SNAKES, ROTATORS, SERPENTS, AND THE OCTAHEDRAL GROUP†

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Specific configurations of horizontal and vertical bending magnets are given that, when acting on the spin polarization vector of a particle beam, generate a group of 24 operators isomorphic to the group of rotational symmetries of a cube, known as the octahedral group. Some of these configurations have the feature of converting transversely polarized beams to longitudinally polarized beams (or vice versa) at the midpoint of the configuration for, in principle, all beam energies. Since the first-order optical transfer matrix for each half of these configurations is nearly that of a drift region, the external geometry remains unchanged and midpoint dispersion is not introduced. Changing field strengths and/or polarities allows a configuration to serve as either a snake (first or second kind) or a rotator, where in both cases the spin polarization is longitudinal at the midpoint. Practical applications using these configurations for electron or proton beams can be envisioned. The requisite high integrated field strengths will limit these applications for electrons.

1. INTRODUCTION

The concept of inserting into a ring accelerator the spin rotating devices that were subsequently called Siberian snakes was first proposed by the Soviet authors Y. A. Derbenev and A. M. Kondratenko of Novosibirsk, USSR.¹ There are several known types of Siberian snakes^{2,3} for manipulation of a beam polarization vector.

The snake of the first kind rotates the polarization vector about the longitudinal axis of the beam (a rotation of 90° for each one-half of a snake). A particular magnet sequence representing a snake of the first kind was proposed by K. Steffen.⁴ It has the elegance of not introducing dispersion nor affecting the beam trajectory external to the system. It has a wide range of operating energies but, unfortunately, is not useful by itself in converting transverse polarization into longitudinal polarization.

The snake of the second kind rotates the polarization vector about the transverse axis (this axis is horizontal in Ref. 2 and, again, the rotation is 90° for one-half of a snake). This snake does rotate transverse polarization into longitudinal polarization, making it more interesting to those doing polarized beam experiments. However, known versions of this snake do introduce midpoint dispersion and may have a limited range of operating energies and/or variable geometry.

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Two other novel versions of the Siberian snake have been proposed.⁵ They are the left- and right-pointed snakes which represent rotations of 180° about axes lying in the plane containing the horizontal axis and the axis of the beam direction. The direction cosines of this rotation axis have a magnitude of $\pm\sqrt{2}/2$ with respect to the coordinate axes mentioned.

In general then, a Siberian snake has been defined in Ref. 2 to consist of a sequence of magnets that rotate the spin vector by 180° about an arbitrary axis lying in a plane containing the horizontal axis and the axis of the beam direction. This definition requires that a snake always invert the vertical component of the spin polarization vector.

The importance of Snakes in circular machines has been extensively described, but we will refer only to a review article by Montague.⁶ One of the uses described in this review and attributed to Derbenev and Kondratenko applies to electron storage rings. It is explained that two snakes placed in a ring at diametrically opposing positions, one of the first kind and the other of the second kind, can be used to achieve a spin tune of 0.5, with the vertical component of the spin vector parallel to the field in one half of the machine and antiparallel in the other half, independent of energy. Such a procedure, according to Montague, reduces "substantially the effects of large energy spread and imperfection resonances at high energies, permitting polarized beams to be obtained up to perhaps 100 GeV." In his review, Montague develops an elegant method that uses spinor algebra and unitary transformations for describing spin transformations and calculating spin tune. This method is used in proving that the configuration described above has a spin tune of 0.5, which means, he points out, that "any arbitrary spin vector closes upon itself after two revolutions" around the ring.

There are also systems called rotators.³ This name has been used to classify systems of magnets which have the property of rotating the vertical component of the polarization vector into the longitudinal direction at the midpoint (interaction point) and then restoring the original direction. These systems are useful for polarized beam physics. Montague has shown that, in general, if such a system is constrained to be fully antisymmetric about the midpoint, then the overall spin transformation is the identity, independent of beam energy or the details of the field strengths. This is an important point to which we will again refer. Most rotators operate at only a specified beam energy or with changing geometry.^{3,6}

One of the configurations of magnets which we will describe is identical in appearance to that of the snake of the first kind as given in Ref. 4. Either snakes or rotators can be generated with this configuration. In addition, its function can be easily changed by adjusting field strengths or polarities. Rather than having several names for a single system, depending upon which purpose it serves, we have chosen to simply use the name, "serpent" when referring to this configuration. We will point out when this configuration is serving as a snake or rotator.

Another half-snake or half-rotator we have named the half up-down snake. It shares many of the properties of the serpent but is sufficiently different that it requires another name. Now, we will describe the serpent and return to a description of the up-down snake later.

Figure 1 represents the configuration of horizontal and vertical bend magnets

FIGURE 1 Configuration of horizontal and vertical bending magnets for *one-half* of a serpent.

that will generate one-half of a serpent. This configuration is identical to that of one-half of the snake of the first kind described in Ref. 4, *except that for the serpent the magnitudes of the fields are to be doubled*. It retains the nice features of the first kind in that it has an extremely wide range of operating energies and acceptance, does not introduce dispersion (at the end of half of system), and the beam entering the snake (head) is collinear with the beam exiting (tail) so geometrically it can replace a drift region. The first-order optical transfer matrix of this configuration is also essentially that of a drift region except for the small focusing effects and momentum compaction factor produced by the dipole magnets. The magnetic fields are determined by the desired precession angles independently of beam energy. Hence, the deviation from the drift matrix decreases with increasing beam energy. Corrections can compensate for these effects and may or may not be necessary, depending upon the application.

For the remainder of this discussion, we limit ourselves to only those configurations of magnets that do not introduce dispersion or change external geometry as a function of beam energy.

2. COORDINATE SYSTEM AND NOTATION

Before proceeding further, we will comment on notation, write some useful relationships, and define a coordinate system.

