to the polarization vectors for different fermion lines to be different from each other by a gauge transformation (which is irrelevant) and a complex phase, which has to be compensated by an explicit phase factor in the result for the amplitude. We now show how this situation can be improved upon by employing a complex normalization of the polarization vector. We shall use a form for ϵ , the contraction of the polarization vector with a Dirac gamma matrix³ as follows:

³ Strictly speaking this expression also contains an axial part, but since this is proportional to the photon momentum k it drops out in massless QED because of gauge invariance. See ref. [9].

$$\mathbb{E}_\lambda(k, p_1, p_2) = N_\lambda(p_1, p_2) [\omega_\lambda \not{k} \not{p}_1 \not{p}_2 - \omega_{-\lambda} \not{p}_1 \not{p}_2 \not{k}],$$

$$N_+(p_1, p_2)^{-1} = s_+(k, p_1) s_-(p_1, p_2) s_+(p_2, k) / \sqrt{2},$$

$$N_-(p_1, p_2) = N_+(p_1, p_2)^*,$$

where k is the photon momentum and λ its helicity, and the vectors p_1 and p_2 are arbitrary massless four-momenta, not proportional to k or to each other. It is easily checked that this object has the correct properties for a polarization vector corresponding to a definite helicity state, i.e. $k \cdot \mathbb{E}_\pm = 0$, $\mathbb{E}_\pm \cdot \mathbb{E}_\pm = 0$, $\mathbb{E}_- = (\mathbb{E}_+)^*$, $\mathbb{E}_+ \cdot \mathbb{E}_- = -1$; moreover, using the spinor products defined above we can calculate the product of two DIFFERENT polarization vectors as the trace of their two $\mathbb{E}$'s :

$$\begin{aligned} & 4\mathbb{E}_+(k, p_1, p_2) \cdot \mathbb{E}_-(k, q_1, q_2) \\ &= N_+(p_1, p_2) N_-(q_1, q_2) \times \\ & \text{Tr}[(\omega_+ \not{k} \not{p}_1 \not{p}_2 - \omega_- \not{p}_1 \not{p}_2 \not{k})(\omega_- \not{k} \not{q}_1 \not{q}_2 - \omega_+ \not{q}_1 \not{q}_2 \not{k})] \\ &= 2 N_+(p_1, p_2) N_-(q_1, q_2) \bar{u}_+(k) \not{p}_1 \not{p}_2 \not{k} \not{q}_1 \not{q}_2 u_+(k) \\ &= 2 N_+(p_1, p_2) N_-(q_1, q_2) \times \\ & s_+(k, p_1) s_-(p_1, p_2) s_+(p_2, k) s_-(k, q_1) s_+(q_1, q_2) s_-(q_2, k) \\ &= -4 \end{aligned}$$

It is seen that all the polarization vectors actually have the same complex phase, even if they are based on different sets of auxiliary momenta. This

means that we can use different polarizations in different gauge-invariant subsets of Feynman diagrams without having to worry about compensating phase factors, while retaining the great simplifications present in the original work [9]. Using Eq.(2.4) we can write $\mathbb{E}$ in an even more compact form:

$$\mathbb{E}_\lambda(k, p_1, p_2) = \lambda \sqrt{2} [u_\lambda(k) \bar{u}_\lambda(p_2) / s_\lambda(p_2, k) - u_{-\lambda}(p_1) \bar{u}_{-\lambda}(k) / s_\lambda(k, p_1)] .$$

This is the definition which we shall use in the following. We are now able to evaluate the bremsstrahlung amplitudes, which is the subject of the next section.

3. Helicity amplitudes for hard bremsstrahlung

In this section we shall present the helicity amplitudes for

1. radiative production of a pair of massless fermions:

$$e^+(p_1, \lambda_1) e^-(p_2, \lambda_2) \rightarrow f(q_1, \nu_1) \bar{f}(q_2, \nu_2) \gamma(k, \lambda),$$

where p_1, p_2, q_1, q_2 and k are the momenta of the particles, and $\lambda_1, \lambda_2, \nu_1, \nu_2$ and λ are the corresponding helicities;

2. radiative Bhabha scattering:

$$e^+(p_1, \lambda_1) e^-(p_2, \lambda_2) \rightarrow e^-(q_1, \nu_1) e^+(q_2, \nu_2) \gamma(k, \lambda),$$

with the same notation.

The first process is described by 8 Feynman diagrams and gives rise to 8 nonvanishing helicity amplitudes; the second process is described by 16 similar diagrams and leads to 12 nonzero amplitudes. This makes a systematic treatment in terms of GENERIC DIAGRAMS [13] attractive. The two diagrams depicted in fig.1 form a gauge-invariant set with respect to the emitted bremsstrahlung photon (In this figure the internal boson line represents the combination of the propagators of the photon and the Z^0). The momenta k_1, k_2, k_3 and k_4 are generic momenta, to which we shall later assign specific values, and $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are the corresponding helicities. Using the Feynman rules we find, after some simple manipulations, the following expression for the generic diagrams:

$$\begin{aligned}
 G(k_1, k_2, k_3, k_4) = & iQ_1 \delta_{\lambda_1 \lambda_2} \delta_{\lambda_3 \lambda_4} \times \\
 & \bar{u}_{\lambda_1}(k_1) [\not{\epsilon} (b_2 k_2 + b_3 k_3 + b_4 k_4) \not{\gamma}^\mu / (2b_1 k_1 \cdot k) \\
 & - \not{\gamma}_\mu (b_2 k_1 + b_3 k_3 + b_4 k_4) \not{\epsilon} / (2b_2 k_2 \cdot k)] u_{\lambda_1}(k_2) \\
 & \times \bar{u}_{\lambda_3}(k_3) \not{\gamma}_\mu u_{\lambda_3}(k_4) \\
 & \times [Q_1 Q_3 / (2b_3 b_4 k_3 \cdot k_4) \\
 & + (v_1 - \lambda_1 a_1)(v_3 - \lambda_3 a_3) / (2b_3 b_4 k_3 \cdot k_4 - M^2 + iM\Gamma)] .
 \end{aligned}$$

