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MODEL ON CURVED BACKGROUND AND STRINGS

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CONFORMAL ANOMALY IN TWO-DIMENSIONAL SIGMA-MODEL
ON CURVED BACKGROUND AND STRINGS

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ABSTRACT

The expression for the Weyl anomaly in a generalized bosonic sigma-model defined on a curved 2-space is derived. The conditions on "g-functions" implying the vanishing of the anomaly are discussed in connection with string-theory effective equations.

1. INTRODUCTION. A string-theory ground state should be an extremum of the effective action (EA) for massless fields resulting from integration over massive fields in a string field theory action. A perturbative method for computing EA is based on matching string theory and field theory S-matrices [1]. Thus obtained off-shell EA is ambiguous [2] because of the equivalence theorem EA's which differ by a field redefinition correspond to the same perturbative S-matrix. Since the generating functional for on-shell string scattering amplitudes has a simple path integral representation [3-5] analogous representation may exist for the EA (see also [6]). It appears that a renormalized partition function of a generalized $\mathcal{G}$ -model corresponding to a string interacting with background massless fields coincides with the EA in a particular "scheme" [2]. The partition function must be computed with gauges fixed for all "on-shell" symmetries (Weyl, Möbius, modular, etc.) even though these are not symmetries of an off-shell theory. The crucial consistency requirement is that the effective "mass shell" should be independent of a choice of these gauges, i.e. that the ground-state $\mathcal{G}$ -model should be invariant under the "on-shell" symmetries. Independently, preservation of the conformal (Weyl) symmetry is generally considered as a necessary requirement on consistent (tree-level) string vacua [7-10]. The Weyl-invariance (WI) condition is usually assumed to be that of vanishing of "g-functions". It is thus natural to anticipate [9-11] that " $\beta = 0$ " should be equivalent to the (tree-level) equations derived from the EA (see also [2, 12-14]). There exists, however, a confusion about the definition of these "g-functions". While the vanishing of the ordinary RG g-functions (up to a structure induced by a diffeomorphism [15]) is indeed the condition of on-shell scale-invariance of a $\mathcal{G}$ -model (i.e. the vanishing of the integral of the trace T^m_m of the energy-momentum tensor) the WI-condition ($T^m_m = 0$) is obviously stronger [2] (see also [16]). It is the aim of this paper to derive precise conditions of WI of a $\mathcal{G}$ -model on a curved 2-surface corresponding to a closed Bose string theory. We shall also remark on how our results may help to understand the structure of the tree-level EA.

2. RENORMALIZATION OF $\mathcal{G}$ -MODEL IN FLAT SPACE. Our derivation of

the trace anomaly in the G -model will follow the general method developed in [17-20] on the example of the φ^4 -theory. A first step is to discuss renormalization of relevant composite operators in the flat space theory with the bare action ^{1/}

$$I_0 = \frac{1}{4\pi\lambda} \int d^d z G_{\mu\nu}(x) \partial_\mu x^\mu \partial_\nu x^\nu \quad (1)$$

Here $G_\mu = \mu^2 (G + \sum_{n=1}^{\infty} \frac{1}{\epsilon^n} T_n)$ is a bare metric. I_0 is a power counting renormalisable theory with an infinite set of couplings

$$\begin{aligned} \text{represented by } G \\ I_0 = \frac{1}{4\pi\lambda} \int d^d y G_{\mu\nu}(y) \int d^d z \delta^3(y - x(z)) \partial_\mu x^\mu \partial_\nu x^\nu = \\ = \frac{1}{4\pi\lambda} G_0 \cdot \int d^d z (\delta \partial x \partial x)_0, \quad f \cdot g = \int d^d y f(y) g(y) \end{aligned} \quad (2)$$

To discuss renormalization of composite operators of dimension zero $(x^*) = \sum_k C_{nk} [x^k]$, $C = C(G, \epsilon)$, $[...]$ will denote renormalised operators or "normal products", see e.g. [18])

it is useful to introduce the corresponding "source" [15]

$\Phi(x|z) = \sum_n a_n(z) x^n$ in the action (1). It is sufficient to consider only a "homogeneous" source $\Phi(x|z) = \Phi(x(z)) J(z)$, i.e. to add $\int d^d z J(z) \Phi(x) = \Phi_0 \cdot \int d^d z J \delta_0$ (δ_0 will denote the bare operator $\delta^3(y^\mu - x^\mu(z))$). Renormalization of an infinite set of couplings represented by Φ is given by ^{2/}

$$\Phi_0 = Z_1 \Phi, \quad Z_1(\epsilon, G) = \mu^\epsilon \left(1 + \sum_{n=1}^{\infty} \frac{\zeta_n}{\epsilon^n} \right) \quad (3)$$

^{1/} We shall use dimensional regularization ($d=2+\epsilon$) and minimal subtraction. $\lambda = \lambda'$ is the loop counting parameter. Bare quantities will be denoted by a subscript "0". We shall adopt the following dimension assignments: $\lambda \sim G \sim x \sim cm^0$, $Z \sim cm$, $\mu \sim cm^{-1}$ (μ is a renormalization mass parameter).

