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MODULAR INVARIANCE AND COMPACTIFICATION
OF THE MODULI SPACE

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ABSTRACT

The first Chern class for the ratio of determinant line bundles defining the string measure in the functional integral approach, is computed for the bosonic and for the type II and the heterotic string. The conditions under which the non vanishing of this first Chern class can imply global modular anomalies are discussed.

Modular invariance appears to be crucial to the consistency of the string theory. In this letter we address the problem of modular invariance of superstrings when they are defined on the compact moduli space $\tilde{R}_g$.

In the Polyakov first quantized approach to string theory [1], the string generating functional at each loop level requires to integrate over the space $\mathcal{M}_g$ of metrics. In the critical dimension ($D = 26$ or 10) this integral reduces to an integral over the Teichmüller space, which for genus $g > 1$ can be written like:

$$T_g = \frac{\mathcal{M}_g^c}{Diff_0 \cdot Conf} \quad (1)$$

with $\mathcal{M}_g^c$ the space of metrics with constant curvature equal -1 and $Diff_0$ the space of diffeomorphism connected to the identity. Invariance under the residual modular invariance, $Mod_g = \frac{Diff}{Diff_0}$, will allow us to define the string generating functional as an integral over the Riemann space $R_g = \frac{T_g}{Mod_g}$. More precisely the vacuum to vacuum amplitude reduces to the product of the determinants obtained by gaussian integration over the whole set of field, including the ghosts fields. These determinants depend on the complex structure of the world sheet, and in this way will define line fiber bundles on R_g . In this framework, modular invariance can be studied just computing the total first Chern class of the determinant line bundle which defines the string measure. In fact T_g is contractible [2], therefore any fiber bundle on T_g is trivial, thus the first Chern class of the determinant line bundle on $R_g = \frac{T_g}{Mod_g}$ can give us a topological obstruction to the realization of modular invariance. In this letter, we will compute this first Chern class for the supersymmetric case on the compact Riemann space $\tilde{R}_g$ obtained by adding to R_g surfaces with nodes [3].

Let us start recalling some properties of R_g . A complex structure on R_g can be defined by embedding T_g into the Bers space $B_g(L) \sim C^{3g-3}$ of quadratic differentials defined over the lower complex plane [2]. Moreover, a natural set of coordinates on R_g are given by the twist-length (τ^i, l^i) Fenchel-Nielsen coordinates associated to a "pant" decomposition of the surface [3]. In terms of this set of coordinates the modular invariant Weil-Peterson $(1, 1)$ Kähler form can be written like [4]:

$$\omega_{wp} = \sum_i d\tau_i \wedge dl_i \quad (2)$$

Now a determinant line bundle on R_g for any operator D_c defined on a Riemann surface C of genus g can be obtained patching together the one dimensional vector spaces:

$$L \equiv \bigwedge^{max} (Ker D_c)^* \otimes \bigwedge^{max} (Ker D_c^*) \quad (3)$$

on the space $\mathbb{C}_g$ of surfaces (algebraic curves) of genus g and after that, projecting on R_g by:

$$\pi : \mathbb{C}_g \rightarrow R_g \quad (4)$$

$$\pi^{-1}(x) = C_x$$

We will denote the resulting line bundle on $R_g, \pi^*(L)$. The relevant thing for string theory is that the whole set of determinant line bundles which appear in the measure are, for the non compact case R_g , powers of the determinant line bundle defined by the Cauchy-Riemann operator $\bar{\partial}$. Moreover the first Chern class of this determinant line bundle is determined using Quillen's theorem [5], by the $(1,1)$ Kähler form ω_{wp} defined in equation (2).

Let us briefly quote the result of ref. [5] and its physical interpretation as Schwinger terms for the operator product expansion of the energy-momentum tensor of the string coordinates fields. For the Cauchy-Riemann operator $\bar{\partial}$ we have that the determinant line bundle is given by:

$$L = \bigwedge^{max} (Ker \bar{\partial})^* \otimes \bigwedge^{max} (Ker \bar{\partial}^*) \quad (5)$$

which can be written, using Hopf's index theorem, like:

$$L = \bigwedge^g H^0(C, K_c) \quad (6)$$

with K_c , the canonical line bundle on C defined by: $K_c = \Theta_c^*$, the dual to the tangent bundle; ($H^0(C, K_c)$ is just the space of 1-holomorphic differentials).

