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THE FERMIONIC ENERGY-MOMENTUM
TENSOR IN TERMS OF CURRENTS
IN AN EXTERNAL GAUGE FIELD*

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ABSTRACT

It is shown that for two-dimensional massless Dirac fields interacting with external gauge fields, the energy-momentum tensor can be expressed in terms of the currents via the Sugawara-Sommerfield formula.

It was shown almost two decades ago¹ that the energy-momentum tensor of a system of two-dimensional free massless Dirac fields can be written as a sum of traceless, symmetric products of currents – a properly regularized and normalized version of a formula proposed by Sugawara and Sommerfield.² Recent research in two-dimensional conformally invariant field theory has revived interest in this construction. It is now realized* that it applies to chiral conformal theories in two dimensions, including, for instance, chirally symmetric non-linear sigma models.

The purpose of this paper is to prove that for a set of two-dimensional massless Dirac fields interacting with external gauge fields A_μ the energy-momentum tensor is again given by a Sugawara-Sommerfield formula, suitably interpreted in the spirit of Ref. [1]. Obviously, this theory is not conformally (or even translationally) invariant.

That this result is plausible is shown by the following argument.⁴ The divergence of the Sugawara-Sommerfield formula can be expressed in terms of the covariant divergence and curl of the currents:

$$\partial^\mu \text{tr} \left(j_\mu j_\nu - \frac{1}{2} g_{\mu\nu} j^2 \right) = \text{tr} \{ (D^\mu j_\mu) j_\nu - j^\mu \epsilon_{\mu\nu} \epsilon^{\rho\sigma} D_\rho j_\sigma \} . \quad (1)$$

Now $D^\mu j_\mu = 0$ and, owing to the chiral anomaly, the covariant curl is proportional to the external field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$. $\epsilon^{\rho\sigma} D_\rho j_\sigma \propto \epsilon^{\rho\sigma} F_{\rho\sigma}$. Hence Eq. (1) is proportional to $j^\mu F_{\mu\nu}$, which is the divergence of the fermionic energy-momentum tensor. However, this naive argument must be made rigorous, as in Ref. [1], by taking into account subtleties of operator products.

Consider an N -component column vector of Dirac fields ψ , and a matrix of vector fields $A_\mu = \sum_a A_\mu^a T^a$, where T^a , $a = 0, 1, \dots, N^2 - 1$ are Hermitian generators of $U(N)$, normalized such that $\text{tr} T^a T^b = \delta_{ab}$, with $T^0 = \frac{1}{\sqrt{N}} \mathbb{1}$ and $[T^a, T^b] = i f_{abc} T^c$. The action is

$$S = \int d^2 x \bar{\psi}(x) (i \not{\partial} + \not{A}(x)) \psi(x) . \quad (2)$$

* See e.g., Ref. [3] and references therein.

In terms of the chiral projections of ψ , the Dirac equation reads*

$$\left(i\partial_{\pm} + A_{\pm}(x)\right) \frac{1 \mp i\gamma^5}{2} \psi(x) = 0. \quad (3)$$

Let $R_{\pm}(x)$ be $U(N)$ valued functions such that

$$A_{\pm}(x) = -i(\partial_{\pm} R_{\pm}(x)) R_{\pm}^{-1}(x); \quad (4)$$

then Eq. (3) can be rewritten as

$$i\partial(U(x)\psi(x)) = 0 \quad (5)$$

with

$$U(x) = R_{+}^{-1}(x) \frac{1 - i\gamma^5}{2} + R_{-}^{-1}(x) \frac{1 + i\gamma^5}{2}. \quad (6)$$

Eq. (5) is a free Dirac equation for the field

$$\psi'(x) \equiv U(x)\psi(x); \quad (7)$$

moreover, U is unitary and leaves the equal time anticommutation relations invariant. As a result, ψ' is a genuine free field. The strategy in the following will be to establish properties of objects involving ψ by expressing them in terms of ψ' .