One-half of a serpent rotates the spin polarization vector about an axis that can be graphically represented by a vector connecting opposite corners of a cube (the rotation angle is 120° for one-half of a serpent). It may be seen immediately that such an operation can perform an even or cyclic permutation of the coordinates of the polarization vector and thus permute transverse and longitudinal coordinates. Four such non-orthogonal axes are chosen for our representation. They will

be defined later as vectors having direction cosines all equal in magnitude but with varying signs. We define directions for these axes and the magnetic fields such that for an electron a positive rotation about these axes obeys the right-hand rule. The reader will note that in general such a rotation applied three times results in an identity. If represented operationally, $A^3(120^\circ) = I$, or $A^2(120^\circ) = A^{-1}(120^\circ) = A(-120^\circ)$. We will choose to write $A^{-1}(120^\circ)$ instead of $A^2(120^\circ)$.

The precession angle of the spin polarization vector is given by the relationship

$$\psi_p = a\gamma\phi, \quad (1)$$

in which ϕ is the bending angle of the beam in the transverse magnetic field, ψ_p is the precession angle for the polarization vector of an on-momentum particle about the direction of the field in the coordinate system following the beam (orbit frame), γ is the Lorentz factor, and a is a measure of the particle's anomalous magnetic moment (see Ref. 6 for a discussion of the Thomas-BMT equation and detailed references). For electrons

$$a = \left(\frac{g-2}{2} \right) = 1.159652 \times 10^{-3}, \quad (2a)$$

whereas for protons

$$a = 1.792846. \quad (2b)$$

It is also useful to express ψ_p in terms of the $\int B dl$ of the applied field since for ultrarelativistic particles the energy dependence factors out. For electrons

$$\psi_p = 0.680 \int B dl \quad (\text{radians, T-m}). \quad (3)$$

For protons the coefficient becomes 0.573. Note that an integrated field of 2.31(2.74) T-m will precess the electron(proton) spin polarization vector by $\pi/2$ or 90° , independent of beam energy. This fact will be referred to later.

At and above the beam energies of the Stanford Linear Collider (≥ 50 GeV), the bending angle of the beam is small ($< 1^\circ$) in traversing such a field. Hence, the spin precession angle when measured in laboratory coordinates or beam coordinates is nearly the same.

Figure 2 shows the coordinate system following the beam and the definitions of certain vectors. In this coordinate system, the electron spin polarization vector is expressed in terms of its initial coordinates, and it is assumed that its magnitude has been normalized to a value between 0.0 and 1.0. This vector is expressed as

$$\mathbf{p} = H\hat{\mathbf{i}} + V\hat{\mathbf{j}} + S\hat{\mathbf{k}},$$

or alternatively as the column vector

$$\mathbf{p} = \begin{pmatrix} H \\ V \\ S \end{pmatrix}.$$

FIGURE 2 Beam-following coordinate system with certain vectors defined.

For an electron traversing a horizontal bend magnet, the spin polarization vector will precess through an angle θ , where positive θ is defined by the right-hand rule representing rotation about the y axis. The resultant polarization vector will be given by

$$\mathbf{p}' = \mathbf{H}(+\theta)\mathbf{p},$$

where $\mathbf{H}(+\theta)$ is an orthogonal matrix operator defined by

$$\mathbf{p}' = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} H \\ V \\ S \end{pmatrix} \quad (\text{horizontal bend}). \quad (4)$$

For a vertical bend magnet, the precession angle ψ is again defined as positive by the right-hand rule about the x axis, and

$$\mathbf{p}' = \mathbf{V}(+\psi)\mathbf{p},$$

where the orthogonal matrix $\mathbf{V}(+\psi)$ is defined by

$$\mathbf{p}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \psi & -\sin \psi \\ 0 & \sin \psi & \cos \psi \end{pmatrix} \begin{pmatrix} H \\ V \\ S \end{pmatrix} \quad (\text{vertical bend}). \quad (5)$$

With these definitions, we now represent the effect on the spin polarization vector of the combined vertical and horizontal bend magnets that were shown in Fig. 1 by

$$\mathbf{p}' = \mathbf{H}(+\theta)\mathbf{V}(+\psi)\mathbf{H}(-2\theta)\mathbf{V}(-2\psi)\mathbf{H}(+\theta)\mathbf{V}(+\psi)\mathbf{p}. \quad (6)$$

Note that the beam sees a vertical bending magnet first in this configuration, so the first matrix operating on $\mathbf{p}$ is $\mathbf{V}(+\psi)$; therefore, one reads the matrices from right to left to reconstruct a configuration.

We will represent this configuration by the notation $\mathbf{V}(+\psi, +\theta)$, which indicates that the first magnet seen by the beam is a vertical bending magnet with a positive precession angle, and the second magnet (horizontal) also has positive precession. Other configurations will follow the same patterns, so $\mathbf{H}(+\psi, -\theta)$, where the arguments are *not* transposed, would represent a horizontal magnet

first, with negative precession angle, followed by a vertical magnet with positive precession angle.

We write Eq. (6) as

$$\mathbf{p}' = \mathbf{V}(+\psi, +\theta)\mathbf{p}.$$

3. SERPENT

If we now select field strengths such that $\psi = +90^\circ$ and $\theta = +90^\circ$ (recalling that the required integrated fields are independent of beam energy, having values given by Eq. (3)), then

$$\mathbf{p}' = \mathbf{V}(+90, +90)\mathbf{p}.$$

Or, in the expanded form of the equation,

$$\mathbf{p}' = \mathbf{H}(+90)\mathbf{V}(+90)\mathbf{H}(-180)\mathbf{V}(-180)\mathbf{H}(+90)\mathbf{V}(+90)\mathbf{p}.$$

Calculation using Eqs. (4) and (5) gives

$$\mathbf{p}' = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} H \\ V \\ S \end{pmatrix}$$

or

$$\mathbf{p}' = \begin{pmatrix} S \\ H \\ V \end{pmatrix}.$$

We see that the vertical component of the initial polarization vector has now been rotated into the longitudinal direction. This configuration performs a cyclic permutation of initial coordinates with no changes of sign. It can be represented by a positive rotation of 120° about the axis $\mathbf{a}_1$, given by

$$\mathbf{a}_1 = 1/3(+\sqrt{3}\hat{\mathbf{i}} + \sqrt{3}\hat{\mathbf{j}} + \sqrt{3}\hat{\mathbf{k}}).$$

