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THE WEAK HAMILTONIAN ON THE LATTICE⁺⁾

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The renormalized octet weak Hamiltonian on the lattice is constructed in a form suitable for Monte Carlo calculation. A viable method for computing so-called "eye" diagrams is proposed.

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Octet enhancement in $\Delta S = \pm 1$ weak transitions is a long standing problem in particle physics.

The realization that strong interactions are described by an asymptotically free gauge field theory has permitted to make considerable progress, in that one has been able to show that short distance effects do indeed favour $\Delta I = 1/2$ versus $\Delta I = 3/2$ amplitudes⁽¹⁾. A further enhancement should be provided by the so-called "penguin" operators⁽²⁾ but a quantitative understanding has not been achieved yet, lacking a reliable calculation of the matrix elements of the relevant operators.

Preliminary results for the matrix elements of the weak hamiltonian in lattice QCD have been presented⁽³⁾⁽⁴⁾⁽⁵⁾, for $\Delta I = 3/2$ amplitudes or, equivalently, the $\Delta S = 2$, $K_0 - \bar{K}_0$ transition. The computation of the $\Delta I = 1/2$ transition amplitude is considerably more difficult, however, due the severe (power like) ultraviolet divergences displayed by the relevant operators, in the limit of vanishing lattice spacing ($a \rightarrow 0$). The divergences can be traced back to the explicit chiral symmetry breaking induced by the Wilson term in the QCD action^(f). Indeed, the explicit construction of the renormalized lattice operators corresponding to the octet weak hamiltonian has not been given yet. The aim of the present paper is to fill this theoretical

(f) The Wilson term is needed to eliminate fermion doubling. It is very unclear that one can study this problem in the Kogut-Susskind formulation which suffers from an inextricable mixing between spin and flavour quantum numbers.

gap, along the lines indicated in ref.(6). More details of the analysis are given elsewhere⁽⁷⁾. Our results are as follows.

- i) We show that one can reconstruct the octet part of the $K \rightarrow \pi \pi$ weak transition, to lowest order in quark mass, from a Monte Carlo determination of the $K - \pi$ matrix elements of regularized, chiral, lattice operators, provided the lattice spacing is such as to allow propagation of charmed quarks ($a^{-1} > m_c$).
- ii) In the frozen charm case ($m_c > a^{-1}$), due to more severe divergences in the parity conserving sector, the knowledge of the $K - \pi$ matrix elements is not sufficient; however a direct calculation of the $K \rightarrow \pi \pi$ matrix elements is still possible.
- iii) A related and important issue is that of the CP-violating parameter ϵ' ⁽⁸⁾; in the realistic situation $m_t \gg a^{-1}$, the calculation of ϵ' involves the matrix elements of operators which are, on the lattice, as singular as those found in case ii). Therefore one may obtain ϵ' only by a direct computation of the $K \rightarrow \pi \pi$ matrix elements.

For simplicity, we restrict ourselves in the following to the dynamical charm case, $a^{-1} > m_c$.

We start by recalling that the Wilson expansion allows us to express the weak hamiltonian as a combination of local operators. For the case at hand, we have:

$$H_{eff} = \frac{G}{\sqrt{2}} \sin\theta \cos\theta \sum_{i=\pm} C^{(i)}(\mu, M_W) \mathcal{O}^{(i)}(\mu) \quad (1)$$

where μ is a subtraction point (we assume $M_W \gg a^{-1} > \mu > m_c$) $C^{(\pm)}$ are the well known Wilson coefficients⁽¹⁾ and:

$$\mathcal{O}^{(\pm)}(\mu) = Z^{(\pm)}(\mu) \mathcal{O}^{(\pm)} \quad (2)$$

$$\begin{aligned} \mathcal{O}^{(+)} = \frac{1}{5} [& (\bar{s}_L \gamma_\mu d_L)(\bar{u}_L \gamma_\mu u_L) + (\bar{s}_L \gamma_\mu u_L)(\bar{u}_L \gamma_\mu d_L) + \\ & + 2(\bar{s}_L \gamma_\mu d_L)(\bar{d}_L \gamma_\mu d_L) + 2(\bar{s}_L \gamma_\mu s_L)(\bar{s}_L \gamma_\mu d_L)] + \\ & - [(\bar{s}_L \gamma_\mu d_L)(\bar{c}_L \gamma_\mu c_L) + (\bar{s}_L \gamma_\mu c_L)(\bar{c}_L \gamma_\mu d_L)] \end{aligned} \quad (3a)$$

$$\mathcal{O}^{(-)} = [(\bar{s}_L \gamma_\mu d_L)(\bar{u}_L \gamma_\mu u_L) - (\bar{s}_L \gamma_\mu u_L)(\bar{u}_L \gamma_\mu d_L)] - (u \rightarrow c) \quad (3b)$$