Consider first the $U(N)$ currents j_{μ}^a , which have the generic form

$$j_{\mu}^a(x) = \bar{\psi}(x) \gamma_{\mu} T^a \psi(x). \quad (8)$$

This naive expression contains a singular product of interacting fields at the same point in space-time. A better approach is to define the currents by covariant point splitting. Let

$$\mathbf{P}(x_1, x_2) = P \exp \left[i \int_{x_2}^{x_1} dy \cdot A(y) \right]; \quad (9)$$

under a gauge transformation $A_{\mu}(x) \rightarrow g(x) A_{\mu}(x) g(x)^{-1} - i\partial_{\mu} g(x) g(x)^{-1}$, $\mathbf{P}(x_1, x_2)$ becomes $g(x_1) \mathbf{P}(x_1, x_2) g(x_2)^{-1}$. Thus, the definition

$$j_{\mu}^a(x) = \lim_{\delta \rightarrow 0} \left[\bar{\psi} \left(x + \frac{\delta}{2} \right) \mathbf{P} \left(x + \frac{\delta}{2}, x \right) \gamma_{\mu} T^a \mathbf{P} \left(x, x - \frac{\delta}{2} \right) \psi \left(x - \frac{\delta}{2} \right) + \text{VEV} \left[\right]_{A=0} \right] \quad (10)$$

insures that the currents transform according to the adjoint representation of $U(N)$ under gauge transformations of the theory. The subtraction of the vacuum expectation

* We use $\gamma^5 = i\gamma^0\gamma^1$, $\epsilon^{01} = 1$: light-cone components are defined by $a_{\pm} = \frac{1}{\sqrt{2}}(a_0 \pm a_1)$.

value guarantees that the definition reduces to the customary one in the free field case: the vacuum referred to is that of the Fock space of ψ' .

The limit in Eq. (10) can be evaluated by substitution of Eq. (7) and use of

$$\bar{\psi}'_{\alpha}(x) \psi'_{\beta}(y) =: \bar{\psi}'_{\alpha}(x) \psi'_{\beta}(y) : + S_{\beta\alpha}^{(-)}(x-y), \quad (11)$$

where the colons denote normal ordering of free fields and $S^{(-)}$ is given by

$$S_{\beta\alpha}^{(-)}(x) = -\frac{i}{2\pi} \frac{\not{x}_{\beta\alpha}}{x^2 - ix^0}. \quad (12)$$

With Eq. (11) in Eq. (10), j_{μ} becomes a sum of two terms. For the one involving the normal-ordered product the limit is trivial: it is

$$j_{\mu}^a(x) \equiv: \bar{\psi}'(x) \gamma_{\mu} U(x) T^a U^{\dagger}(x) \psi'(x) :. \quad (13)$$

The other term is the current induced in the vacuum:

$$\begin{aligned} c_{\mu}^a(x) &\equiv \lim_{\delta \rightarrow 0} \text{tr} S^{(-)}(\delta) \gamma_{\mu} U \left(x + \frac{\delta}{2} \right) \mathbf{P} \left(x + \frac{\delta}{2}, x \right) T^a \mathbf{P} \left(x, x - \frac{\delta}{2} \right) U^{\dagger} \left(x - \frac{\delta}{2} \right) \\ &= \frac{1}{2\pi} \text{tr} T^a \left[A_{\mu}(x) + \frac{i}{2} (g_{\mu\nu} + \epsilon_{\mu\nu}) \partial^{\nu} R_{+}(x) R_{+}^{-1}(x) \right. \\ &\quad \left. + \frac{i}{2} (g_{\mu\nu} - \epsilon_{\mu\nu}) \partial^{\nu} R_{-}(x) \cdot R_{-}^{-1}(x) \right]. \end{aligned} \quad (14)$$

Here (and elsewhere in this paper) the limit $\delta \rightarrow 0$ is evaluated symmetrically, such that

$$\lim_{\delta \rightarrow 0} \frac{\delta_{\mu}\delta_{\nu}\delta_{\rho}\delta_{\sigma}}{(\delta^2)^2} = \frac{1}{8} (g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}). \quad (15)$$

The vector current is covariantly conserved, and for the axial vector current the chiral anomaly is recovered:

$$D^{\mu} j_{\mu} \equiv \partial^{\mu} j_{\mu} - i[A^{\mu}, j_{\mu}] = 0 \quad (16)$$

$$D^{\mu} j_{\mu 5} \equiv \epsilon^{\mu\nu} (\partial_{\mu} j_{\nu} - i[A_{\mu}, j_{\nu}]) = \frac{1}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu}. \quad (17)$$

Next, we need the energy-momentum tensor. It is the gauge-invariant extension of the corresponding object in the free Dirac theory, naively written as

$$\theta_{\mu\nu}(x) = \frac{i}{4} \bar{\psi}(x) \gamma_{\mu} (\partial_{\nu} - iA_{\nu}(x)) \psi(x) + (\mu \leftrightarrow \nu) + \text{h.c.} \quad (18)$$