The Kronecker δ 's arise from conservation of helicity along massless fermion lines. The form for the fermion propagators is obtained using momentum conservation, expressed as

$$b_1 k_1^\mu + b_2 k_2^\mu + b_3 k_3^\mu + b_4 k_4^\mu + k^\mu = 0 ,$$

where the coefficient b_i is $+1(-1)$ if the momentum k_i is outgoing(incoming). The coupling Q_i is the electromagnetic charge of the particle with momentum

k_i , and v_i and a_i denote its vector and axial-vector coupling to the Z_0 . M and Γ denote the Z_0 mass and total width. Since these two Feynman diagrams form a gauge-invariant set (with respect to the bremsstrahlung photon), we may choose at will any $\mathbb{E}$ of the form introduced in the previous section: we shall use

$$\mathbb{E} = \mathbb{E}_\lambda(k, k_1, k_2)$$

With the spinor techniques and this representation for $\mathbb{E}$ it becomes extremely straightforward to work out the amplitudes for a given assignment of the momenta and helicities. In fact this can conveniently be done using REDUCE [16]. We now present the results. The helicity amplitude for radiative fermion pair production can be expressed in terms of the generic diagram as follows:

$$M_f(\lambda_1, \lambda_2, v_1, v_2, \lambda) = G(p_1, p_2, q_1, q_2) + G(q_1, q_2, p_1, p_2)$$

The corresponding nonvanishing amplitudes are given in table 1. In these expressions we have introduced two RADIATION FACTORS, corresponding to 'initial-' and 'final-state' radiation, respectively:

$$\begin{aligned} v_p &= s_-(q_1, q_2) / [s_+(k, p_1) s_+(p_2, k)] , \\ v_q &= s_-(p_1, p_2) / [s_+(k, q_1) s_+(q_2, k)] . \end{aligned}$$

Furthermore, $-e$ stands for the electric charge of the electron, and v and a are its vector and axial-vector couplings to the Z_0 ; the corresponding coupling constants for the outgoing fermions are e' , v' and a' , respec-

tively. The kinematic invariants are defined by $s_0=(p_1+p_2)^2$ and $s_1=(q_1+q_2)^2$.

Table 1: Helicity amplitudes for $e^+e^- \rightarrow f\bar{f}\gamma$

λ_1	λ_2	v_1	v_2	λ	$M_f(\lambda_1, \lambda_2, v_1, v_2, \lambda)$
+	+	+	+	+	$s_+(p_1, q_1)^2 [eF_{++}(s_1)v_p - e'F_{++}(s_0)v_q]$
+	+	+	+	-	$s_-(p_2, q_2)^2 [eF_{++}(s_1)v_p^* - e'F_{++}(s_0)v_q^*]$
+	+	-	-	+	$-s_+(p_1, q_2)^2 [eF_{+-}(s_1)v_p - e'F_{+-}(s_0)v_q]$
+	+	-	-	-	$-s_-(p_2, q_1)^2 [eF_{+-}(s_1)v_p^* - e'F_{+-}(s_0)v_q^*]$
-	-	+	+	+	$-s_+(p_2, q_1)^2 [eF_{-+}(s_1)v_p - e'F_{-+}(s_0)v_q]$
-	-	+	+	-	$-s_-(p_1, q_2)^2 [eF_{-+}(s_1)v_p^* - e'F_{-+}(s_0)v_q^*]$
-	-	-	-	+	$s_+(p_2, q_2)^2 [eF_{--}(s_1)v_p - e'F_{--}(s_0)v_q]$
-	-	-	-	-	$s_-(p_1, q_1)^2 [eF_{--}(s_1)v_p^* - e'F_{--}(s_0)v_q^*]$

$$F_{v_1 v_2}(x) = i\sqrt{8} [-ee'/x + (v-v_1a)(v'-v_2a')/(x-M^2+iM\Gamma)]$$

In a similar way, radiative Bhabha scattering can be treated. We now have twice as many diagrams: on the other hand, all fermions have identical coupling constants. The helicity amplitude is given in terms of generic diagrams as follows:

$$M_e(\lambda_1, \lambda_2, \nu_1, \nu_2, \lambda) = G(p_1, p_2, q_1, q_2) + G(q_1, q_2, p_1, p_2) \\ - G(p_1, q_2, q_1, p_2) - G(q_1, p_2, p_1, q_2)$$

where the minus signs are due to Fermi statistics. The result for the nonzero amplitudes is given in table 2, where we have introduced two more radiation factors:

$$\nu_+ = s_-(q_1, p_2) / [s_+(k, p_1) s_+(q_2, k)] , \\ \nu_- = s_-(p_1, q_2) / [s_+(k, q_1) s_+(p_2, k)] ,$$

which correspond to bremsstrahlung from the positron line (ν_+) or the electron line (ν_-) in the diagrams with t-channel exchange of photon or Z_0 . The two corresponding kinematic invariants are defined as $t_0 = (p_1 - q_2)^2$ and $t_1 = (p_2 - q_1)^2$.

This finishes our presentation of the helicity amplitudes. The results in tables 1 and 2 are extremely compact and easy to program. Their evaluation by complex computer arithmetic is very fast, and the relative complex phases of the amplitudes are implicit. Moreover, the effects of any number of additional neutral currents can easily be incorporated. As for the complexity of our results, they should be compared with, for instance, the results of the CALKUL group [9] for these two processes, where in particu-