We now define a matrix operator, $\mathbf{A}_1(+120)$ where

$$\mathbf{A}_1(+120) = \mathbf{V}(+90, +90) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

As noted earlier, $\mathbf{A}_1^3(+120) = \mathbf{I}$ and

$$\mathbf{A}_1^2(+120) = \mathbf{A}_1^{-1}(+120) = \mathbf{A}_1(-120).$$

Since the rotation angle will be understood to be 120° for this and three following operators, we shorten the notation further to

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{A}_1^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We now select three other axes of rotation given by

$$\mathbf{a}_2 = 1/3(+\sqrt{3}\hat{\mathbf{i}} - \sqrt{3}\hat{\mathbf{j}} - \sqrt{3}\hat{\mathbf{k}}),$$

$$\mathbf{a}_3 = 1/3(-\sqrt{3}\hat{\mathbf{i}} + \sqrt{3}\hat{\mathbf{j}} - \sqrt{3}\hat{\mathbf{k}}),$$

and

$$\mathbf{a}_4 = 1/3(-\sqrt{3}\hat{\mathbf{i}} - \sqrt{3}\hat{\mathbf{j}} + \sqrt{3}\hat{\mathbf{k}}).$$

These will be eigenvectors (rotation axes) for corresponding operators

$$\mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4,$$

and their inverses

$$\mathbf{A}_2^{-1}, \mathbf{A}_3^{-1}, \mathbf{A}_4^{-1}.$$

With these definitions, we find the following correspondence between configurations and operators for half-serpents:

$$\begin{aligned} \mathbf{V}(+90, +90) &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \mathbf{A}_1 & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} S \\ H \\ V \end{pmatrix} \\ \mathbf{V}(+90, -90) &= \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \mathbf{A}_2 & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} -S \\ -H \\ V \end{pmatrix} \\ \mathbf{V}(-90, +90) &= \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = \mathbf{A}_3 & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} S \\ -H \\ -V \end{pmatrix} \\ \mathbf{V}(-90, -90) &= \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = \mathbf{A}_4 & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} -S \\ H \\ -V \end{pmatrix} \\ \mathbf{H}(+90, +90) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} = \mathbf{A}_4^{-1} & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} V \\ -S \\ -H \end{pmatrix} \\ \mathbf{H}(+90, -90) &= \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} = \mathbf{A}_3^{-1} & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} -V \\ -S \\ H \end{pmatrix} \\ \mathbf{H}(-90, +90) &= \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} = \mathbf{A}_2^{-1} & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} -V \\ S \\ -H \end{pmatrix} \\ \mathbf{H}(-90, -90) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \mathbf{A}_1^{-1} & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} V \\ S \\ H \end{pmatrix} \end{aligned}$$

And, of course,

$$\mathbf{V}(0, 0) = \mathbf{H}(0, 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \quad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} H \\ V \\ S \end{pmatrix}.$$

With one-half of a serpent, there are eight interesting final states where the transverse polarization has been rotated to longitudinal polarization. They are all cyclic (even) permutations of coordinates with some changes of sign. The total number of such permutations and sign changes is 24 (3 even permutations and 8 possible assignments of sign). If both even and odd permutations are included, the total is 48. Of that set, a subset of 24 would have a determinant of +1 and represent rotations. Both even and odd permutations can be generated by rotations. The subset of 24 operators having a determinant -1 would include a reflection of right-handed coordinates to left-handed coordinates. The nine operators we have found thus far do not form a group.

Reversing direction of the longitudinal polarization requires that the fields of at least half of the magnets reverse polarity. For electrons the total $\int B \, dl$ required is 18.4 T-m. This high value, coupled with the need for low fields to limit synchrotron radiation, will require long magnets and, hence, large energy-dependent beam excursions within the system. Spin depolarization effects may also be enhanced. Applications for electron beams are limited by these considerations. For protons the required integrated field is only slightly more, but shorter magnets can be used. The purpose here is to proceed to investigate other interesting properties of these configurations which, at least in principle, may have applications.

We now seek additional final states by combining two one-half serpents, end to end. This gives us three new operators, $\mathbf{C}_1$, $\mathbf{C}_2$, and $\mathbf{C}_3$ (see Table I), that can be made in a number of ways. They do not change transverse to longitudinal polarization. In fact, $\mathbf{C}_1$ is the operation defining a snake of the first kind, whereas the operation $\mathbf{C}_2$ is that defining a snake of the second kind. These two operators and all operators corresponding to a snake will, by definition, invert the vertical component.

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{C}_1 \quad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} -H \\ -V \\ S \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \mathbf{C}_2 \quad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} H \\ -V \\ -S \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \mathbf{C}_3 \quad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} -H \\ V \\ -S \end{pmatrix}$$

TABLE I
Multiplication table for combinations of two half-serpents.

	A_1	A_2	A_3	A_4	A_4^{-1}	A_3^{-1}	A_2^{-1}	A_1^{-1}
A_1^{-1}	I	C_3	C_1	C_2	A_2	A_4	A_3	A_1
A_2^{-1}	C_3	I	C_2	C_1	A_3	A_1	A_2	A_4
A_3^{-1}	C_1	C_2	I	C_3	A_1	A_3	A_4	A_2
A_4^{-1}	C_2	C_1	C_3	I	A_4	A_2	A_1	A_3
A_4	A_2^{-1}	A_3^{-1}	A_1^{-1}	A_4^{-1}	I	C_1	C_2	C_3
A_3	A_4^{-1}	A_1^{-1}	A_3^{-1}	A_2^{-1}	C_1	I	C_3	C_2
A_2	A_3^{-1}	A_2^{-1}	A_4^{-1}	A_1^{-1}	C_2	C_3	I	C_1
A_1	A_1^{-1}	A_4^{-1}	A_2^{-1}	A_3^{-1}	C_3	C_2	C_1	I

The operators in the top row of Table I represent the first half-serpent as seen by the beam (or the right-hand matrix operator). Those in the left column are for the second half-serpent (or the left-hand matrix operator). The identity operator has been omitted as a multiplier.