^{2/} Homogeneous renormalization of Φ is independent of a particular choice of $J(z)$. If $J = \sqrt{g}$ (g_{mn} is a metric on a curved world sheet) Φ has a natural interpretation as a tachyon field [4,5]; if $J = \sqrt{g} R^{(1)}$ Φ is a dilaton field [4,5]; if $J = \sqrt{g} (R^{(1)})^n$ Φ corresponds to a higher level massive "Stueckelberg" scalar [6]. Inhomogeneous term in (3) is absent (in dimensional regularization) if Φ is a tachyon but is present if Φ is a dilaton (see below).

The RG β -functions corresponding to G and Φ are defined as usual ($t = \ln \mu$)

$$\hat{\beta}_{G\mu\nu} = \frac{dG_{\mu\nu}}{dt} = -\epsilon G_{\mu\nu} + \beta_{G\mu\nu}, \quad (4)$$

$$\beta_G = -T_1 + G \cdot \frac{\partial}{\partial G} T_1 = -\lambda \frac{\partial}{\partial \lambda} T_1, \quad T_1(\lambda, G) = \lambda f(\lambda^{-1} G).$$

$$\hat{\beta}_\Phi = \frac{d\Phi}{dt} = -\epsilon \Phi + \beta_\Phi = -\gamma \Phi, \quad \gamma = Z_1^{-1} \frac{d}{dt} Z_1, \quad (5)$$

$$\hat{\gamma} = \epsilon + \gamma, \quad \beta_\Phi = -\gamma \Phi, \quad \gamma = -G \cdot \frac{\partial}{\partial G} \zeta_1 = \lambda \frac{\partial}{\partial \lambda} \zeta_1(\lambda^{-1} G) \quad (6)$$

Here we used that $\frac{d}{dt}(G_0, \Phi_0) = 0$ and introduced the anomalous dimension operator γ which is a differential operator acting on functions of y [15] (β_G and γ are obviously independent of Φ). As is well-known [15]

$$\beta_{G\mu\nu} = \lambda R_{\mu\nu} + \frac{1}{2} \lambda^2 R_{\mu\alpha\beta\gamma} R_{\nu}{}^{\alpha\beta\gamma} + O(\lambda^3), \quad (7)$$

$$\gamma = \frac{1}{2} \lambda G^{\mu\nu} \partial_\mu \partial_\nu + O(\lambda^2) \quad (8)$$

To compute higher-loop terms in (8) we may use the normal coordinate expansion of (1) and Φ , $\Phi(\bar{x}, t) = \sum_{n=1}^{\infty} \frac{1}{n!} (\partial_{\mu_1} \dots \partial_{\mu_n} \Phi) \bar{x}^{\mu_1} \dots \bar{x}^{\mu_n}$

The general structure of ζ_1 or γ is

$$\gamma = \frac{1}{2} \Omega_2^{\mu\nu} \partial_\mu \partial_\nu + \sum_{n=3}^{\infty} \Omega_n^{\mu_1 \dots \mu_n} \partial_{\mu_1} \dots \partial_{\mu_n} = \gamma^\mu \partial_\mu, \quad (9)$$

$$\Omega_2^{\mu\nu} = \lambda G^{\mu\nu} + c_1 \lambda^2 R^{\mu\nu} + c_2 \lambda^3 R_{\alpha\beta\gamma}{}^\mu R^{\nu\alpha\beta\gamma} + O(\lambda^4),$$

$$\Omega_3^{\mu\nu\sigma} = \lambda^4 \epsilon_{\mu\nu\sigma} R_{\rho}{}^{\mu\nu} H^{\alpha\beta\gamma\rho} + \dots, \quad \Omega_4^{\mu\nu\rho\sigma} = \lambda^5 \epsilon_{\mu\nu\rho\sigma} R_{\alpha\beta}{}^{\mu\nu} R^{\rho\alpha\beta\gamma} + \dots$$

The 2-loop coefficient c_1 vanishes in the minimal subtraction scheme ^{3/}. $c_1 = \rho(\gamma_0 - \ln 4\pi)$ (ρ is a rational number and γ_0

is Euler's constant) is determined by the 3-loop diagram with two $R_{\dots} \{ \} \{ \}$ and one $\partial \partial \Phi \{ \}$ vertices. It is clear that all c_n -coefficients are renormalisation scheme dependent since they change under a redefinition of $G_{\mu\nu}$. In particular,