$Z^{(\pm)}$ are such that $\mathcal{O}^{(\pm)}(\mu)$ are normalized to unity in the four quark matrix elements at μ .

To obtain multiplicatively renormalizable operators from the naive lattice transcription of those in eqs.(3), we have to perform a number of new subtractions, due to the presence of the Wilson term. The relevant operators to subtract have equal flavour and space-time properties, equal or lower dimension and different naive chirality from those of the operators we started with:

$$\tilde{\mathcal{O}}^{(\pm)} = \mathcal{O}^{(\pm)} + \delta_6 \mathcal{O}^{(\pm)} + \delta_5 \mathcal{O}^{(\pm)} + \delta_3 \mathcal{O}^{(\pm)} \quad (4)$$

Dimension six operators appearing the r.h.s. of eq.(4) have been already classified^{(3),(5),(9)}. The corresponding subtraction, $\delta_6 \mathcal{O}^{(\pm)}$, can be reliably taken from lattice perturbation theory,⁽⁷⁾ as the coefficients depend only upon the bare coupling constant, g_0 (and the Wilson coefficient r).

As for the lower dimension operators, after a straightforward analysis,⁽⁷⁾ we find:

- i) dimension five. There is only one operator which, because of the GIM cancellation, enters with a finite, (i.e. g_0 and r dependent), coefficient:

$$\delta_5 \mathcal{O}^{(\pm)} = d_5^{(\pm)} (m_c - m_u) g_0 (\bar{s} \sigma_{\mu\nu} F^{\mu\nu} d) \quad (5)$$

where $F^{\mu\nu}$ is the non abelian gluon field strength.

Similarly to the previous case, $d_5^{(\pm)}$ can be reliably computed in perturbation theory (non vanishing contributions arise at two loops but an explicit of computation of $d_5^{(\pm)}$ is still lacking).

- ii) dimension three. The subtraction contains two operators:

$$\delta_3 \mathcal{O}^{(\pm)} = d_s^{(\pm)} (\bar{s} d) + d_p^{(\pm)} (\bar{s} \gamma_5 d) \quad (6)$$

with:

$$d_s^{(\pm)} = \frac{m_c - m_u}{a^2} C_0^{(\pm)} + \dots$$

$$d_p^{(\pm)} = \frac{(m_c - m_u)(m_s - m_d)}{a} D_0^{(\pm)} + \dots \quad (7)$$

Due to severe divergences, the coefficients $d_{s,p}$ must be determined in a non-perturbative way.

Summing up:

$$\begin{aligned} \tilde{\theta}^{(\pm)} = & \theta^{(\pm)} + d_6 \theta^{(\pm)} + d_5 \theta^{(\pm)} + \\ & + d_s^{(\pm)} (\bar{s} d) + d_p^{(\pm)} (\bar{s} \gamma_5 d) \end{aligned} \quad (8)$$

The lattice operators which correspond to the continuum operators in eq.(2), are obtained with a further rescaling:

$$\hat{\theta}^{(\pm)}(\mu) = Z_{LATT}^{(\pm)}(\mu a) \tilde{\theta}^{(\pm)} \quad (9)$$

where Z_{LATT} is chosen so that the four quark matrix elements of $\hat{\theta}$ are normalized to unity at the momentum scale $\mu < a^{-1}$ and can be taken from perturbation theory. With this choice, $\hat{\theta}^{(\pm)}(\mu)$ coincide with the continuum operators $\theta^{(\pm)}(\mu)$ and:

$$H_{eff} = \frac{G}{\sqrt{2}} \sin \theta \cos \theta \sum_{i=\pm} C^{(i)}(\mu, M_W) Z_{LATT}^{(i)}(\mu a) \hat{\theta}^{(i)} \quad (10)$$

The coefficients $d_{s,p}$, can be determined in a non-perturbative way⁽⁶⁾ requiring that the lattice Ward identities which involve the axial current and $\hat{\theta}$ coincide with those of the continuum theory.