For example,

$$A_4 A_1 = A_2^{-1}$$

and

$$A_1 A_4 = A_3^{-1}.$$

The set of 12 operators

$$\{A_1, A_2, A_3, A_4, A_4^{-1}, A_3^{-1}, A_2^{-1}, A_1^{-1}, C_1, C_2, C_3, I\}$$

forms a noncommutative group with respect to matrix multiplication. *Therefore, combining three or more half-serpents will not generate any additional final states or new operators.*

Note that $C_1^2 = C_2^2 = C_3^2 = I$, and $C_i C_j = C_j C_i = C_k$ for $i \neq j \neq k$.

The set of four operators $\{C_1, C_2, C_3, I\}$ forms a commutative group with respect to matrix multiplication.

The rotation angle is 180° for operators C_1 , C_2 , and C_4 about eigenvectors

$$c_1 = \pm \hat{k},$$

$$c_2 = \pm \hat{i},$$

and

$$c_3 = \pm \hat{j},$$

respectively.

As mentioned, it is shown in Table I that either the snake of the first kind or the snake of the second kind can be generated by the proper combination of half-serpents. There is an important difference, however, between these new combinations and those known earlier. This difference stems from the fact that half-serpents do *not* rotate about eigenvectors confined to the plane containing

the unit vectors $\hat{i}$ and $\hat{k}$. If the first half-serpent as seen by the beam corresponds to one of the operators $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, or $\mathbf{A}_4$, there is the bonus of having rotated the vertical component into the longitudinal direction at the interaction region (midpoint). For the first time, *we can obtain snakes of either the first or second kind with midpoint longitudinal polarization for all energies above a lower limit determined by magnet apertures.*

Combinations $\mathbf{A}_i^{-1}\mathbf{A}_i = \mathbf{I}$, where $i = 1, 2, 3, 4$ could be used as rotators. These rotators would retain their properties for all beam energies above a lower limit. Notice also that these rotators are fully antisymmetric with respect to the midpoint, a property that Montague has shown will always result in an identity spin transformation. We can, therefore, reverse the direction of the longitudinal polarization at the midpoint by ramping magnets from one configuration to another while maintaining this antisymmetry.

4. AN EXAMPLE

To illustrate how, in principle, these devices could be used in a circular machine, we will emulate the example cited by Montague. As in his example, an electron ring is assumed, but the new features important to this article can be applied equally well to proton machines. We will use in this example a circular machine having four symmetrically placed straight sections or drifts and assume an interaction point (IP) at the midpoint of each. Each drift will be equipped with an identical assemblage of magnets, as shown in Fig. 3.

The configuration chosen for illustration in Fig. 3 represents the fully antisymmetric rotator given by $\mathbf{A}_1^{-1}\mathbf{A}_1 = \mathbf{I}$. It can be made apparent by pairing these magnets, starting with the two adjacent to the IP, that the antisymmetry ensures the identity transformation for the spin. Furthermore, if care is taken to preserve the correct field relationships within each of the four triplets of magnets, the overall optical transfer matrix (that of a drift) is preserved for all field strengths. It follows that these magnets (forming a rotator) can be ramped if this

FIGURE 3 Antisymmetric rotator, $\mathbf{A}_1^{-1}\mathbf{A}_1 = \mathbf{I}$.

symmetry and field strength relationship are maintained. This will not be true for the snakes that can also be formed by this assemblage, as the spin transformation changes during ramping. In Fig. 3, the first half-serpent as seen by the beam would always be represented by an operator $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$, or $\mathbf{I}$, selected by choosing the proper fields and polarities. The first four of these operators would allow longitudinal polarization at the IP. The second half-serpent can generate operators $\mathbf{A}_1^{-1}, \mathbf{A}_2^{-1}, \mathbf{A}_3^{-1}, \mathbf{A}_4^{-1}$, or $\mathbf{I}$. Other operators that can be formed by these assemblages are not included in this example and will be discussed later.

Prior to a particular machine running period, a selection is made, designating two diametrically opposed interaction regions to serve as snakes. The experiments at these IPs can still benefit by having longitudinally polarized beams available, but change of direction from parallel to antiparallel cannot be readily made.

We number the IPs 1 through 4 counter-clockwise and arbitrarily select IP 1 and IP 3 to serve as snakes. The fields and polarities of IP 1 are adjusted to perform the operation $\mathbf{A}_4^{-1}\mathbf{A}_1 = \mathbf{C}_2$ (snake of the second kind) and those of IP 3 to perform the operation $\mathbf{A}_3^{-1}\mathbf{A}_1 = \mathbf{C}_1$ (snake of the second kind). For now, the magnets at IP 2 and IP 4 are left with zeroed fields.

The configuration just described is shown in Fig. 4a, where the direction of a

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FIGURE 4 (a) Circular machine with two snakes (first and second kinds). Vertical polarization becomes longitudinal at two IPs. (b) Rotators now turned on, providing reversible longitudinal polarization at the other two IPs.

vertically polarized spin vector is indicated in various regions. This vector or an antiparallel vector becomes longitudinal at IP 1 and IP 3. The initial direction (up or down) of this vector is arbitrary as there is no preferred polarization direction in this configuration. That is because the presence of the snakes will cause any vertical component to be parallel to the bend fields in one half of the machine and antiparallel to the fields in the other half. The Sokolov–Ternov polarizing mechanism (see Ref. 6) is thus turned off. As pointed out by Montague, an alternate polarizing mechanism such as wigglers would have to be provided. This configuration has a spin tune of 0.5, as pointed out by Derbenev and Kondratenko, which means that after two revolutions the spin polarization vector will close upon itself.