Table 2: Helicity amplitudes for $e^+e^- \rightarrow e^+e^-\gamma$

λ_1	λ_2	v_1	v_2	λ	$M_e(\lambda_1, \lambda_2, v_1, v_2, \lambda)$
+	+	+	+	+	$s_+(p_1, q_1)^2 [E_{++}(s_1)v_p + E_{++}(s_0)v_q + E_{++}(t_1)v_+ + E_{++}(t_0)v_-]$
+	+	+	+	-	$s_-(p_2, q_2)^2 [E_{++}(s_1)v_p^* + E_{++}(s_0)v_q^* + E_{++}(t_1)v_+^* + E_{++}(t_0)v_-^*]$
+	+	-	-	+	$-s_+(p_1, q_2)^2 [E_{+-}(s_1)v_p + E_{+-}(s_0)v_q]$
+	+	-	-	-	$-s_-(p_2, q_1)^2 [E_{+-}(s_1)v_p^* + E_{+-}(s_0)v_q^*]$
+	-	-	+	+	$s_+(p_1, p_2)^2 [E_{+-}(t_1)v_+ + E_{+-}(t_0)v_-]$
+	-	-	+	-	$s_-(q_1, q_2)^2 [E_{+-}(t_1)v_+^* + E_{+-}(t_0)v_-^*]$
-	+	+	-	+	$s_+(q_1, q_2)^2 [E_{-+}(t_1)v_+ + E_{-+}(t_0)v_-]$
-	+	+	-	-	$s_-(p_1, p_2)^2 [E_{-+}(t_1)v_+^* + E_{-+}(t_0)v_-^*]$
-	-	+	+	+	$-s_+(p_2, q_1)^2 [E_{-+}(s_1)v_p + E_{-+}(s_0)v_q]$
-	-	+	+	-	$-s_-(p_1, q_2)^2 [E_{-+}(s_1)v_p^* + E_{-+}(s_0)v_q^*]$
-	-	-	-	+	$s_+(p_2, q_2)^2 [E_{--}(s_1)v_p + E_{--}(s_0)v_q + E_{--}(t_1)v_+ + E_{--}(t_0)v_-]$
-	-	-	-	-	$s_-(p_1, q_1)^2 [E_{--}(s_1)v_p^* + E_{--}(s_0)v_q^* + E_{--}(t_1)v_+^* + E_{--}(t_0)v_-^*]$

$$E_{v_1 v_2}(x) = i\sqrt{8} [e^3/x - e(v-v_1 a)(v-v_2 a)/(x-M^2+iM\Gamma)]$$

lar the effects of a finite Z width causes the expressions to be quite opaque.

Since in the above derivation we have not neglected the complex phase differences between the various helicity amplitudes, we can make complex linear combinations of the amplitudes which will then describe states with non-longitudinal polarization. This we shall discuss in the next section.

4. Arbitrary polarization from helicity states

We shall now discuss how to obtain fermion states with a more general polarization, defined by a polarization vector s^μ with the following properties:

$$s \cdot s = -1 \quad , \quad p \cdot s = 0 \quad .$$

The polarization vector $\vec{s}$ can be pointing in any direction except that of $\pm\vec{p}$; if it is orthogonal to $\vec{p}$ in some reference frame, we call it transverse in that frame (note that, in contrast to helicity, this is not a Lorentz-invariant concept). The massless Dirac spinor $u(p,s)$ corresponding to a state with the above polarization is defined by its projection operator :

$$u(p,s)u(p,s) = (1 + \gamma^5 \not{s}) \not{p} / 2 \quad ,$$

and, in analogy to the treatment in section 2, we see that a valid definition of this spinor is given by

$$u(p,s) = (1 + \gamma^5 \not{s}) u_+(p) / \sqrt{2} ,$$

where we have related it to the positive-helicity spinor with the same momentum. This allows us to evaluate the above definition with the use of spinor techniques as follows:

$$\begin{aligned} u(p,s) &= [u_+(p) + \not{s} \gamma^5 u_-(k_0) / \sqrt{2p \cdot k_0}] / \sqrt{2} \\ &= [u_+(p) + \not{s} u_-(k_0) / \sqrt{2p \cdot k_0}] / \sqrt{2} \\ &= [u_+(p) + \Phi(p,s) u_-(p)] / \sqrt{2} \end{aligned}$$

where the complex phase factor $\Phi(p,s)$ is defined as

$$\begin{aligned} \Phi(p,s) &= \bar{u}_-(p) \not{s} u_-(k_0) / \sqrt{2p \cdot k_0} \\ &= (s^2 - is^3) - (s^0 - s^1)(p^2 - ip^3) / (p^0 - p^1) \end{aligned}$$

In the last line of this result we have made use of the trace techniques and the choice for the vectors k_0^μ and k_1^μ discussed in section 2. An illustrative case is that of a fermion with momentum $p^\mu = (p^0, p^1, p^2, p^3) = (E, 0, 0, vE)$ and polarization vector $s^\mu = (s^0, s^1, s^2, s^3) = (0, v', 0, 0)$, where v and v' are 1 or -1. In that case the phase factor is just $\Phi(p,s) = -vv'i$. In principle, the above result is sufficient to describe arbitrarily polarized states of massless fermions. Before we proceed to apply this to the computation of actual cross sections, however, we want to make a few remarks. In the first place, the above decomposition is not symmetric with respect to the two spinors $u_+(p)$ and $u_-(p)$. This is of course due to the fact that we have based our definition of $u(p,s)$ on $u_+(p)$; had we used $u_-(p)$ instead, an analogous complex phase $\Phi'(p,s)$ would occur in front of $u_+(p)$ instead

of in front of $u_-(p)$. In the second place, since $|\Phi|^2=1$ both helicity states always occur in the decomposition, no matter how close the direction of the spin $\vec{s}$ is to $\vec{p}$ or $-\vec{p}$. This is due to the fact that we have assumed the mass to be zero: in appendix A we shall show how we can sensibly take the longitudinal limit for nonzero mass, and recover longitudinally polarized states that go over to pure helicity states in a smooth way.

The above results hold for a spinor in a pure polarization state. In practice, however, we have to deal with electron and positron beams with partial polarization, that can be considered as incoherent superpositions of pure states. To be more precise, a high-energy $e^\pm$ beam can be characterized by the following operator:

$$\Omega(P_{||}, P_{\perp}) = (1 + P_{||}\gamma^5 + P_{\perp}\gamma^5 \not{\epsilon}_{\perp}) \not{\epsilon}_{\perp} / 2 ,$$

where $P_{||}$ and $P_{\perp}$ are the degrees of longitudinal and transverse polarization, respectively. The proof that the operator for a beam with arbitrary polarization can always be written in the above form is presented for completeness in Appendix B. In principle, we can write down a polarization vector s_p^μ such that Ω can be written as a simple weighted sum over the two opposite-polarization states:

$$\begin{aligned} \Omega(P_{||}, P_{\perp}) = & (1+P)/2 \, u(p, s_p) \bar{u}(p, s_p) + \\ & (1-P)/2 \, u(p, -s_p) \bar{u}(p, -s_p) , \end{aligned}$$

where P is the total degree of polarization:

$$P = \sqrt{(P_{||}^2 + P_{\perp}^2)} ,$$

The particular expression for the vector s_p^μ can be found from appendix B, with an angle θ between s_p and p given by

$$\sin\theta = (P_\perp/P) \sqrt{(1-\beta^2)} / \sqrt{[1-\beta^2(P_\perp/P)^2]},$$