If d_s, p , are so determined, the matrix elements of $\hat{\theta}$ do satisfy, in the limit $a \rightarrow 0$, the usual soft-pion relations⁽⁷⁾.

At this point we may turn the argument around, and use the latter condition to determine the coefficients themselves.

To lowest order in chiral symmetry breaking, the soft pion theorems imply the following relations⁽¹⁰⁾:

$$\langle 0 | \mathcal{O}_{(\mu)}^{(\pm)} | K^0 \rangle = i (m_K^2 - m_\pi^2) \delta^{(\pm)} \quad (11a)$$

$$\langle \pi^+(p) | \mathcal{O}_{(\mu)}^{(\pm)} | K^+(k) \rangle = - \frac{m_K^2}{f_\pi} \delta^{(\pm)} + (p \cdot k) \gamma^{(\pm)} \quad (11b)$$

$$\langle \pi^+ \pi^- | \mathcal{O}_{(\mu)}^{(\pm)} | K^0 \rangle = i \frac{m_K^2 - m_\pi^2}{f_\pi} \gamma^{(\pm)} \quad (11c)$$

(the last matrix element is taken for vanishing four momentum transfer). We can compute on the lattice the matrix elements of all the operators in eq.(8). For small momentum transfer, we define^(f):

$$\langle 0 | \theta^{(\pm)} + \delta_6 \mathcal{O}^{(\pm)} + \delta_5 \theta^{(\pm)} | K^0 \rangle = i \delta_1^{(\pm)}$$

$$\langle \pi^+(p) | \theta^{(\pm)} + \delta_6 \mathcal{O}^{(\pm)} + \delta_5 \theta^{(\pm)} | K^+(k) \rangle = \delta_2^{(\pm)} + (p \cdot k) \gamma_2^{(\pm)} + \dots \quad (12)$$

(f) The factors of i in eq.(12) and following ones do not arise, of course, in the euclidean lattice calculation; they are put in by hand to compare with the corresponding Minkowski space relations, eqs.(11).

and, similarly:

$$\langle 0 | \bar{s} \gamma_5 d | k^0 \rangle = i \delta_P$$

$$\langle \pi^+(p) | \bar{s} d | k^+(k) \rangle = \delta_S + (p \cdot k) \gamma_S + \dots \quad (13)$$

where dots denote terms of higher order in momentum transfer. Enforcing the equality of $\hat{\sigma}_{(\mu)}^{(i)}$ and $\sigma_{(\mu)}^{(i)}$ gives:

$$Z_{LATT}^{(i)} (\delta_1^{(i)} + d_P^{(i)} \delta_P) = (m_K^2 - m_\pi^2) \delta^{(i)} \quad (14a)$$

$$Z_{LATT}^{(i)} (\delta_2^{(i)} + d_S^{(i)} \delta_S) = - \frac{m_K^2}{f_\pi} \delta^{(i)} \quad (14b)$$

$$Z_{LATT}^{(i)} (\gamma_2^{(i)} + d_S^{(i)} \gamma_S) = \gamma^{(i)} \quad (14c)$$

We may use eqs.(12) to (14) to compute $d_{s,p}$, and express γ according to:

$$\gamma^{(i)} = Z_{LATT}^{(i)} \left(\gamma_2^{(i)} - \delta_2^{(i)} \frac{\gamma_S}{\delta_S} \right) - \frac{m_K^2}{f_\pi} \delta^{(i)} \frac{\gamma_S}{\delta_S} \quad (15)$$

Note that $\delta_{1,2}^{(i)}$ and $\gamma_2^{(i)}$ contain quadratic and linear divergences which cancel in the expression of γ ; $\delta_{s,p}$ and γ_S can be renormalized by a common logarithmically divergent factor, so

that their ratio is finite. We are interested in the value of γ in the chiral limit. Eq.(15) shows that this limit does not depend upon δ . In fact, δ is completely arbitrary, due to the ambiguities inherent to the definition of renormalized Ward identities⁽⁷⁾, and could be taken to vanish, to obtain definite values of d_s, p . Finally, we obtain:

$$\langle \pi^+ \pi^- | H_{eff} | K^0 \rangle = \frac{G}{\sqrt{2}} \sin \theta \cos \theta \ i \frac{m_K^2 - m_\pi^2}{f_\pi} \cdot \sum_{i=\pm} C^{(i)}(\mu, M_W) \bar{\gamma}^{(i)} \quad (16)$$

$$\bar{\gamma}^{(i)} = \lim_{m_{d,s} \rightarrow 0} Z_{LATT}^{(i)}(\mu a) \left(\gamma_2^{(i)} - \delta_2^{(i)} \frac{\gamma_5}{\delta_5} \right) \quad (17)$$