Putting together Quillen's theorem and Wolpert's result (eq. (2)), we get that the curvature of $\pi^*(L)$:

$$\Theta = \bar{\partial} \partial \log |s|^2 \quad (7)$$

with s any local holomorphic section and $|s|^2 = \det \Delta$ ($\Delta = \bar{\partial}^* \bar{\partial}$), defined by the zeta function regularization, is given by the Kähler form ω_{wp} . The first Chern class of $\pi^*(L)$ is [6]:

$$C_1(\pi^*(L)) = \frac{i}{2\pi} \Theta \quad (8)$$

This Chern class can be interpreted as the Schwinger term for the energy-momentum tensor of the string coordinates fields [7], just noticing that $\log|s|^2$ is the effective action obtained by integration over the string coordinates. Once we know the first Chern class of the determinant line bundle π^*L , the next step will be to write the other set of determinants, which appear in the string measure in terms of L . This was done by Mumford [8] using Grothendieck's- Riemann-Roch theorem [9]. In order to show this result, we will introduce some new piece of notation. Firstly, spin $\frac{1}{2}$ fields are defined as sections of a fiber bundle L_α defined by a solution of $L_\alpha \otimes L_\alpha = K_C$; these solutions are the theta characteristics of the surface C [10, 11], its number for genus g is 2^{2g} . Defining now tensorial products of L_α , we consider the covariant derivatives for each spin structure α in harmonic coordinates [12]:

$$\nabla_\alpha^{\frac{n}{2}} : L_\alpha^n = \otimes^n L_\alpha \rightarrow L_\alpha^{n+2} \quad (9)$$

with the adjoint defined by:

$$\nabla_{\alpha, \frac{n+2}{2}} : L_\alpha^{n+2} \rightarrow L_\alpha^n \quad (10)$$

It is clear that these operators are defined on R_g , namely because each point in R_g defines a different set of harmonic coordinates through the corresponding Beltrami differential [2]. Associated with these operators, we have the corresponding determinant line bundles:

$$\mathcal{L}_m \equiv \bigwedge^{\max} (Ker \nabla_\alpha^{\frac{n}{2}})^* \otimes \bigwedge^{\max} (Ker \nabla_{\alpha, \frac{n+2}{2}}) \quad (11)$$

($m = \frac{n}{2}$) which can be written like:

$$\mathcal{L}_m = \bigwedge^{\max} H_0(C, \otimes^{2m} L_\alpha) \quad (12)$$

For m integer $\pi^*(\mathcal{L}_m)$ are defined on R_g , otherways they are defined on a 2^{2g} covering of R_g . The relevant fiber bundles for the string are: $\mathcal{L}_1 = L(\text{eq.}(6))$ for the string coordinates, $\mathcal{L}_{\frac{1}{2}}$ for the fermionic coordinates, $\mathcal{L}_2$ for the bosonic Faddeev Poppov ghost, and $\mathcal{L}_{\frac{3}{2}}$ for the Rarita-Schwinger ghosts.

After introducing the notation, we can briefly describe how the Grothendieck's- Riemann-Roch theorem works. Let us start recalling the Chern and Todd character of an arbitrary fiber bundle $\mathcal{F}$ [9]:

$$ch(\mathcal{F}) = rk(\mathcal{F}) + c_1(\mathcal{F}) + \frac{1}{2}c_1(\mathcal{F})^2 - c_2(\mathcal{F}) + \dots \quad (13)$$

$$td(\mathcal{F}) = 1 - \frac{1}{2}c_1(\mathcal{F}) + \frac{1}{12}(c_1(\mathcal{F})^2 + c_2(\mathcal{F})) + \dots \quad (13, a)$$

Now let us call Ω^1 the space of 1-holomorphic differentials ($\Omega^1 \equiv H_0(C, K_c)$). We know from the theory of Teichmüller space [2, 13] that the cotangent space $\Theta^*_{R_g}$ to R_g is the space of 2-holomorphic differentials, the ghost zero modes for $g > 1$ [14]. Therefore we have:

$$\pi^*(\otimes^2 \Omega^1) = \Theta^*_{R_g} \quad (14)$$

Now using (13) and general properties of Chern classes [9] we get:

$$c_1(\pi^*(\otimes^2 \Omega^1)) = c_1(\bigwedge^{max} \pi^*(\otimes^2 \Omega^1)) = c_1(K_{R_g}) \quad (15)$$

where K_{R_g} is the canonical fiber bundle on R_g . Using a most compact notation and generalizing (15) to any tensor product of Ω^1 's we get:

$$ch(\pi^*(\otimes^m \Omega^1))_1 = c_1(\bigwedge^{max} (\pi^*(\otimes^m \Omega^1))) \quad (16)$$