^{3/} For a related observation see [14]. Note that there are $\frac{1}{\epsilon^2}$ 2-loop infinities ($\sim R^{\mu\nu} \partial_\mu \partial_\nu \Phi$ and $\partial^\mu \Phi$) connected by RG with the 1-loop renormalisations of G (7) and Φ (8).

we may redefine $G_{\mu\nu}$ so that $\Omega_2^{\mu\nu} = \lambda G^{\mu\nu}$. In addition to the renormalization scheme ambiguity there is an ambiguity in β_G and β_Φ implied by a possibility to redefine χ^μ [15]:
 $\beta_G \rightarrow \beta_G + 2 \partial_\mu b_\mu$, $\beta_\Phi \rightarrow \beta_\Phi + b^\mu \partial_\mu \Phi$, i.e. $\chi^\mu \rightarrow \chi^\mu - b^\mu$
 (see (9))^{4/}.

To determine renormalization of the composite operators of canonical dimension 0 it is sufficient to consider only the basic operator $\delta = \delta^3(y - x(z))$ and to use the "renormalized action principle" stating that derivative of a finite quantity with respect to a finite parameter is finite (see e.g. [21]).

Let

$$e^{-W[G, \Phi, \lambda]} = \int d\chi e^{-I_0[G, \Phi, \lambda]} \quad (10)$$

Then (we shall sometimes use $\partial/\partial \dots$ to denote functional derivatives)

$$\frac{\partial W}{\partial \Phi(y)} = \langle \int d^4x J(x) [\delta] \rangle = \langle \int d^4x J(x) \frac{\partial \Phi_0}{\partial \Phi} \cdot \delta_0 \rangle, \quad (11)$$

where $\langle i \rangle = 1$ and $[\delta]$ is the renormalized operator (we used that $\partial G_0 / \partial \Phi = 0$). Since the dimension 0 operators cannot mix with a total derivative operator we find using (3)

$$[\delta] = \frac{\partial \Phi_0}{\partial \Phi} \cdot \delta_0 = Z_1^T \delta_0, \quad \Phi \cdot [\delta] = \Phi_0 \cdot \delta_0, \quad (12)$$

where Z_1^T is adjoint to the operator Z_1 with respect to the scalar product $f \cdot g$ in (2). Renormalization of δ is thus analogous to renormalization of φ^2 in a φ^4 -theory (Φ plays the role of a mass). The corresponding anomalous dimension is adjoint

^{4/} Computing the path integral by expanding near a classical solution $\bar{\chi}$ of $\partial_\mu \partial_\mu \bar{\chi}^\mu = 2\pi\lambda \partial^\mu \Phi_0 J$ we find that simultaneous shifting of the counterterms $T_{1\mu\nu}$ by $2 \partial_\mu a_\nu$ and $\mathcal{L}_1$ by $a^\mu \partial_\mu$ leaves the counter-action unchanged. Note that χ^μ being a differential operator cannot be completely canceled out by the function b^μ .

to $\tilde{g}$ in (5).

Dimension 2 operators form two independent sets: $\{x^\mu \partial x^\mu \partial x^\mu\}$ and $\{x^\mu \square x^\mu\}$ which can be represented by $\mathcal{O}_1^{\mu\nu} = \delta^{\mu\nu}(y-x) \partial_\mu x^\mu \partial_\nu x^\mu$ and $\mathcal{O}_2^{\mu\nu} = \delta^{\mu\nu}(y-x) \square x^\mu$. In general, $\mathcal{O}_1$ and $\mathcal{O}_2$ mix under renormalization. To study this mixing it is convenient to replace $\mathcal{O}_1$ by $\mathcal{O}_3 = \partial_\mu (\delta^{\mu\nu}(y-x) \partial_\nu x^\mu)$ (which is a linear combination of $\mathcal{O}_1$ and $\mathcal{O}_2$), to consider scalar products $H_{\mu\nu} \cdot \mathcal{O}_1^{\mu\nu} = H_{\mu\nu}(x) \partial_\mu x^\mu \partial_\nu x^\mu$ and $N_\mu \cdot \mathcal{O}_2^{\mu\nu} = \partial_\mu (N_\mu(x) \partial_\nu x^\mu) = \partial_\mu N_\mu \partial_\nu x^\mu + N_\mu \square x^\mu$ and to compute the "adjoint" renormalization of $H_{\mu\nu}$ and N_μ