$$\text{sign}(\cos\theta) = \text{sign}(P_\parallel) ,$$

where β is the fermions' relativistic velocity. With this approach, however, we introduce into our problem a vector whose components are of $\mathcal{O}(E/m_e)$, and which makes an angle of $\mathcal{O}(m_e/E)$ with respect to the momentum (E is the electron energy, and m_e its mass). Clearly, in this way cancellations over many orders of magnitude will be introduced in our computation. A better idea seems to be to let the summation over the spins run over three terms instead of over two, representing both longitudinally and transversely polarized states:

$$\begin{aligned} \Omega(P_\parallel, P_\perp) = & P_\perp u(p, s) u(p, s) + \\ & (1-P_\perp+P_\parallel)/2 u_+(p) u_+(p) + \\ & (1-P_\perp-P_\parallel)/2 u_-(p) u_-(p) , \end{aligned}$$

where s^μ is the transverse polarization vector introduced above. Using the decomposition rule for arbitrarily polarized pure states derived in the beginning of this section, we find the following decomposition rule for a beam:

$$\begin{aligned} \Omega(P_\parallel, P_\perp) = & (1+P_\parallel) u_+(p) u_-(p) + \\ & (1-P_\parallel) u_-(p) u_-(p) + \\ & P_\perp \Phi(p, s) u_-(p) u_+(p) + \end{aligned}$$

$$P_{\perp} \Phi(p, s)^* u_+(p) u_-(p) .$$

Here, we have reduced the effect of transverse polarization to a coherent summation of states with different helicities. The above decomposition is easily generalized to the case of actual interest, namely that of two colliding beams with arbitrary polarization. Let us denote by $p^{\pm}$, $s^{\pm}$, $P_{\parallel}^{\pm}$ and $P_{\perp}^{\pm}$ the momentum, transverse spin direction, and degrees of longitudinal and transverse polarization for the $e^{\pm}$ beams, respectively. Let us denote by $M(\lambda_1, \lambda_2)$ a helicity amplitude for a process under consideration, where λ_1 is the positron helicity, and λ_2 is the electron helicity (for simplicity we suppress all dependence on possible other helicities for the moment; these are summed over incoherently if we do not consider final state polarization). By taking the beam decomposition formula twice we arrive at the formula for the POLARIZED SUM over the matrix element square:

$$\begin{aligned} \Sigma |M|^2 = 1/4 \times [& (1+P_{\parallel}^+)(1+P_{\perp}^-) |M(+,+)|^2 + \\ & (1+P_{\parallel}^+)(1-P_{\perp}^-) |M(+,-)|^2 + \\ & (1-P_{\parallel}^+)(1+P_{\perp}^-) |M(-,+)|^2 + \\ & (1-P_{\parallel}^+)(1-P_{\perp}^-) |M(-,-)|^2 + \\ & 2(1+P_{\parallel}^+)P_{\perp}^- \operatorname{Re}(\Phi_- M(+,-)M(+,+)^*) + \\ & 2(1-P_{\parallel}^+)P_{\perp}^- \operatorname{Re}(\Phi_- M(-,-)M(-,+)^*) + \\ & 2(1+P_{\parallel}^-)P_{\perp}^+ \operatorname{Re}(\Phi_+ M(+,+)M(+,-)^*) + \\ & 2(1-P_{\parallel}^-)P_{\perp}^+ \operatorname{Re}(\Phi_+ M(+,-)M(-,-)^*) + \\ & 2P_{\perp}^+P_{\perp}^- \operatorname{Re}(\Phi_+ \Phi_- M(+,-)M(-,+)^*) + \\ & 2P_{\perp}^+P_{\perp}^- \operatorname{Re}(\Phi_+ \Phi_-^* M(+,+)M(-,-)^*)] , \end{aligned}$$

where Φ_+ and Φ_- are the complex phase factors discussed above for the e^+ and e^- , respectively. As already indicated before, in the physically interesting case where the $e^\pm$ beams travel in the $\pm z$ -direction, and have the transverse components of their spins oriented along the $\pm x$ -direction, respectively, the two phase factors Φ_+ and Φ_- are equal⁴ to $-i$. Using this decomposition rule and the results for the helicity amplitudes presented in tables 1 and 2 we can calculate the cross section for arbitrary spin configurations of the beams in the processes $e^+e^- \rightarrow e^+e^-\gamma$ and $e^+e^- \rightarrow f\bar{f}\gamma$, in a manner appropriate for those kinematical configurations where the masses of the particles can safely be neglected. We also like to point out that in the present approach the expressions for polarized and unpolarized are of equal complexity, the only difference being in the way the spin sum and average is performed numerically; in contrast, in the traditional approach using trace techniques the introduction of polarization tends to lead to much lengthier expressions.

As in the case of unpolarized beams, special attention now has to be given to those situations where the particle masses cannot be neglected, in particular when the photon is emitted at small angles to one of the charged particles.⁵ In such situations, in the result for the cross section the square of a small four-momentum product will appear in the denominator, while the numerators of the Feynman diagrams exhibit cancellations that

⁴ again, this holds provided we choose k_0^μ and k_1^μ as before.

⁵ We shall not consider the case of two charged fermions being collinear (as in zero-angle Bhabha scattering), where also a finite mass must be kept.

soften this 'double pole' into a 'single pole'. In such cancellations the masses of the fermions may not always be taken to be zero. In reference [10] it has been shown how to tackle this problem when the beams are unpolarized. We now have to redo this analysis, keeping arbitrary polarization of the beams. This is the subject of the next section. Before finishing this section, we like to remark that the above weighted spin sum formula is of course valid independently of the presence of bremsstrahlung. In particular the nonradiative polarized cross section can also be calculated in this way. For completeness we therefore give the helicity amplitudes for $e^+e^- \rightarrow e^+e^-$ and $e^+e^- \rightarrow f\bar{f}$ in appendix D.

5. Finite-mass effects in collinear bremsstrahlung

In order to arrive at a prescription for the polarization decomposition which is also valid for small emission angles of the photon, we shall first review the case of unpolarized beams. A similar, although not completely identical discussion can be found in ref. [10]. Then we shall redo the derivation for polarized beams. In this section we shall only give the various results: a discussion of their derivation is given in appendix C.