The Grothendieck's-Riemann-Roch theorem gives us these fiber bundles on R_g in terms of second order Chern classes on the space $\mathbb{C}$ of curves (notice that $dim_{\mathbb{C}} \mathbb{C} = dim_{\mathbb{C}} R_g + 1$) [8]:

$$c_1(\pi^* \mathcal{L}_m) = ch(\pi^*(\otimes^m \Omega^1))_1 = \pi^*(ch(\otimes^m \Omega^1).td\Omega^1)_2 \quad (17)$$

Using (13) and (13.a) we get after some algebra:

$$c_1(\pi^* \mathcal{L}_m) = (6m(m-1) + 1).c_1(\pi^* \mathcal{L}_1) \quad (18)$$

in agreement with Alvarez's result [15]. The characteristic classes for the string determinants are: $c_1(\mathcal{L}_{\frac{1}{2}}) = -\frac{1}{2}c_1(\mathcal{L}_1)$, $c_1(\mathcal{L}_2) = 13c_1(\mathcal{L}_1)$, $c_1(\mathcal{L}_{\frac{3}{2}}) = \frac{11}{2}c_1(\mathcal{L}_1)$. For m half integer the equation that really make sense on the 2^{2g} covering of R_g is : $(\pi^* \mathcal{L}_m)^2 = ((\pi^* \mathcal{L}_1)^{6m(m-1)+1})^2$. Using the functional integral representation for n -loop vacuum to vacuum amplitudes for the bosonic [16], type II [17], and heterotic [12], we get that the values for the total Chern class are equal to zero, independently in the right and left moving sectors, for the bosonic in $D = 26$ and for type II and heterotic in $D = 10$.

In the second part of this letter, we wil extend these computations to the compac moduli space. The compact moduli space $\tilde{R}_g$ is defined [3] by $\frac{\hat{T}_g}{Mod_g}$ where $\hat{T}_g$ is the augmented Teichmüller space, defined by adding Riemann surfaces on which a finite number of simple

loops have been contracted to points. This compactification of R_g can be identified with the compactification $\tilde{M}_g$ of the moduli space of stable algebraic curves of genus g [8]. The boundary $\tilde{M}_g - M_g$ is the union of components $\Delta_0 \dots \Delta_{\frac{g}{2}}$ containing curves with nodes of degree i (a node which divides the curve in two connected pieces with genus i and $g - i$). The component Δ_0 of irreducible curves corresponds to pinching loops which are not dividing geodesics.

The representation of $\tilde{M}_g$ by $\frac{\hat{T}_g}{Mod_g}$ should in principle allow us to interpret first Chern classes of the extended line determinant bundles to $\tilde{M}_g$ as obstructions to modular invariance, if the space $\hat{T}$ is contractible. Before considering this problem, we will study how equation (18) change as an effect of the boundary of M_g . The first modification we must take into account is that the space of 1-holomorphic differentials must be redefined when we are working on a surface with nodes, more explicitly, on singular curves the space of 1-holomorphic forms must be defined as the space of forms which has poles of order 1 at the punctures [18]. Let us call this space ω^1 , which will coincide with Ω^1 on regular curves. Equation (17) now becomes:

$$c_1(\pi^* \mathcal{L}_m) = ch(\pi^*(\otimes^m \omega^1))_1 = \pi^*(ch(\otimes^m \omega^1).td\Omega^1)_2 \quad (19)$$

where now $\pi : \tilde{\mathcal{C}} \rightarrow \tilde{M}_g = \hat{R}_g$. The main difference between (19) and (17) is that now Ω^1 have a second Chern class different from zero, given by [8]:

$$\pi^* c_2(\Omega^1) = \delta \quad (20)$$

where δ is the fiber bundle defined by the divisor [19]:

$$\Delta_0 + \frac{1}{2}\Delta_1 + (\Delta_2 + \dots \Delta_{\frac{g}{2}}) \quad (21)$$

Notice that $c_2(\Omega^1) = 0$ on M_g . Using (20) and (19), equation (18) becomes [8, 19]:

$$c_1(\pi^* \mathcal{L}_m) = c_1(((\pi^* \mathcal{L}_1)^{12} \otimes \delta^{-1})^{\binom{m}{2}} \otimes (\pi^* \mathcal{L}_1)) = (6m(m-1)+1)c_1(\pi^* \mathcal{L}_1) - \binom{m}{2} c_1(\delta) \quad (22)$$