In gauge-covariant point splitting Eq. (18) is replaced by

$$\begin{aligned} \theta_{\mu\nu} = \lim_{\delta \rightarrow 0} \left[\frac{i}{4} \bar{\psi} \left(x + \frac{\delta}{2} \right) \mathbf{P} \left(x + \frac{\delta}{2}, x \right) \gamma_\mu (\partial_\nu - i A_\nu(x)) \mathbf{P} \left(x, x - \frac{\delta}{2} \right) \psi \left(x - \frac{\delta}{2} \right) \right. \\ \left. + (\mu \leftrightarrow \nu) + \text{h.c.} - \text{VEV} \right]_{A=0}. \end{aligned} \quad (19)$$

The evaluation of this expression proceeds as with the currents. The terms with the A 's from the covariant derivations add up to $\frac{1}{2} \text{tr} [j_\mu(x) A_\nu(x) + j_\nu(x) A_\mu(x)]$. Terms with derivatives are evaluated by substitution of Eq. (7), as a result of which three kinds of contributions appear, depending on whether the derivative acts on ψ' or $\bar{\psi}'$ (I), U or $U^\dagger$ (II), or a path-ordered exponential (III). The type III terms are $\lim_{\delta \rightarrow 0} \frac{1}{4} \psi' \left(x + \frac{\delta}{2} \right) \gamma_\mu U \left(x + \frac{\delta}{2} \right) \left\{ A_\nu \left(x + \frac{\delta}{2} \right) \mathbf{P} \left(x + \frac{\delta}{2}, x - \frac{\delta}{2} \right) - 2 \mathbf{P} \left(x + \frac{\delta}{2}, x \right) \times \right.$

$A_\nu(x) \mathbf{P} \left(x, x - \frac{\delta}{2} \right) + \mathbf{P} \left(x + \frac{\delta}{2}, x - \frac{\delta}{2} \right) A_\nu \left(x - \frac{\delta}{2} \right) \left. \right\} U^\dagger \left(x - \frac{\delta}{2} \right) \psi' \left(x - \frac{\delta}{2} \right) + (\mu \leftrightarrow \nu)$. The quantity inside the curly brackets is $O(\delta^2)$ and Eq. (12) implies that the other factors diverge linearly in the limit; hence, the above vanishes with δ . The type II terms become, after some reshuffling, $\lim_{\delta \rightarrow 0} \frac{1}{2} \bar{\psi}' \left(x + \frac{\delta}{2} \right) \gamma_\mu \mathbf{P} \left(x + \frac{\delta}{2}, x \right) \times \{ U(x) \partial_\nu U^\dagger(x) + O(\delta^2) \} \mathbf{P} \left(x, x - \frac{\delta}{2} \right) \psi' \left(x - \frac{\delta}{2} \right) + (\mu \leftrightarrow \nu)$, which develops into $\text{tr} \left[-\frac{1}{2} (j_\mu A_\nu + j_\nu A_\mu) + \pi (j_\mu c_\nu + j_\nu c_\mu - g_{\mu\nu} j_\lambda c^\lambda) \right]$. The type I terms, finally, give after substitution of Eq. (11) $\frac{i}{4} : \bar{\psi}'(x) \gamma_\mu \partial_\nu \psi'(x) : + (\mu \leftrightarrow \nu) + \text{h.c.}$, as well as $\lim_{\delta \rightarrow 0} (-\frac{1}{2}) \text{tr} \partial_\mu S^{(-)}(\delta) \gamma_\mu U \left(x - \frac{\delta}{2} \right) \mathbf{P} \left(x + \frac{\delta}{2}, x - \frac{\delta}{2} \right) U^\dagger \left(x - \frac{\delta}{2} \right) + (\mu \leftrightarrow \nu)$, which becomes $-\pi \text{tr} (c_\mu c_\nu - \frac{1}{2} g_{\mu\nu} c_\lambda c^\lambda)$. Addition of the above contributions leads to

$$\begin{aligned} \theta_{\mu\nu} = \frac{i}{4} : \bar{\psi}' (\gamma_\mu \partial_\nu + \gamma_\nu \partial_\mu - \gamma_\mu \bar{\partial}_\nu - \gamma_\nu \bar{\partial}_\mu) \psi' : \\ + \pi \text{tr} \left(j_\mu c_\nu + j_\nu c_\mu - g_{\mu\nu} j_\lambda c^\lambda - c_\mu c_\nu + \frac{1}{2} g_{\mu\nu} c_\lambda c^\lambda \right). \end{aligned} \quad (20)$$