(cf. (3), (12))

$$(H_{\mu\nu}(x) \partial_\mu x^\mu \partial_\nu x^\mu)_0 = [\tilde{H}_{\mu\nu}(x) \partial_\mu x^\mu \partial_\nu x^\mu] + [\partial_\mu (\tilde{N}_\mu(x) \partial_\nu x^\mu)], \quad (13)$$

where $\tilde{H}_{\mu\nu} = \Lambda^{\alpha\beta}_{\mu\nu} H_{\alpha\beta}$, $\tilde{N}_\mu = \Lambda^{\alpha\beta}_{\mu} H_{\alpha\beta}$ and Λ 's depend on G and ϵ . Observing that $\frac{\partial W}{\partial G_{\mu\nu}} = \langle \int \frac{d^d x}{4\pi\lambda} [\mathcal{O}_1^{\mu\nu}] \rangle = \frac{\partial G_{\alpha\beta}}{\partial G_{\mu\nu}} \cdot \langle \int \frac{d^d x}{4\pi\lambda} (\mathcal{O}_1^{\alpha\beta})_0 \rangle$ we conclude that $(\Lambda^{-1})^{\mu\nu}_{\alpha\beta} = \frac{\partial G_{\alpha\beta}}{\partial G_{\mu\nu}}$. To find the total derivative term in (13) we shall expand the bare action and $(H_{\mu\nu} \partial x^\mu \partial x^\nu)_0$ in normal coordinates near a classical solution $\bar{x}$ ^{5/} and compute

$$\begin{aligned} \langle (H_{\mu\nu}(x) \partial_\mu x^\mu \partial_\nu x^\mu)_0 \rangle &= \tilde{H}_{\mu\nu}(\bar{x}) \partial_\mu \bar{x}^\mu \partial_\nu \bar{x}^\mu + \partial_\mu (\tilde{N}_\mu(\bar{x}) \partial_\nu \bar{x}^\mu) + \\ &+ \dots = \tilde{H}_{\mu\nu}(\bar{x}) \partial_\mu \bar{x}^\mu \partial_\nu \bar{x}^\mu + \dots \end{aligned} \quad (14)$$

where dots stand for non-local terms and $\tilde{H}_{\mu\nu} = \tilde{H}_{\mu\nu} + \partial_\mu \tilde{N}_\nu$ (we used that $\partial_\mu \partial_\mu \bar{x} = 0$). Direct computation gives us $\tilde{H}_{\mu\nu}$; independently we know $\partial G_0 / \partial G$ and hence $\tilde{H} = H - \frac{1}{\epsilon} \frac{\partial T_2}{\partial G} \cdot H + O(1/\epsilon^2)$. The difference $\tilde{H} - \tilde{H}$ must be representable as $\partial_\mu \tilde{N}_\nu$. At the one-loop order we find

^{5/} The necessary formulas are given in [22], e.g.

$$\begin{aligned} H_{\mu\nu} &= H_{\mu\nu}(\bar{x}) + \partial_\lambda H_{\mu\nu} \bar{x}^\lambda + \frac{1}{2} (\partial_\lambda \partial_\rho H_{\mu\nu} + \frac{2}{3} R_{\lambda\rho\mu\nu} H_{\lambda\lambda}) \bar{x}^\lambda \bar{x}^\rho + \dots \\ \partial_\mu x^\mu &= \partial_\mu \bar{x}^\mu + \partial_\mu \bar{x}^\mu + \frac{1}{3} R_{\alpha\beta\gamma} \bar{x}^\alpha \bar{x}^\beta + \dots \end{aligned}$$

$$\bar{H}_{\mu\nu} = H_{\mu\nu} + \frac{\lambda}{2\epsilon} \left(-\partial^2 H_{\mu\nu} + 2 R_{\sigma\mu} H_{\nu\sigma} + 2 R_{\sigma\nu} H_{\mu\sigma} \right) + O(\lambda^2) \quad (15)$$

$$\tilde{H}_{\mu\nu} = H_{\mu\nu} + \frac{\lambda}{2\epsilon} \left(-\partial^2 H_{\mu\nu} + 2 \partial_\mu \partial_\nu H - \partial_\mu \partial_\nu H \right) + O(\lambda^2)$$

and hence $(H = H_{\mu}^{\mu})$

$$\tilde{N}_{\mu} = -\frac{\lambda}{\epsilon} \partial_\mu (H_{\rho\rho} - \frac{1}{2} G_{\rho\rho} H) + O(\lambda^2) \quad (16)$$

The two-loop computation gives the following $\frac{\lambda}{\epsilon}$ -contribution 6/

$$\begin{aligned} \bar{H}_{\mu\nu}^{(2)} &= \tilde{H}_{\mu\nu}^{(2)} + \partial_\mu \tilde{N}_\nu^{(2)} \\ \tilde{H}_{\mu\nu}^{(2)} &= \frac{\lambda^2}{4\epsilon} \left(R_{\nu\alpha\beta\gamma} \partial_\alpha \partial_\beta H_{\gamma\mu} - R_{\nu\alpha\beta\gamma} \partial_\mu \partial_\beta H_{\gamma\alpha} - \right. \\ &\quad \left. - \frac{1}{2} R_{\alpha\beta\gamma\mu} R_{\beta\gamma\delta\nu} H_{\alpha\delta} \right) + (\mu \leftrightarrow \nu) \quad (17) \\ \tilde{N}_\mu^{(2)} &= -\frac{\lambda^2}{2\epsilon} R_{\mu\alpha\beta\gamma} \partial_\gamma H_{\alpha\beta} \end{aligned}$$

We conclude that there is a non-trivial mixing of the $\delta \partial x \partial x$ and $\partial(\delta x)$ -operators.