Assuming that spin structures can be defined on the kind of singular curves we are dealing with [20], we will use (22) for m integer or half integer, and we will give sense to (22) by considering some power N of the left and right part of the equation. It is worthwhile to point out here that the new contribution (eq. (20)) only depends on Ω^1 , and it is in

principle independent of the way spin structures are defined on singular curves. Thus the new Chern classes are:

$$c_1(\mathcal{L}_{\frac{1}{2}}) = -\frac{1}{2}c_1(\mathcal{L}_1) + \frac{1}{8}c_1(\delta) \quad (23)$$

$$c_1(\mathcal{L}_{\frac{3}{2}}) = \frac{11}{2}c_1(\mathcal{L}_1) - \frac{3}{8}c_1(\delta) \quad (24)$$

$$c_1(\mathcal{L}_2) = 13c_1(\mathcal{L}_1) - c_1(\delta) \quad (25)$$

and $c_1(\mathcal{L}_1)$ unchanged as a result of Wolpert's theorem, which extends ω_{wp} given by (2) to the whole $\tilde{M}_g$ [13]. With the values (23) to (25) we can easily check that the first Chern class for the bosonic string is $-c_1(\delta)$. For the superstring of type II is zero [21] in ten dimensions, and for the heterotic string zero for the right moving sector and $c_1(\delta)$ for the left moving sector.

To interpret these Chern classes as some kind of topological obstruction to modular invariance can be, in principle, the wrong interpretation. In fact for the bosonic string the Chern class we get is the Chern class of a line fiber bundle defined on $\frac{\hat{T}_g}{Mod_g}$. The argument that this first Chern class represents an obstruction to modular invariance is based on the fact that any line determinant bundle on $\hat{T}_g$ is trivial. This is true for T_g , but may be not necessarily true for $\hat{T}_g$, which is a contractible space but *not* a manifold [2]. In the superstring case the situation is even less clear, since the line fiber bundle is defined on a 2^{2g} covering of $\hat{T}_g$. To define this covering requires to define theta characteristic on singular curves, which is not always possible [20]. Moreover a general proof of modular invariance has been recently given by Witten [22]. In this proof, the triviality of the spin cobordism group in three dimensions is used in order to compute the eta invariants through the Atiyah-Patodi-Singer theorem [23]. The four dimensional densities which appear in this index theorem should be equal to the four dimensional density, which appears in the family index theorem for a two dimensional family of Riemann surfaces, namely $(ch\mathcal{F}.td\Omega^1)_2$, with $\mathcal{F}$ replaced by the determinant line bundles under study. The additional contribution to $td\Omega^1$ on $\tilde{R}_g$ in equation (20) [8], seems to us a potential problem for the extension of the proof in ref. [22], to the case of Riemann surfaces with nodes.

For the superstring case it is well known that modular invariance forces us to sum over different spin structures with well defined coefficients [24, 12, 17]. Before doing this sum, the ratio of determinants which define the string measure for the left and right moving sectors, say $S(\alpha, \dots)$ with $\alpha, \dots$ representing the spin structures, will be the section of a line

bundle on $\tilde{R}_g$ with transition functions defined by the ratio $\frac{S(\alpha, \dots)}{S(\alpha', \dots)}$ with $\alpha', \dots$ a modular transformation of α . Our previous analysis shows us that *this* line bundle is trivial for the superstring of type II and non trivial for the left moving sector of the heterotic string.

It is important to notice that a non vanishing Chern class implies the existence of a pole on the boundary of $\tilde{R}_g$, which in principle, can be interpreted as signaling the existence of a tachion. However, the interpretation of this Chern class as an obstruction to modular invariance on the boundary, can be, at least at the level of the bosonic string, just considered as implying that it is not possible to regularize the theory in a modular invariant way. In summary we just point out that, if the fact that line fiber bundles are trivial on contractible spaces extends to spaces like $\hat{T}_g$, which is a well defined topological space but not a manifold, then the non vanishing first Chern class, we computed in this letter, should imply a global anomaly with respect to modular transformations.

Finally we would like to make some comments on a possible mathematical relation between the functional integral approach to string theory in terms of determinant line bundles on the moduli space R_g and Witten's approach in terms of non commutative differential geometry [25]. Non commutative differential geometry is the natural framework to study the topology of spaces like R_g which are the orbit spaces of a discrete group, in this particular case the modular group. The associated C^* algebra is then non commutative [26]. Family index theorems can be then extended to these spaces using the K-theory of the corresponding C^* algebra [26]. It is amusing to speculate that some kind of BRS operator can define the cohomology of the non commutative $*$ -algebra associated to the moduli space. In such a case we should be able to understand the kind of topological obstructions we consider in this letter from a different point of view.

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