Rotators are now introduced as shown in Fig. 4b. The half-serpents at IP 2 and IP 4 are ramped up as previously described to act as rotators. Both interaction regions have been given a configuration identical to that shown in Fig. 3. The resulting longitudinal polarization in one region is the reverse of that in the other. When desired, the polarization at either of these regions can be reversed independently of all others. For instance, by ramping only the vertical magnets in IP 2 to the opposite polarity, the operation $\mathbf{A}_1^{-1}\mathbf{A}_1 = \mathbf{I}$ becomes $\mathbf{A}_3^{-1}\mathbf{A}_3 = \mathbf{I}$ and at the interaction point the longitudinal polarization is reversed.

5. COMBINATIONS WITH HALF-SNAKES

Since a given half-serpent can be converted to a half-snake of the first kind by just halving the field strength, we may consider configurations where these two are combined end to end. As it turns out, eight new rotation operators are found in this way. They are not cyclic or even permutations. Instead they represent an odd (1 or 3) number of transpositions of initial coordinates; they also change coordinate signs. These operators are interesting in that they also rotate transverse to longitudinal polarization.

First, we calculate the operators $\mathbf{S}_3$ and $\mathbf{S}_3^{-1}$, which are obtained by one half of a snake of the first kind (90° rotation about eigenvector $\mathbf{s}_3 = \hat{\mathbf{k}}$).

$$\begin{aligned} \mathbf{V}(\pm 45, \pm 45) = \mathbf{H}(\mp 45, \pm 45) &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{S}_3^{-1} & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} V \\ -H \\ S \end{pmatrix} \\ \mathbf{V}(\pm 45, \mp 45) = \mathbf{H}(\pm 45, \pm 45) &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{S}_3 & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} -V \\ H \\ S \end{pmatrix} \end{aligned}$$

As noted earlier,

$$\mathbf{S}_3^2 = (\mathbf{S}_3^{-1})^2 = \mathbf{C}_1.$$

Combining $\mathbf{S}_3$ and $\mathbf{S}_3^{-1}$ with the operators for one half of a serpent

($\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$ and inverses, see Table II), gives the new operators.

$$\begin{array}{ll}
 \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \mathbf{D}_1 & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} -S \\ -V \\ -H \end{pmatrix} \\
 \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \mathbf{D}_2 & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} S \\ -V \\ H \end{pmatrix} \\
 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \mathbf{D}_3 & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} S \\ V \\ -H \end{pmatrix} \\
 \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \mathbf{D}_3^{-1} & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} -S \\ V \\ H \end{pmatrix} \\
 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} = \mathbf{E}_1 & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} -H \\ -S \\ -V \end{pmatrix} \\
 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \mathbf{E}_2 & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} -H \\ S \\ V \end{pmatrix} \\
 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \mathbf{E}_3 & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} H \\ -S \\ V \end{pmatrix} \\
 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \mathbf{E}_3^{-1} & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} H \\ S \\ -V \end{pmatrix}
 \end{array}$$

These operators all rotate transverse to longitudinal polarization. Also note that $\mathbf{D}_1^2 = \mathbf{D}_2^2 = \mathbf{E}_1^2 = \mathbf{E}_2^2 = \mathbf{I}$. $\mathbf{D}_1$ and $\mathbf{D}_2$ invert the vertical component.

The rotation angle is 180° for operators $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{E}_1$, and $\mathbf{E}_2$ about eigenvectors

$$\mathbf{d}_1 = 1/2(\pm\sqrt{2}\hat{\mathbf{i}} \mp \sqrt{2}\hat{\mathbf{k}}),$$

$$\mathbf{d}_2 = 1/2(\pm\sqrt{2}\hat{\mathbf{i}} \pm \sqrt{2}\hat{\mathbf{k}}),$$

$$\mathbf{e}_1 = 1/2(\pm\sqrt{2}\hat{\mathbf{j}} \mp \sqrt{2}\hat{\mathbf{k}}),$$

and

$$\mathbf{e}_2 = 1/2(\pm\sqrt{2}\hat{\mathbf{j}} \pm \sqrt{2}\hat{\mathbf{k}}),$$

respectively.

The rotation angle for $\mathbf{D}_3$ and $\mathbf{E}_3$ is 90° , like that of $\mathbf{S}_3$, with their respective

TABLE II
Multiplication table for combinations of two half snakes of the first kind
and/or half serpents.

	A_1	A_2	A_3	A_4	A_4^{-1}	A_3^{-1}	A_2^{-1}	A_1^{-1}	S_3	S_3^{-1}
S_3^{-1}	E_3	E_2	E_1	E_3^{-1}	D_1	D_3^{-1}	D_3	D_2	I	C_1
S_3	E_2	E_3	E_3^{-1}	E_1	D_3	D_2	D_1	D_3^{-1}	C_1	I
A_1^{-1}	I	C_3	C_1	C_2	A_2	A_4	A_3	A_1	E_3^{-1}	E_2
A_2^{-1}	C_3	I	C_2	C_1	A_3	A_1	A_2	A_4	E_2	E_3^{-1}
A_3^{-1}	C_1	C_2	I	C_3	A_1	A_3	A_4	A_2	E_1	E_3
A_4^{-1}	C_2	C_1	C_3	I	A_4	A_2	A_1	A_3	E_3	E_1
A_4	A_2^{-1}	A_3^{-1}	A_1^{-1}	A_4^{-1}	I	C_1	C_2	C_3	D_1	D_3^{-1}
A_3	A_4^{-1}	A_1^{-1}	A_3^{-1}	A_2^{-1}	C_1	I	C_3	C_2	D_3	D_2
A_2	A_3^{-1}	A_2^{-1}	A_4^{-1}	A_1^{-1}	C_2	C_3	I	C_1	D_3^{-1}	D_1
A_1	A_1^{-1}	A_4^{-1}	A_2^{-1}	A_3^{-1}	C_3	C_2	C_1	I	D_2	D_3

The operators in the top row of Table II represent the first half-serpent or half-snake as seen by the beam (the right-hand matrix operator). Those in the left column are for the second half-serpent or half-snake (the left-hand matrix operator). The identity operator has been omitted as a multiplier.