Let us now specialize to the case of the operator $-\epsilon G_{\mu\nu} \cdot (\delta x^\mu \partial x^\nu)$, i.e. $H_{\mu\nu} = -\epsilon G_{\mu\nu}$. Since $\partial_\mu G_{\mu\rho} = 0$ we find that the finite part of $\tilde{N}_\mu$ then vanishes at one and two loops and that in 2-loop approximation: $-\epsilon G_{\mu\nu} \cdot (\delta \partial x \partial x)_0 = \hat{\beta}_{\epsilon\mu\nu} \cdot [\delta \partial x^\mu \partial x^\nu]$, where $\hat{\beta}_\epsilon$ is given in (4), (7). In general, $\tilde{H}_{\mu\nu} = \hat{\beta}_{\epsilon\mu\nu}$ because (as follows from $\frac{d}{dt} G_0 = 0$) $-\epsilon G_{\mu\nu} = \hat{\beta}_{\epsilon\alpha\beta} \cdot \frac{\partial G_{0\mu\nu}}{\partial G_{\alpha\beta}}$ and hence

6/ The basic 2-loop diagrams correspond to the correlators $\langle (R \{ \partial/\partial \bar{x} \}) (H R \{ \partial/\partial \bar{x} \}) \rangle$, $\langle (R \{ \partial/\partial \bar{x} \}) (\partial^2 H + \frac{1}{2} R H) \{ \partial/\partial \bar{x} \} \rangle$ and $\langle (R \{ \partial/\partial \bar{x} \}) (\partial H \{ \partial/\partial \bar{x} \}) \rangle$. The only complication (as compared to the 2-loop renormalization of G) is to correctly account for the contribution of the third diagram (which naively is linearly divergent). The basic integral is $\int d^d z (D(z, z'))^2 \partial_n \partial_m D(z, z') = -\frac{\delta_{nm}}{d} I^2$, $I = D(z, z)$, $D(z, z')$ is the free propagator.

$$- \epsilon G_{\mu\nu} \cdot (\delta \partial_\mu x^\mu \partial_\nu x^\nu)_0 = \hat{\beta}_{\mu\nu} \cdot [\delta \partial_\mu x^\mu \partial_\nu x^\nu] + \hat{N}_\mu \cdot [\partial_\mu (\delta \partial_\mu x^\mu)] \quad (18)$$

where $\hat{N}_\mu = \tilde{N}_\mu (H \rightarrow -\epsilon G_0)$ may be non-vanishing at three and higher loops. (The integrated form of (18) easily follows from $0 = \frac{dW}{dt} = \frac{\partial W}{\partial t} + \hat{\beta}_G \cdot \frac{\partial W}{\partial G}$, $\frac{\partial W}{\partial G_{\mu\nu}} = \langle \int \frac{d^d x}{4\pi\lambda} [\delta \partial_\mu x^\mu \partial_\nu x^\nu] \rangle$, $\frac{\partial W}{\partial t} = \langle \int \frac{d^d x}{4\pi\lambda} \epsilon G_0 \cdot (\delta \partial_\mu x^\mu \partial_\nu x^\nu)_0 \rangle$). In general, $\hat{N}_\mu = \text{finite} + \text{poles}$. In the above operator relations we ignored contribution of the operator $E = [\phi'(x) \frac{\delta I_0}{\delta x^\mu}]$ which has zero expectation value and is not renormalized (the rhs. of (18) contains E with $\phi' = \gamma'$, where γ' is defined in (9)).

3. WEYL ANOMALY IN CURVED SPACE To preserve renormalizability of the modelⁱⁿ curved 2-space the bare action must contain additional dimension 2 non-total derivative even parity operator

$$R^{(d)} \phi(x) \quad [2]^{7/}$$

$$I_0 = \frac{1}{4\pi\lambda} \int d^d x \sqrt{g} G_{\mu\nu} \partial_\mu x^\mu \partial_\nu x^\nu + \frac{1}{4\pi} \int d^d x \sqrt{g} R^{(d)} \phi_0(x). \quad (19)$$