Consider an electron with momentum p^μ and mass m in the initial state of some scattering process. Keeping the momentum dependence explicit, we can denote the cross section (i.e. the matrix element squared and summed over the spins) for the process where no bremsstrahlung is emitted, by

$$\Sigma |M_0|^2 = G(p) ,$$

which is to be understood in the high-energy limit, where m is negligible compared to p^0 . Upon emission of a bremsstrahlung photon at very small angle with respect to the beam, so that $p.k$ is a small number, the peaking behaviour of the radiative cross section in the limit $m=0$ is given by

$$\Sigma |M_{\text{rad}, m=0}|^2 = e^2 [2-2x+x^2]/[x(p.k)] \times G((1-x)p) ,$$

where one can recognise the Altarelli-Parisi splitting function of x , which is defined as k^0/p^0 . If we let $p.k$ go down to $\mathcal{O}(m^2)$ we have to include a term with $m^2/(p.k)^2$ as follows:

$$\Sigma |M_{\text{rad}, m \neq 0}|^2 = e^2 \{ [2-2x+x^2]/[x(p.k)] - m^2(1-x)/(p.k)^2 \} \times G((1-x)p) .$$

This leads to the following result for unpolarized states: the finite-mass effect can be taken into account by multiplying the result for a massless electron by the following MASS EFFECT FACTOR:

$$1 - T(p,k) = 1 - (m^2/p.k) x(1-x)/[2-2x+x^2] .$$

At high energies, this overall factor is indistinguishable from 1 if the photon makes a large angle with the electron, and we reproduce the massless-electron result: for small angles, the factor is different from 1, and the correct peaking behaviour is obtained. A similar factor $1-T(p',k)$ holds for a positron with momentum p' in the initial state; for bremsstrahlung off final-state fermions the corresponding factor is easily derived to be

$$1 - T'(q,k) = 1 - (m'^2/q.k) x(1+x)/[2+2x+x^2] ,$$

where q^μ and m' are the fermion's momentum and mass, and x' is the ratio k^0/q^0 .

We now turn to the case of an electron beam with degrees $P_{||}$ of right-handed longitudinal polarization and $P_{\perp}$ of transverse polarization, with transverse polarization vector s^μ , as in the previous section. The polarized spin sum in the nonradiative case can now be denoted by

$$\Sigma |M_0|^2 = G(p, P_{||}, P_{\perp}, s) ,$$

and in the case of bremsstrahlung at small angles the peaking behaviour of the cross section is given by

$$\Sigma |M_{\text{rad}, m=0}|^2 = e^2 [2-2x+x^2]/[x(p \cdot k)] \times G(p, P_{||}, V(p, k) P_{\perp}, s) ,$$

$$V(p, k) = 2(1-x)/[2-2x+x^2] .$$

It is seen that the effective degree of longitudinal polarization is not affected by the emission of the photon, but the degree of transverse polarization is decreased. As before, we now proceed to include the effect of a small but finite mass. It turns out that the terms in the beam operator with $P_{||}$ and $P_{\perp}$ get mass-effect terms that are different than those for the unpolarized case: there is no simple factorization. This can be cured by introducing effective degrees of longitudinal and transverse polarization:

$$P_{||}(\text{eff}) = P_{||} \times [1-R(p, k)T(p, k)]/[1-T(p, k)] ,$$

$$P_{\perp}(\text{eff}) = P_{\perp} \times [1-S(p, k)T(p, k)]/[1-T(p, k)] ,$$

$$R(p,k) = [1-x+x^2]/(1-x) ,$$

$$S(p,k) = [1+(1-x)^2]/(2-2x) .$$

We see that in collinear photon bremsstrahlung both degrees of polarization are changed. It is amusing to notice that if the radiation is strictly collinear, i.e. the angle between the electron and photon three-momenta is zero, $P_{||}(\text{eff})$ and $P_{\perp}(\text{eff})$ go exactly to $-P_{||}$ and 0, respectively. In analogy with the unpolarized case, the finite-mass effects of the electron can now be written as

$$\sum |M_{\text{rad}, m \neq 0}|^2 = [1-T(p,k)] \sum |M_{\text{rad}, m=0}|^2 P \rightarrow P(\text{eff}) .$$

For the positron, identical formulae hold. For the final-state particles we only need the factors $1-T'$ since we assume that their polarization is not measured.

Before we come to the concluding remarks of this section, we have to discuss yet another source of 'finite-mass effects' of rather more subtle origin. This is related to our use of spinor techniques to calculate the helicity amplitudes, and the problem of the assignment of actual numerical values to the four-momenta of the particles. The spinor products $s_{\pm}$ that were introduced in section 2 are defined for massless momenta. Of course, the value of a spinor product $s(p,q)$ is changed only infinitesimally if the numerical values of the momenta p^{μ} and q^{μ} are changed infinitesimally from strictly massless to nearly, but not completely massless. If the particles are

all produced at large angles with respect to each other and the beams, we can therefore safely use the spinor-product expressions of tables 1 and 2. However, in the peaking situations of collinear bremsstrahlung we have to be more careful. Consider the spinor product $s_+(k, p_1)$, which occurs in the denominators of the radiation factors v_p and v_+ . It is this product that makes the result for the cross section peak whenever $\vec{k}$ and $\vec{p}_1$ are collinear. To be precise: the absolute value of this product is given by

$$|s_+(k, p_1)|^2 = B^\mu k_\mu ,$$

$$B^\mu = 2 p_1^\mu - m_1^2 k_0^\mu / (p_1 \cdot k_0) , \quad m_1^2 = p_1 \cdot p_1 .$$

as can be seen from the definition of $s_\pm$ in terms of the momenta and k_0 in section 2. If the angle between $\vec{p}_1$ and $\vec{k}$ is large the term with m_1^2 in B^μ can be ignored, and we obtain the correct denominator, $2p_1 \cdot k$. If the angle is small, however, we obtain a wrong value. This can be seen from the fact that B.B vanishes exactly, leading to an actual divergence in the cross section for nearly collinear bremsstrahlung, even for massive fermions. To remedy this, we introduce the following MASS COMPENSATION FACTOR, which makes up for the unwanted modification of the cross section that arises from our rewriting the real propagators, consisting of vector products, in terms of spinor products with the actual massive fermion momenta in the arguments:

$$W_{\text{comp}} = W(p_1, k) W(p_2, k) W(q_1, k) W(q_2, k) ,$$

$$W(p, k) = |s_+(p, k)|^2 / (2 p \cdot k) .$$

This factor goes to zero at the position of the unwanted divergences, in such a way as to ensure the desired peaking behaviour of the resulting cross section formula.