FIGURE 5 Conversion of $V(\psi, \theta)$ to $H(\psi, \theta)$.

eigenvectors given by

$$\mathbf{d}_3 = +\hat{\mathbf{j}}$$

and

$$\mathbf{e}_3 = +\hat{\mathbf{i}}.$$

Again, it can be shown that either the set of operators

$$\{\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3, \mathbf{D}_3^{-1}, \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \mathbf{I}\}$$

or the set

$$\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \mathbf{E}_3^{-1}, \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \mathbf{I}\}$$

will form a noncommutative group with respect to matrix multiplication.

As pointed out, conversion between snakes of the first kind and serpents is easy, requiring only a change in field strengths. Also reversing polarity of fields generates different configurations. However, it is difficult to convert a magnet from horizontal to vertical bend (mechanical rotation required), but if configurations are constructed using seven magnets, as shown in Fig. 5, where one magnet is normally turned off, then we can convert from $\mathbf{H}(\psi, \theta)$ to $\mathbf{V}(\psi, \theta)$ configurations if desired.

6. LEFT-RIGHT SNAKES AND HALF UP-DOWN SNAKES

It may have been noticed that the operators $\mathbf{D}_1$ and $\mathbf{D}_2$ correspond to the definitions of left- and right-pointed snakes. These snakes are shown in Fig. 6a and were conceived (see Ref. 5) as a practical means of achieving a spin tune of 0.5. They are efficient in that the total $\int B dl$ required is only 16.1 T-m, whereas for two half-snakes of the first kind, the $\int B dl$ is 18.4 T-m. These snakes rotate about eigenvectors $\mathbf{d}_1$ or $\mathbf{d}_2$ by 180° , operations which we have shown not only invert the vertical component, but also rotate the horizontal component of the spin polarization vector into the longitudinal direction. This fact immediately suggests that if a left-right snake were to be converted to *half* an up-down snake (see Fig. 6b) by mechanically rotating it about the direction of the beam, then the eigenvectors would become $\mathbf{e}_1$ and $\mathbf{e}_2$ with corresponding operators $\mathbf{E}_1$ and $\mathbf{E}_2$, which we have shown will rotate the vertical component of the spin polarization vector into the longitudinal direction.

We proceed to examine these types of snakes using the same methods as before and similar notations, understanding that the configurations are given in Fig. 6.

For the left-right snakes, we write

$$\mathbf{p}' = \mathbf{LR}(\psi, \theta)\mathbf{p},$$

and for *half* of an up-down snake, we write

$$\mathbf{p}' = \mathbf{UD}(\psi, \theta)\mathbf{p}.$$

FIGURE 6 (a) Left-right snake. (b) Half up-down snake.

It will be shown later that snakes can be made by combining two half up-down snakes.

With these definitions, we find the following correspondence between configurations and operators for L-R snakes and half U-D snakes:

$$\begin{aligned}
 \mathbf{LR}(+90, +45) &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \mathbf{D}_1 & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} -S \\ -V \\ -H \end{pmatrix} \\
 \mathbf{LR}(-90, +45) &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \mathbf{D}_1 & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} -S \\ -V \\ -H \end{pmatrix} \\
 \mathbf{LR}(+90, -45) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \mathbf{D}_2 & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} S \\ -V \\ H \end{pmatrix} \\
 \mathbf{LR}(-90, -45) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \mathbf{D}_2 & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} S \\ -V \\ H \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{UD}(+45, +90) &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \mathbf{E}_2 & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} -H \\ S \\ V \end{pmatrix} \\
 \mathbf{UD}(+45, -90) &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \mathbf{E}_2 & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} -H \\ S \\ V \end{pmatrix} \\
 \mathbf{UD}(-45, +90) &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} = \mathbf{E}_1 & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} -H \\ -S \\ -V \end{pmatrix} \\
 \mathbf{UD}(-45, -90) &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} = \mathbf{E}_1 & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} -H \\ -S \\ -V \end{pmatrix}
 \end{aligned}$$

With L-R snakes (as expected) or half U-D snakes, we find that they duplicate operations which heretofore required a combination of two half-snakes. Again, we inquire about combining these half U-D snakes with others. We limit these combinations to only those in which the *vertical component of the polarization vector is rotated to longitudinal at the midpoint*. The intention is to summarize such combinations that can be used either as snakes or rotators (i.e., operators $\mathbf{C}_1, \mathbf{C}_2, \mathbf{D}_1, \mathbf{D}_2$, or $\mathbf{I}$). Combinations such as these are shown in Table III.

Again, we find that combining these configurations gives two new operators $\mathbf{S}_1$ and $\mathbf{S}_2$. This brings the total number of operators found to 24. All have a determinant of +1, as expected for rotations, and they represent half of the 48 operators which would include all permutations and all changes of sign.

$$\begin{aligned}
 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} &= \mathbf{S}_1 & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} V \\ H \\ -S \end{pmatrix} \\
 \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} &= \mathbf{S}_2 & \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Rightarrow \begin{pmatrix} -V \\ -H \\ -S \end{pmatrix}
 \end{aligned}$$

Note that $\mathbf{S}_1^2 = \mathbf{S}_2^2 = \mathbf{I}$. Again, it can be shown that the set of operators

$$\{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \mathbf{S}_3^{-1}, \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \mathbf{I}\}$$

forms a noncommutative group with respect to matrix multiplication.

The rotation angle is 180° for operators $\mathbf{S}_1$ and $\mathbf{S}_2$ about eigenvectors

$$\mathbf{s}_1 = 1/2(\pm\sqrt{2}\hat{\mathbf{i}} \pm \sqrt{2}\hat{\mathbf{j}})$$

and

$$\mathbf{s}_2 = 1/2(\pm\sqrt{2}\hat{\mathbf{i}} \mp \sqrt{2}\hat{\mathbf{j}}).$$

TABLE III

Multiplication table for combinations of two half-serpents or half-snakes, where vertical spin becomes longitudinal at midpoint.