ϕ can be interpreted as a condensate of the dilaton field [4,5 2,9,23]. Renormalization of G is the same as in flat space while

$$\phi_0 = Z_1 \phi + Z_2, \quad Z_1 = \mu^\epsilon \left(1 + \sum_{n=1}^{\infty} \frac{\varphi_n}{\epsilon^n}\right), \quad Z_2 = \mu^\epsilon \sum_{n=1}^{\infty} \frac{\alpha_n}{\epsilon^n}, \quad (20)$$

where Z_2 is the universal renormalization operator for a scalar coupling (3) and Z_1 is a function depending on G . Thus we have (cf. (5), (6))

$$\hat{\beta}_\phi \equiv \frac{d\phi}{dt} = -\epsilon \phi + \beta_\phi = -\gamma \phi + \omega, \quad \beta_\phi = -\gamma \phi + \omega, \quad (21)$$

$$\gamma = \lambda \frac{\partial}{\partial \lambda} \varphi_1, \quad \omega = -\alpha_1 + G \cdot \frac{\partial}{\partial G} \alpha_1 = -\frac{\partial}{\partial \lambda} (\lambda \alpha_1 (\lambda^{-1} G)) \quad (22)$$

Here γ and ω depend only on the renormalized metric G (the

^{7/} Dimensional regularization breaks the superficial Weyl invariance of the d=2 theory so non-Weyl invariant counterterms are to be expected. This counter-term is the analogue of the $\eta R \varphi^2$ term in the φ^4 -theory (ϕ is analogue of η -coupling) [17-20].

dependence of β_ϕ on ϕ is shown explicitly). The perturbative structure of χ was already discussed above (see (8), (9)). To compute ω one may use the normal coordinate expansion of (19) near a non-constant classical solution $\bar{\chi}$ of $\partial_m \partial^m \bar{\chi} = \frac{1}{2} R^{(d)} \partial^\mu \phi_0$. Then the infinities $\sim \sqrt{g} R^{(d)}$ can be unambiguously identified ($Z_2(\bar{\chi}) \neq \text{const}$) and we find (expanding near a flat space, $R^{(d)} = \partial_m \partial_n h_{mn} - \square h$, and computing the two-point correlators with one h_{mn} - and one $Z_2(\bar{\chi})$ - leg) [4-5, 2]

$$\omega = a_0 + a_1 \lambda R + a_2 \lambda^2 R^2_{\alpha\beta\delta} + a_3 \lambda^2 R^2_{\alpha\beta} + O(\lambda^3), \quad (23)$$

$$a_0 = \frac{1}{6} D, \quad a_1 = 0, \quad a_2 = \frac{1}{16}, \quad a_3 = 0$$

(it is easy to see from the structure of the normal coordinate expansion that (23) cannot contain other terms depending on the scalar R except the second one).

The λ -independence of $W = -\ln \int dx e^{-I_0}$ implies $\frac{\partial W}{\partial \lambda} + \beta_0 \cdot \frac{\partial W}{\partial G} + \beta_\phi \cdot \frac{\partial W}{\partial \phi} = 0$ and hence the scale anomaly relation

$$\int d^d x \sqrt{g} \langle T^m_m \rangle = \int \frac{d^d x}{4\pi\lambda} \left\{ \beta_{0,\mu} \cdot \langle [\delta \partial_\mu x^\mu x^\mu] \rangle + \lambda \beta_\phi \cdot \langle [\delta] R^{(d)} \rangle \right\} \quad (24)$$

$$\text{where } \langle T^m_m \rangle = \frac{2}{\lambda} g^{mn} \frac{\delta W}{\delta g^{mn}}$$

and we used that

$$\frac{\partial W}{\partial G} = \int \frac{d^d x}{4\pi\lambda} \sqrt{g} \langle [\delta \partial x \partial x] \rangle, \quad \frac{\partial W}{\partial \phi} = \int \frac{d^d x}{4\pi} \sqrt{g} R^{(d)} \langle [\delta] \rangle$$

(cf. (11)). The corresponding operator anomaly relation is

$$\int d^d x \sqrt{g} T^m_m = \int \frac{d^d x}{4\pi\lambda} \sqrt{g} \left\{ \beta_0 \cdot [\delta \partial x \partial x] + \lambda \beta_\phi \cdot [\delta] R^{(d)} + E \right\} \quad (25)$$

where $E = [\chi^\mu \frac{\delta I_0}{\delta \chi^\mu}]$, $\langle E \rangle = 0$ ^{8/}. Note that (25) remains

^{8/} To demonstrate the presence of E in (25) let us temporarily include the tachyon term $\frac{1}{2} \phi^2$ in the bare action (19). Then the integrand in the r.h.s. of (24) receives additional contribution: $\beta_\phi \cdot \langle [\delta] \rangle$. Since $\chi = \chi^\mu \partial_\mu$ and the new classical equations are $\delta I_0 / \delta \chi^\mu = -\partial_\mu \phi$ we find that $\beta_\phi \cdot \langle [\delta] \rangle$ can be replaced by $\langle E \rangle$.

invariant under $\beta_{G\mu\nu} \rightarrow \beta_{G\mu\nu} + 2\partial_\mu \phi_\nu$, $\beta_\phi \rightarrow \beta_\phi + \phi^\mu \partial_\mu \phi$, $\gamma^\mu \rightarrow \gamma^\mu - \phi^\mu$. We may freely omit E being interested only in "on-shell" relations.