This completes our discussion of the finite-mass effects. We are now able to correctly describe the cross section for all those momenta configurations in which all the charged particles make large angles with one another, without any constraint on the photon momentum. We finish this section by summarizing our recipe for obtaining from a set of four-momenta p_1^μ , p_2^μ , q_1^μ , p_2^μ and k^μ and polarization parameters $P_{||}^\pm$, $P_\perp^\pm$, and $s_\pm^\mu$ (for instance given by a Monte Carlo event generator) the corresponding cross section:

1. Calculate the appropriate spinor products, and the helicity amplitudes according to table 1 or 2;
2. Replace the degrees of transverse and longitudinal polarizations $P_{||}$ and $P_\perp$ for the two beams by the effective degrees $P_{||}(\text{eff})$ and $P_\perp(\text{eff})$, as indicated above;
3. Perform the weighted spin sum over the helicity amplitudes as described in the previous section;
4. Multiply the result by a factor $1-T$ for each of the incoming fermions, and by a factor $1-T'$ for each of the outgoing fermions;
5. Multiply the result by the mass compensation factor W_{comp} .

Of course the finite-mass effects only have to be taken into account if (nearly) collinear radiation takes place.

APPENDIX A

POLARIZATION DECOMPOSITION FOR MASSIVE FERMIONS

We can consistently construct massive Dirac spinors for the various polarization states. Let the momentum p^μ and s^μ of the fermion satisfy the following relation:

$$p \cdot p = m^2, \quad p \cdot s = 0, \quad s \cdot s = -1$$

where m is the mass of the particle⁶. We can then define

$$u_+(p) = (\not{p} + m)u_-(k_0)/\sqrt{2p \cdot k_0},$$

$$u_-(p) = (\not{p} + m)\not{\epsilon}_1 u_-(k_0)/\sqrt{2p \cdot k_0},$$

$$u(p, s) = C(p, s) (\not{p} + m)(1 + \gamma^5 \not{s})u_-(k_0),$$

$$C(p, s) = 1/\sqrt{4p \cdot k_0 + 4ms \cdot k_0}.$$

In the limit $m \rightarrow 0$ these definitions reduce in a smooth way to those in sections 2 and 4. We can again decompose $u(p, s)$ in terms of $u_+(p)$ and $u_-(p)$ as follows:

⁶ We only treat the case of a particle here; the results for antiparticles can be obtained by letting $m \rightarrow -m$.

$$u(p,s) = A_+(p,s)u_+(p) + A_-(p,s)u_-(p) ,$$

where the two coefficients $A_{\pm}$ are

$$\begin{aligned} A_+(p,s) &= C(p,s) \bar{u}_+(p) (1 + \gamma^5 \not{s}) u_-(k_0) \\ &= C(p,s) \text{Tr}[\omega_- \not{k}_0 (\not{p} + m) (1 + \not{s})] / \sqrt{2p \cdot k_0} \\ &= \sqrt{[1/2 + ms \cdot k_0 / (2p \cdot k_0)]} , \end{aligned}$$

$$\begin{aligned} A_-(p,s) &= C(p,s) \bar{u}_-(p) (1 + \gamma^5 \not{s}) u_-(k_0) \\ &= C(p,s) \text{Tr}[\omega_- \not{k}_0 \not{k}_1 (\not{p} + m) (1 + \not{s})] / \sqrt{2p \cdot k_0} \\ &= C(p,s) [(p \cdot k_0)(s \cdot k_1) - (p \cdot k_1)(s \cdot k_0) - i\epsilon(k_0, k_1, p, s)] / \sqrt{2p \cdot k_0} \end{aligned}$$

where the ϵ -term denotes the contraction of the Levi-Civita tensor $\epsilon^{\mu\nu\rho\sigma}$ with the four four-vectors. The fact that the spinors $u_{\pm}(p)$ and $u(p,s)$ are all normalized to give $\bar{u}u=2m$ implies that

$$|A_+(p,s)|^2 + |A_-(p,s)|^2 = 1 ,$$

which can of course also be checked by simple algebra from their explicit forms. If we take the limit $m \rightarrow 0$ while keeping the angle between the vectors $\vec{s}$ and $\vec{p}$ fixed, we recover the decomposition rule of section 4. In order to see what happens in the high-energy, longitudinal-polarization case, we use the fact that we can always write the polarization vector s^{μ} as follows:

$$s^{\mu} = \lambda p^{\mu}/m + m r^{\mu} ,$$

with $\lambda = +1$ or -1 and $r \cdot r = 0$. In fact, two choices for r^μ are possible. For $\lambda = +1$, we can take the limit $m \rightarrow 0$, $r^\mu = \mathcal{O}(1)$ so as to recover the massless spinor $u_+(p)$; for $\lambda = -1$, we can take the limit $m \rightarrow 0$, $r^\mu = \mathcal{O}(1)$ such that the massless spinor $u_-(p)$ is produced. Let us consider $A_+(p, s)$. Using the above representation for s^μ we can rewrite it as

$$A_+(p, s) = \sqrt{[(1+\lambda)/2 + m^2 r \cdot k_0 / (2p \cdot k_0)]}.$$

Evaluating A_+ in the above-mentioned two limits for longitudinal polarization, we find that A_+ goes to 1 for $\lambda = +1$, and to 0 for $\lambda = -1$ (of course, A_- then goes to 0 and 1, respectively). We conclude that the limits of longitudinal polarization can sensibly be taken if we keep the mass of the fermion finite. If we subsequently let the mass go to zero, a pure helicity state is obtained. This solves the problem of the apparent inconsistency when s becomes parallel to $\pm p$ in the decomposition formula for massless fermions presented in section 4.