	A_1	A_2	A_3	A_4	E_1	E_2
E_2	D_3^{-1}	D_3	D_1	D_2	C_2	I
E_1	D_1	D_2	D_3^{-1}	D_3	I	C_2
A_4	A_2^{-1}	A_3^{-1}	A_1^{-1}	A_4^{-1}	S_3^{-1}	S_2
A_3	A_4^{-1}	A_1^{-1}	A_3^{-1}	A_2^{-1}	S_3	S_1
A_2	A_3^{-1}	A_2^{-1}	A_4^{-1}	A_1^{-1}	S_1	S_3
A_1	A_1^{-1}	A_4^{-1}	A_2^{-1}	A_3^{-1}	S_2	S_3^{-1}
A_4^{-1}	C_2	C_1	C_3	I	D_3^{-1}	D_2
A_3^{-1}	C_1	C_2	I	C_3	D_3	D_1
A_2^{-1}	C_3	I	C_2	C_1	D_2	D_3^{-1}
A_1^{-1}	I	C_3	C_1	C_2	D_1	D_3
D_1	S_2	S_3	S_1	S_3^{-1}	A_1^{-1}	A_3^{-1}
D_2	S_3^{-1}	S_1	S_3	S_2	A_2^{-1}	A_4^{-1}
S_3	E_2	E_3	E_3^{-1}	E_1	A_3	A_2
S_3^{-1}	E_3	E_2	E_1	E_3^{-1}	A_4	A_1

The operators in the top row of Table III represent the first half-serpent or half-snake as seen by the beam. Those in the left column are for the second half-serpent or half-snake.

Combinations resulting in C_1 , C_2 , D_1 , or D_2 are snakes. Combinations that result in I are rotators. For example, $E_1E_2 = C_2$ is a combination that forms a snake of the second kind; whereas $E_2E_2 = I$ is a rotator.

In Table IV, we summarize the 24 operators that have been found by combining half-serpents or half-snakes of various kinds. This set of operators forms a group with respect to matrix multiplication. The group is isomorphic to the *group of rotational symmetries of a cube*, known as the *octahedral group*.⁷

It is interesting to note that the operators D_3 , E_3 , and S_3 , along with their inverses, are but special cases of the spin operations performed by single magnets: horizontal bending [Eq. (4)], vertical bending [Eq. (5)], or a solenoidal field magnet, respectively. It is not surprising then that it is simple to show that these operators, representing rotations about the fourfold symmetry axes, generate all operators of the octahedral group. The isomorphism of the operators generated by various configurations to the octahedral group, which represents rotations about *all* the axes of rotational symmetry for a cube, furnishes a useful tool for the visualization of spin manipulation and the action of this manipulation on each component of the spin vector. In an example below, we will use this tool to generate two *new* types of snake, each with *midpoint longitudinal polarization*. Before doing so, we first provide an operational representation of a general snake.

In Section 1, the definition of a snake was cited as the single operation of rotating the spin polarization vector by 180° about an arbitrary axis in the horizontal plane. An alternate way of describing this operation as the product of

TABLE IV
Summary of 24 operators with respective rotation angles and eigenvectors (rotation axes).

$\mathbf{A}_1, \mathbf{A}_1^{-1}$	$+120^\circ, -120^\circ$	$\mathbf{a}_1 = 1/3(+\sqrt{3}\hat{i} + \sqrt{3}\hat{j} + \sqrt{3}\hat{k})$
$\mathbf{A}_2, \mathbf{A}_2^{-1}$	$+120^\circ, -120^\circ$	$\mathbf{a}_2 = 1/3(+\sqrt{3}\hat{i} - \sqrt{3}\hat{j} - \sqrt{3}\hat{k})$
$\mathbf{A}_3, \mathbf{A}_3^{-1}$	$+120^\circ, -120^\circ$	$\mathbf{a}_3 = 1/3(-\sqrt{3}\hat{i} + \sqrt{3}\hat{j} - \sqrt{3}\hat{k})$
$\mathbf{A}_4, \mathbf{A}_4^{-1}$	$+120^\circ, -120^\circ$	$\mathbf{a}_4 = 1/3(-\sqrt{3}\hat{i} - \sqrt{3}\hat{j} + \sqrt{3}\hat{k})$
$\mathbf{C}_1$	180°	$\mathbf{c}_1 = \pm\hat{k}$
$\mathbf{C}_2$	180°	$\mathbf{c}_2 = \pm\hat{i}$
$\mathbf{C}_3$	180°	$\mathbf{c}_3 = \pm\hat{j}$
$\mathbf{I}$	0°	
$\mathbf{D}_1$	180°	$\mathbf{d}_1 = 1/2(\pm\sqrt{2}\hat{i} \mp \sqrt{2}\hat{k})$
$\mathbf{D}_2$	180°	$\mathbf{d}_2 = 1/2(\pm\sqrt{2}\hat{i} \pm \sqrt{2}\hat{k})$
$\mathbf{D}_3, \mathbf{D}_3^{-1}$	$+90^\circ, -90^\circ$	$\mathbf{d}_3 = +\hat{j}$
$\mathbf{E}_1$	180°	$\mathbf{e}_1 = 1/2(\pm\sqrt{2}\hat{j} \mp \sqrt{2}\hat{k})$
$\mathbf{E}_2$	180°	$\mathbf{e}_2 = 1/2(\pm\sqrt{2}\hat{j} \pm \sqrt{2}\hat{k})$
$\mathbf{E}_3, \mathbf{E}_3^{-1}$	$+90^\circ, -90^\circ$	$\mathbf{e}_3 = +\hat{i}$
$\mathbf{S}_1$	180°	$\mathbf{s}_1 = 1/2(\pm\sqrt{2}\hat{i} \pm \sqrt{2}\hat{j})$
$\mathbf{S}_2$	180°	$\mathbf{s}_2 = 1/2(\pm\sqrt{2}\hat{i} \mp \sqrt{2}\hat{j})$
$\mathbf{S}_3, \mathbf{S}_3^{-1}$	$+90^\circ, -90^\circ$	$\mathbf{s}_3 = +\hat{k}$

two operations is also given in Ref. 2. This description can be expressed using operators which have been defined here. The first operation represents a rotation of 180° about the axis $+\hat{i}$ [corresponding to operators $\mathbf{E}_3^2$, $(\mathbf{E}_3^{-1})^2$, or $\mathbf{C}_2$ given in Table IV], followed by an arbitrary spin precession through an angle α about $+\hat{j}$ [corresponding to the operator $\mathbf{H}(\alpha)$ for a horizontal bend magnet given by Eq. (4)].