This result is most elegantly expressed in light-cone coordinates:

$$\theta_{\pm\pm} = \frac{i}{2} : \bar{\psi}' (\gamma_\pm \partial_\pm - \gamma_\pm \bar{\partial}_\pm) \psi' : + \pi \text{tr} (2j_\pm c_\pm - c_\pm c_\pm) \quad (21)$$

$$\theta_{+-} = \theta_{-+} = 0. \quad (22)$$

A Sugawara-Sommerfield ansatz relating Eqs. (10) and (19) is constructed along the lines of Ref. [1]:

$$\begin{aligned} \theta_{\mu\nu}^S = \lim_{\epsilon \rightarrow 0} \left[\sum_{a=0}^i \beta_a \left\{ \left[\mathbf{P} \left(x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2} \right) j_\mu \left(x + \frac{\epsilon}{2} \right) \mathbf{P} \left(x + \frac{\epsilon}{2}, x - \frac{\epsilon}{2} \right) \right]^a j_\nu \left(x - \frac{\epsilon}{2} \right) \right. \right. \\ \left. \left. + (\mu \leftrightarrow \nu) - g_{\mu\nu} (\text{trace}) \right\} - \text{VEV} \right]_{A=0}. \end{aligned} \quad (23)$$

The main difference with Eq. (4.5) of Ref. [1] is the presence of the regulators $\mathbf{P}$. The constants β_a are fixed by requiring that $\theta_{\mu\nu}^S|_{A=0}$ be what we know it should be, namely, $\beta_0 = \frac{\pi}{2}$, $\beta_a (a > 0) = \frac{\pi}{2(N+1)}$.

To prove that $\theta_{\mu\nu}^S = \theta_{\mu\nu}$, we note that $\theta_{+-}^S = \theta_{-+}^S = 0$ and concentrate on θ_{++}^S (can be treated similarly). Substitute $j = j_+ + c$ in Eq. (23). For the terms containing j_+ at most linearly, the limit $\epsilon \rightarrow 0$ is trivial and leads to $\pi(2j_+^0 c_+^0 - c_+^0 c_+^0) + \frac{\pi}{N+1} \sum_{a>0} (2j_+^a c_+^a - c_+^a c_+^a)$. With the $\mathbf{P}$ factors expanded to $O(\epsilon^2)$, the remainder becomes

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \left[\pi j_+^0 \left(x + \frac{\epsilon}{2} \right) j_+^0 \left(x - \frac{\epsilon}{2} \right) + \frac{\pi}{N+1} \left(\delta_{ab} + \epsilon^\mu A_\mu^c f_{abc} - \frac{1}{2} \epsilon^\mu \epsilon^\nu A_\mu^c A_\nu^d f_{acd} f_{edb} \right) \right. \\ \left. \times j_+^a \left(x + \frac{\epsilon}{2} \right) j_+^b \left(x - \frac{\epsilon}{2} \right) - \text{VEV} \right]_{A=0}, \end{aligned}$$

where sums over repeated indices in the second term extend over positive values only. The products of currents are rewritten as in Ref. [1], using a modified version of Wick's theorem: that will yield terms quartic, quadratic and of zeroth order in ψ' . The sum of the quartic terms, which have a trivially well-defined limit, vanishes because of the anti-commutativity of the ψ' 's, as in the free case. The quadratic terms are linearly divergent as $\epsilon \rightarrow 0$; their contribution involves both the $O(1)$ and the $O(\epsilon)$ terms from the expansion of the $\mathbf{P}$'s, totalling $\frac{1}{2} : \bar{\psi}' (\gamma_+ \partial_+ - \gamma_+ \bar{\partial}_+) \psi' : + \frac{2N}{N+1} \sum_{a>0} (j_+^a - c_+^a) c_+^a$. The N in the numerator of the last term stems from $f_{abc} f_{abd} = 2N \delta_{cd}$. The terms without ψ' 's, finally, require the $\mathbf{P}$'s to $O(\epsilon^2)$, and turn out to be $\frac{N\pi}{N+1} \sum_{a>0} c_+^a c_+^a$. Addition of all contributions leads precisely to the desired result, Eq. (21). This completes the argument.

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