Eq.(25) gives the expression for the Weyl anomaly up to an important total derivative term. To derive a precise expression for T_m^m we shall follow the standard procedure [17-20], expressing the bare operators in

$$T_m^m = \frac{2}{\sqrt{g}} g^{mn} \frac{\delta I_0}{\delta g^{mn}} = -\frac{1}{4\pi\lambda} \varepsilon G_{\mu\nu} (\delta \partial_\mu x^\mu \partial_\nu x^\nu)_0 - \frac{1}{4\pi} \varepsilon \phi_0(x) R^{(d)} + \frac{1}{2H} (\varepsilon+1) \nabla^2 \phi_0(x), \quad \phi_0(x) = \phi_0 \cdot \delta_0, \quad \nabla^2 = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu)$$

in terms of the renormalized ones ($\langle T_m^m \rangle = \frac{2}{\sqrt{g}} g^{mn} \frac{\delta W}{\delta g^{mn}}$ so that T_m^m is a finite operator). In (26) we used that under $\delta g_{mn} = 2\delta g_{mn}$: $\delta(\sqrt{g} R^{(d)}) = \{-2(d-1)\nabla^2 \phi + (d-2)R^{(d)}\phi\}\sqrt{g}$. The easily derivable (from $\frac{dW}{dt} = 0$) generalization of (18) accounts

$$-\varepsilon G_{\mu\nu} (\delta \partial_\mu x^\mu \partial_\nu x^\nu)_0 = \lambda \varepsilon \phi_0 \cdot \delta_0 R^{(d)} = \hat{\beta}_G \cdot [\delta \partial_\mu x^\mu \partial_\nu x^\nu] + \lambda \hat{\beta}_\phi \cdot [\delta] R^{(d)} + \hat{N}_\mu \cdot [\nabla^\mu (\delta \partial_\mu x^\mu)]$$

Using (27), (20), i.e. $\phi_0 = Z_1 (\phi + P)$, $P = Z_1^{-1} Z_2 = \frac{\mathcal{E}_1}{\varepsilon} + \bar{P}$ and $[\delta] = Z_1^{-1} \delta_0$ we find for (26)

$$T_m^m = \frac{1}{4\pi\lambda} \hat{\beta}_G \cdot [\delta \partial_\mu x^\mu \partial_\nu x^\nu] + \frac{1}{4\pi} \hat{\beta}_\phi \cdot [\delta] R^{(d)} + \frac{1}{4\pi\lambda} \hat{N}_\mu \cdot [\nabla^\mu (\delta \partial_\mu x^\mu)] + \frac{1}{2\pi} (\varepsilon+1) \nabla^2 [\delta] \cdot (\phi + P) + E = \frac{1}{4\pi\lambda} [\beta_{G\mu\nu}(x) \partial_\mu x^\mu \partial_\nu x^\nu] + \frac{1}{4\pi} [\hat{\beta}_\phi(x)] R^{(d)} + \frac{1}{2\pi\lambda} [\nabla^\mu (M_\mu \partial_\mu x^\mu)] + E,$$

where $M_\mu = \frac{1}{2} \hat{N}_\mu + \lambda(\varepsilon+1) \partial_\mu (\phi + P)$ should be a finite function in view of the finiteness of T_m^m . Thus if $\hat{N}_\mu = N_\mu + P_\mu$,

$$P_\mu = \sum_{n=1}^{\infty} \frac{1}{\varepsilon^n} P_\mu^{(n)} \quad \text{then} \quad P_\mu = -\frac{(\varepsilon+1)}{2\pi} \partial_\mu \bar{P} - \frac{1}{2\pi\varepsilon} \partial_\mu \mathcal{E}_1 \quad \text{and so} \quad M_\mu = -\frac{1}{2} N_\mu + \lambda \partial_\mu \mathcal{E}_1 + \lambda(\varepsilon+1) \partial_\mu \phi. \quad \text{Observing that} \quad [\nabla^\mu (M_\mu \partial_\mu x^\mu)] = \partial_\mu M_\mu \cdot [\delta \partial_\mu x^\mu \partial_\nu x^\nu] + \frac{1}{2} M_\mu \partial_\mu \phi \cdot [\delta] R^{(d)} + M_\mu \cdot [\delta] (\partial^\mu \partial_\mu x^\mu - \frac{1}{2} R^{(d)} \partial^\mu \phi),$$

we conclude (taking $d=2$)

$$T_m^m = \frac{1}{4\pi\lambda} [\bar{\beta}_{G\mu\nu}(x) \partial_\mu x^\mu \partial_\nu x^\nu] + \frac{1}{4\pi} [\bar{\beta}_\phi(x)] R^{(2)} + \bar{E}, \quad (29)$$