Needless to say, the subtraction in eq.(17) is crucial. Improper handling of divergent terms may explain the anomalous chiral behavior of the amplitudes computed in ref.(5).

In the frozen charm case ($m_c > a^{-1}$) the operator basis is more complicated. However, the analysis goes precisely in the same way, with the only important difference that the operator $g_0 \bar{s} \sigma_{\mu\nu} F^{\mu\nu} d$ has to be subtracted with a linearly divergent coefficient, due to a lack of GIM cancellation:

$$\begin{aligned} \tilde{\theta}^{(i)} = & \theta^{(i)} + \delta_6 \theta^{(i)} + d_\sigma^{(i)} g_0 (\bar{s} \sigma_{\mu\nu} F^{\mu\nu} d) + \\ & + d_s^{(i)} (\bar{s} d) + d_p^{(i)} (\bar{s} \gamma_5 d) \end{aligned} \quad (18)$$

where the index i goes now over the operator basis given in ref.(2). $K - \pi$ matrix elements give only a relation between d_σ and d_s and do not allow a prediction of γ . In this case,

however, we may still use the $K \rightarrow 0$ matrix element to fix the parity violating component, and compute directly $K \rightarrow \pi \pi$ on the lattice^(f). The same conclusion holds for the calculation of the CP-violating parameter ϵ' , which is determined by the matrix elements of the operators $\theta_{5,6}$ of ref.(2).

It is hardly necessary to stress that the weak hamiltonian thus determined from K and π matrix elements, in the case $a^{-1} > m_c$, can be used to compute other weak transitions, like those between baryonic states, in the chiral limit.

We briefly comment about the feasibility of a Monte Carlo calculation of the matrix elements in eqs.(12) and (13). The main difficulty lies in the evaluation of the so-called eye diagram, which has the general form:

$$\langle \pi(p) | \theta | K(k) \rangle_{eye} \propto \lim_{y \rightarrow \infty} \left\langle \sum_x e^{iqx} \right. \\ \left. \cdot \text{Tr} \left[\gamma_5 S(y,x) \Gamma_A S(x,x) \Gamma_A S(x,0) \gamma_5 S(0,y) \right] \right\rangle \quad (19)$$

where $S(x,y)$ is the quark propagator ($S(x,y) = K^{-1}(x,y)$, if $K(x,y)$ is the kernel in the fermion lagrangian). Eq.(19) requires to know the quark propagator between any two points, thus a computer

(f) We cannot avoid making the $K \rightarrow 0$ subtraction. The operator $\bar{s} \gamma_5 d$ is a four divergence only to zeroth order in $a^{(6)}$ and it gives an infinite contribution to $K \rightarrow \pi \pi$, if multiplied by a quadratically divergent coefficient, such as d_p , in the frozen charm case.

time growing as the square of the lattice volume, V . A method to bypass this problem has been used in ref.(5). The method we propose here is to compute $S(x,x)$ with the Langevin or the pseudo-fermion methods⁽¹¹⁾, which both require a computer time of order V . The insertion of the fermion loop is then solved by inverting the equation:

$$\sum_x K(y,x) S'(x,0) = e^{i q y} \Gamma_A S(y,y) \Gamma_A S(y,0) \quad (20)$$

with the Gauss-Seidel method, so that the r.h.s. of eq.(19) becomes:

$$\langle T_2 \gamma_5 S'(y,0) \gamma_5 S(0,y) \rangle$$

We estimate that a $14^3 \times 28$ lattice with $a^{-1} \simeq 3$ GeV is needed to allow both charmed quark propagation and the extrapolation to the chiral limit. In the quenched approximation, this requires a computation time of the order of 1500 hours with a CRAY - XMP, or 200 hours with a dedicated computer such as APE⁽¹²⁾.

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