APPENDIX B

LONGITUDINAL AND TRANSVERSE POLARIZATION

We show here how an arbitrarily polarized state can be split up exactly into a longitudinally and a transversely polarized part. Let the momentum and spin vector of a fermion with mass m be given by

$$p^\mu = (E, \vec{p}) , \quad s^\mu = (s^0, \vec{s}) .$$

The vector $\vec{s}$ can always be split up into a part parallel to $\vec{p}$, and a part orthogonal to it. Using the identities $p \cdot p = m^2$, $s \cdot s = -1$, and $p \cdot s = 0$ we can by simple algebra obtain the following explicit expression for the components of s^μ :

$$s^0 = \beta \cos \theta / \sqrt{(1 - \beta^2 \cos^2 \theta)} ,$$

$$\vec{s} = (\cos \theta \vec{e} + \sin \theta \vec{e}_1) / \sqrt{(1 - \beta^2 \cos^2 \theta)} ,$$

where θ is the angle between $\vec{s}$ and $\vec{p}$, and $\beta = |\vec{p}|/E$ is the fermion's relativistic velocity. The three-vectors $\vec{e}$ and $\vec{e}_1$ are defined to be parallel to $\vec{p}$ and orthogonal to it, respectively. By specializing to the cases $\theta=0$, $\theta=\pi/2$ we find the polarization vectors corresponding to purely longitudinal and purely transverse polarization:

$$s_{||}^\mu = (\beta, \vec{e}) / \sqrt{(1 - \beta^2)} ,$$

$$s_{\perp}^\mu = (0, \vec{e}_1) .$$

We see that the following decomposition holds:

$$s^\mu = \rho_{||}(\theta) s_{||}^\mu + \rho_{\perp}(\theta) s_{\perp}^\mu ,$$

$$\rho_{||}(\theta) = \cos\theta \sqrt{(1-\beta^2)}/\sqrt{(1-\beta^2\cos^2\theta)} ,$$

$$\rho_{\perp}(\theta) = \sin\theta/\sqrt{(1-\beta^2\cos^2\theta)} .$$

This justifies the decomposition into longitudinal and transverse polarization for an $e^\pm$ beam made in section 4: partially polarized beams are obtained by a suitable incoherent superposition of states with opposite polarization. We also see that the above split-up is exact at all energies, i.e. no terms of $\mathcal{O}(m/E)$ have been neglected. Another conclusion is that at high energies the polarization vector for any beam with a sizeable degree of longitudinal polarization must be nearly collinear with its momentum, the angle between the two three-vectors being of $\mathcal{O}(m/E)$, while the components of s^μ become very large, of $\mathcal{O}(E/m)$.

APPENDIX C

DERIVATION OF THE MASS-EFFECT FACTORS

In this appendix we discuss the finite-mass effects arising from collinear photon bremsstrahlung. For simplicity we shall restrict ourselves to initial-state bremsstrahlung; the results for final-state bremsstrahlung can easily be obtained from those presented below. Consider an electron line with momentum p^μ , spin vector s^μ and mass m entering in a Feynman diagram, or set of diagrams, as depicted in fig.2 . In the cross section this line will lead to the operator

$$Z(p) = (\not{p} + m)(1 + \gamma^5 \not{s})/2 ,$$

which is the equivalent of the Ω of section 4. As before we can split s^μ up in a purely transverse part $P_\perp s_\perp$ and a purely longitudinal part $P_\parallel s_\parallel$. Then, $Z(p)$ reduces to $\Omega(P_\parallel, P_\perp)$ in the limit $m \rightarrow 0$. We now proceed to consider collinear bremsstrahlung from this electron line. The corresponding cross section will be dominated by the diagram(s) depicted in fig.3, provided we choose a suitable gauge for the bremsstrahlung photon. To this end we may take some lightlike vector q^μ which is at large angles with both p^μ and the bremsstrahlung momentum k^μ . Since these two momenta are nearly parallel, we then have approximately $x = k^0/p^0 = (k \cdot q)/(p \cdot q)$. A good gauge choice is to take the following polarization sum for the photon:

$$\sum \epsilon^\mu \epsilon^\nu = -g^{\mu\nu} + [k^\mu q^\nu + k^\nu q^\mu]/(q \cdot k) .$$

and the bremsstrahlung equivalent of Z is given by then given by

$$Z'(p,k) = e^2 / (8 p.k^2) \times (\not{p}-\not{k}+m)\gamma_\mu(\not{p}+m)(1+\gamma^5 \not{s})\gamma_\nu(\not{p}-\not{k}+m) \\ \times [-g^{\mu\nu} + (k^\mu q^\nu + k^\nu q^\mu)/(q.k)]$$

We now determine the behaviour of this expression in the (nearly) collinear limit. To define the collinear limit, we split k^μ in a part proportional to p^μ , a purely transverse part $k_\perp^\mu$ and a part k_r^μ , which has zero space-like components:

$$k^\mu = x p^\mu + k_\perp^\mu + k_r^\mu,$$

where x is defined above. The collinear limit is characterized by the fact that $k_\perp$ becomes of order m , and k_r of order m^2 . Then, $p.k$ is of order m^2 , while for instance $k.s_\perp$ and pk are of order m . Due to the factor $(p.k)^2$ in the denominator, the operator Z' has an apparent behaviour as $1/m^4$ in this limit, which is however softened by cancellations in the numerator from order $(p^0)^2$ down to m^2 . Any term in the numerator which is of order higher than m^2 can safely be neglected, since it will not appreciably contribute to the cross section. We proceed to split the polarization vector s^μ in a longitudinal and a transverse part, using the same vector q^μ as above:

$$s^\mu = P_\parallel [p^\mu/m - m q^\mu/(p.q)] + P_\perp s_\perp^\mu,$$

and contracting over the indices μ and ν in Z' , neglecting orders of m^3 and higher in performing the Dirac algebra. In the resulting expression, all terms are of the desired order in m . To this order, the component k_r does not contribute to the cross section, and the component $k_\perp$ only linear-

ly; moreover, every term containing k_1 is proportional to either P_{11} or P_1 . We now make the following observation. The k_1 dependence is only relevant for polarized particles, i.e. the incoming beams (under the assumption that the polarization of the final state is not measured). Photons that are collinear with the incoming beams are not detected: hence two photons which are both emitted at the same tiny angle (of $\mathcal{O}(m/p^0)$) with respect to the beams, but with opposite azimuthal orientation around it, are therefore in practice undistinguishable; but their dependences on k_1 exactly cancel each other. Therefore, any cross section in which we have to integrate over the phase space of the unseen hard bremsstrahlung photons (i.e., any phenomenologically relevant cross section) is therefore independent of k_1 in the collinear limit. Therefore, we may drop the terms with k_1 from the expression for Z' . After doing so, we are left with terms that go either with $m^2/(p.k)^2$ or with $1/(p.k)$, which have the right peaking behaviour. The result is

$$Z'(p,k) = e^2/(2p.k) \times (A_1 + A_2 P_{11} \gamma^5 + A_3 P_1 \gamma^5 \not{k}_1) \not{\epsilon}' ,$$

$$A_1 = [2-2x+x^2]/x - m^2[1-x]/(p.k) ,$$

$$A_2 = [2-2x+x^2]/x - m^2[1-x+x^2]/(p.k) ,$$

$$A_3 = [2-2x]/x - m^2[1-x]/(p.k) .$$

In this result, the various terms in the finite-mass effect formulae in section 5 can easily be recognized.