The product defining a snake is then $\mathbf{H}(\alpha)\mathbf{C}_2$.

The arbitrary axis referred to in the initial definition of a snake is defined as the vector $\mathbf{a}$, which is now given by

$$\mathbf{a} = \pm \cos\left(\frac{\alpha}{2}\right)\hat{i} \pm \sin\left(\frac{\alpha}{2}\right)\hat{k}.$$

For example, the special cases describing L-R snakes are obtained by setting $\alpha = \pm 90^\circ$, since

$$\mathbf{H}(+90)\mathbf{C}_2 = \mathbf{D}_3\mathbf{C}_2 = \mathbf{D}_1 = \mathbf{LR}(\pm 90, +45) \quad (7)$$

and

$$\mathbf{H}(-90)\mathbf{C}_2 = \mathbf{D}_3^{-1}\mathbf{C}_2 = \mathbf{D}_2 = \mathbf{LR}(\pm 90, -45). \quad (8)$$

Similarly, $\mathbf{H}(0)\mathbf{C}_2 = \mathbf{C}_2$ and $\mathbf{H}(180)\mathbf{C}_2 = \mathbf{C}_3\mathbf{C}_2 = \mathbf{C}_1$ define snakes of the second and first kinds, respectively.

In Table III, we show that $\mathbf{D}_1$ or $\mathbf{D}_2$ (the L-R snake operators) appearing in Eqs. (7) and (8) can also be obtained by the product of two operators, where the first operator rotates the vertical spin component into the longitudinal direction.

This suggests that such configurations may be capable of generating snakes with *longitudinal polarization at the midpoint and an arbitrary precession angle* α . See Refs. 2 and 6 for detailed references and discussion of the advantages of achieving an arbitrary precession angle α .

As an example, we assume that the first operator is given by $\mathbf{A}_1$, which provides *midpoint longitudinal polarization*, and the second operator (to be found) is given by $\mathbf{M}$. Equating their product to the product defining a snake gives

$$\mathbf{M}\mathbf{A}_1 = \mathbf{H}(\alpha)\mathbf{C}_2$$

or

$$\mathbf{M} = \mathbf{H}(\alpha)\mathbf{C}_2\mathbf{A}_1^{-1}.$$

Reducing gives

$$\mathbf{M} = \begin{pmatrix} -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & -1 \\ -\cos \alpha & -\sin \alpha & 0 \end{pmatrix}.$$

If we again set $\alpha = +90^\circ$, then $\mathbf{M} = \mathbf{E}_1$ and $\mathbf{M}\mathbf{A}_1 = \mathbf{E}_1\mathbf{A}_1 = \mathbf{D}_1$, as shown in Table III. The operator $\mathbf{E}_1$ is generated by one half of a U-D snake. This suggests that practical solutions for a chosen value of α near 90° may be found by examining $\mathbf{UD}(\psi, \theta)$ configurations for values of ψ and θ that will generate the operator $\mathbf{M}$.

Still assuming that the first operator is given by $\mathbf{A}_1$, similar considerations for α near 0° would indicate examination of the configuration corresponding to operator $\mathbf{A}_4^{-1}$, and for α near 180° , the configuration corresponding to the operator $\mathbf{A}_3^{-1}$. Taking these last two cases as examples and using the aid of a computer, it can be shown that for $\alpha = +20^\circ$

$$\mathbf{H}(+20)\mathbf{C}_2 = \mathbf{H}(+86.67, +90.96)\mathbf{V}(+93.33, +89.52),$$

and for $\alpha = +160^\circ$,

$$\mathbf{H}(+160)\mathbf{C}_2 = \mathbf{H}(+86.67, -90.19)\mathbf{V}(+86.67, +90.19).$$

Both of these snakes with arbitrarily chosen values for α have *longitudinal polarization of 99.8% at the interaction point* assuming vertical polarization elsewhere. In these two cases, the angle α is determined by the change in strength of the vertical bend magnets from the initial value of 90° , whereas the smaller changes in the horizontal bends allowed the simultaneous reduction to zero of four small residual matrix elements.

Using our knowledge of configurations which can perform rotations about all the symmetry axes of a cube, we have easily found two new snakes with special properties.

7. CONCLUSIONS

By combining two half-serpents or half-snakes of various kinds, we have found a group of 24 rotation operators. The isomorphism to the octahedral group suggests

that efficient means of obtaining arbitrary rotations for particular applications are achievable.

Several combinations, as shown in Tables II and III, result in longitudinal polarization at the midpoint (interaction point) and also invert the vertical component. In principle, these or other combinations representing rotators would be useful in circular machines.

For electrons the total $\int B dl$ required for these combinations ranges from 32.2 T-m to 36.8 T-m. These high values may limit the usefulness of these configurations in some applications due to synchrotron radiation, which implies long magnets and large beam excursions. In proton machines, shorter magnets can be used.

One half of a serpent could be used to rotate longitudinal to vertical polarization for transporting and injecting electrons into a high-energy (~ 10 GeV) damping ring of a Super Linear Collider. Another half-serpent would restore the longitudinal polarization in the damped beam. The need for solenoidal fields the strength of which scale as γa_{e-} would be eliminated.

At the Stanford Linear Collider, presently under construction, complications due to the precession of a longitudinally polarized beam in the mile-long arcs could be avoided if the polarization was made to be parallel to those fields. The desired longitudinal polarization at the interaction point then could be achieved by introducing the proper half-serpent. Unfortunately, the required drift space (10 to 20 m) is not presently available.

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