This finishes our derivation. The results for an incoming positron are exactly the same; those for outgoing particles are obtained by changing k^μ into $-k^\mu$ and dropping the terms with $P_\perp$ and $P_\parallel$.

APPENDIX D

HELICITY AMPLITUDES FOR THE NONRADIATIVE PROCESSES

In this appendix we give for completeness the helicity amplitudes for massless fermion pair production and Bhabha scattering without bremsstrahlung. The process $e^+(p_1, \lambda_1) e^-(p_2, \lambda_2) \rightarrow f(q_1, v_1) \bar{f}(q_2, v_2)$ is described by the helicity amplitude $M_f^0(\lambda_1, \lambda_2, v_1, v_2)$ where the superscript denotes the non-radiative case. It can take the following 4 nonzero values:

$$M_f^0(+, +, +, +) = i s_+(p_1, q_1) s_-(q_2, p_2) F_{++}(s_0)/\sqrt{2},$$

$$M_f^0(+, +, -, -) = i s_+(p_1, q_2) s_-(q_1, p_2) F_{+-}(s_0)/\sqrt{2},$$

$$M_f^0(-, -, +, +) = i s_-(p_1, q_2) s_+(q_1, p_2) F_{-+}(s_0)/\sqrt{2},$$

$$M_f^0(-, -, -, -) = i s_-(p_1, q_1) s_+(q_2, p_2) F_{--}(s_0)/\sqrt{2},$$

where the symbols are the same as those occurring in section 3.

In the case of the process $e^+(p_1, \lambda_1) e^-(p_2, \lambda_2) \rightarrow e^-(q_1, v_1) e^+(q_2, v_2)$ we have the helicity amplitude $M_e^0(\lambda_1, \lambda_2, v_1, v_2)$, which has 6 nonvanishing values:

$$M_e^0(+, +, +, +) = i s_+(p_1, q_1) s_-(q_2, p_2) [E_{++}(s_0) + E_{++}(t_0)]/\sqrt{2},$$

$$M_e^0(+, +, -, -) = i s_+(p_1, q_2) s_-(q_1, p_2) E_{+-}(s_0)/\sqrt{2},$$

$$M_e^0(+, -, -, +) = i s_+(p_1, p_2) s_-(q_2, q_1) E_{+-}(t_0)/\sqrt{2},$$

$$M_e^0(-, +, +, -) = i s_-(p_1, p_2) s_+(q_2, q_1) E_{-+}(t_0)/\sqrt{2},$$

$$M_e^0(-, -, +, +) = i s_-(p_1, q_2) s_+(q_1, p_2) E_{-+}(s_0)/\sqrt{2},$$

$$M_e^0(-, -, -, -) = i s_-(p_1, q_1) s_+(q_2, p_2) [E_{--}(s_0) + E_{--}(t_0)]/\sqrt{2}.$$

This finishes our presentation of the nonradiative amplitudes. The same spinor techniques and phase conventions have been used as in the rest of this paper; hence, the results for arbitrary beam polarization can be obtained in exactly the same manner for the nonradiative processes using these formulae.

References

- [1] G. Altarelli et al., 'Precision Tests of the Standard Model', in 'Physics at LEP', CERN Yellow Report 86-02 (1986).
- [2] B.W. Lynn, M.E. Peskin and R.G. Stuart, contribution to CERN Yellow Report 86-02
- [3] K.-I. Hikasa, University of Wisconsin preprint MAD/PH/263 (November 1985)
- [4] H.S. Song, Phys. Rev. D33(1986)1252
- [5] M. Greco, G. Pancheri-Srivastava and Y. Srivastava, Nucl. Phys. B171(1980)118 (E:B197(1982)543)
- [6] M. Bohm and W. Hollik, Nucl. Phys. B204(1982)45
- [7] R. Kleiss, Ph.D. thesis, Leiden University (1982)
- [8] E.A. Kuraev, M.Yu. Lel'chuk, V.S. Panin and Yu.P. Persun'ko, Sov. J. Nucl. Phys. 32(1980)548
- [9] The CALKUL collaboration, Phys. Lett. 103B (1981) 124; Nucl. Phys. B206 (1982) 53; B206 (1982) 61; B239 (1984) 395;
- [10] The CALKUL collaboration, Nucl. Phys. B239 (1984) 382
- [11] R. Kleiss, Nucl. Phys. B241(1984)235
- [12] F.A. Berends, P.H. Daverveldt and R.Kleiss, Nucl. Phys. B253(1985)441
- [13] R. Kleiss and W. J. Stirling, Nucl. Phys. B262(1985)235
- [14] Z. Xu, D.-H. Zhang and L. Chang, Tsinghua preprints TUTP-84/3, 84/4 and 84/5 (1984).
- [15] J.F. Gunion and Z. Kunszt, Phys. Lett. 159B(1985)167 and 161B(1985)333;
Z. Kunszt, CERN preprint CERN-TH.4319/85 (1985);
S.J. Parke and T.R. Taylor, Fermilab-Pub-85/118-T (1985).

- [16] A.C. Hearn, REDUCE 3.2 User's manual.

Figure captions

fig.1

The two generic diagrams for radiative Bhabha scattering and fermion pair production. The actual Feynman diagrams are obtained by assigning momenta and helicity labels to the variables k_i and λ_i .

fig.2

General form of a diagram with an incoming electron.

fig.3

General form of a diagram with photon bremsstrahlung from the incoming electron line.

Fig. 1

Fig. 2

Fig. 3