$$[A(x) \partial x \partial x] = A \cdot [\delta \partial x \partial x], \quad [A(x)] = A[\delta],$$

$$\bar{\beta}_{G\mu\nu} = \beta_{G\mu\nu} + 2 \partial_{(\mu} M_{\nu)} = \beta_{G\mu\nu} + 2 \partial_{(\mu} W_{\nu)} + 2\lambda \partial_\mu \partial_\nu \phi, \quad (30)$$

$$\bar{\beta}_\phi = \beta_\phi + M^\mu \partial_\mu \phi = -\gamma \phi + \omega + W^\mu \partial_\mu \phi + \lambda (\partial_\mu \phi)^2, \quad (31)$$

where $W_\mu = \frac{1}{2} N_\mu + \lambda \partial_\mu \phi$ and $\bar{E} = [(\gamma' - W') \frac{\delta I_0}{\delta x^\mu}]$.

This is the final form of the operator expression for the Weyl anomaly. The dependence on the dilaton field is shown explicitly.

$\beta_{G\mu\nu}$, γ , ω and W_μ are finite computable functions of G , its curvature and covariant derivatives. Eqs. (28) and (29) are obviously consistent with (25). Note that (29) is invariant under $\beta_{G\mu\nu} \rightarrow \beta_{G\mu\nu} + 2 \partial_{(\mu} \epsilon_{\nu)}$, $\beta_\phi \rightarrow \beta_\phi + \epsilon^\mu \partial_\mu \phi$, $\gamma' \rightarrow \gamma' - \epsilon'$, $W' \rightarrow W' - \epsilon'$.

If W_μ were absent in (30), (31) (as it is sometimes thought) $\langle T_m^m \rangle$ were not invariant under reparametrizations of x^μ . Though there exists an abstract parameterization in which $W_\mu = 0$, in a standard parametrization (the one in which (7), (8), (23) are true) W_μ is a definite quantity which is non-zero at three and higher loops.

4. DISCUSSION. Let us first compare the (on-shell) conditions of scale invariance (SI) and Weyl invariance (WI). According to (25) the condition of SI is $\beta_{G\mu\nu} = -2 \partial_{(\mu} \gamma_{\nu)}$, $\beta_\phi = -\gamma' \partial_\mu \phi$, where γ^μ is an arbitrary vector. As follows from (29), the condition of WI is stronger: $\beta_{G\mu\nu} = -2 \partial_{(\mu} M_{\nu)}$, $\beta_\phi = -M^\mu \partial_\mu \phi$, where M^μ is a definite function of G and ϕ . Thus WI-theories form a subclass of SI-theories. Let us call a condition "perturbative" if it is true at each order of loop (λ) expansion. A theory is finite if all counterterms vanish on-shell. It is clear that a perturbative finiteness condition is equivalent to

the perturbative SI-condition ($\beta_G = - \sum_{l=1}^{\infty} l \lambda^l \bar{T}_l^{(l)} =$
 $= \sum_{l=1}^{\infty} \beta_G^{(l)}$ so that $T_{\mu\nu}^{(l)} = \partial_{\mu} \bar{T}_{\nu} \leftrightarrow \beta_{G\mu\nu}^{(l)} = \partial_{\mu} \bar{T}_{\nu}$). In particular, perturbative WI implies perturbative finiteness. A non-perturbatively finite theory (no such theories are known at present) in general has non-trivial β -functions and hence is not SI even at the finiteness point (see also [13]). Also, non-perturbatively SI-theory need not be finite. Let us emphasize that the condition that a G -model is to satisfy in order to describe a consistent string vacuum is WI and not finiteness (2-d UV infinities do not produce any difficulty in a string theory since string-theory objects like effective action or effective equations are in any case expressed in terms of renormalized 2-d couplings [2]).

Now let us compute an expectation value of T_m^m (29) expanding near a classical solution $\bar{x}$. We get

$$\langle T_m^m \rangle = \frac{1}{4\pi\lambda} \tilde{\beta}_{G\mu\nu}(\bar{x}) \partial_{\mu} \bar{x}^{\mu} \partial_{\nu} \bar{x}^{\nu} + \frac{1}{4\pi} \tilde{\beta}_{\phi}(\bar{x}) R^{(2)} + (\text{non-local terms}) \quad (32)$$

The coefficients $\tilde{\beta}_G$ and $\tilde{\beta}_{\phi}$ are the " β -functions" that appeared in [9,13]. In general [2]

$$\tilde{\beta}_G = \bar{\beta}_G + \dots, \quad \tilde{\beta}_{\phi} = \bar{\beta}_{\phi} - \frac{1}{4} \bar{\beta}_{G\mu\nu} \Sigma^{\mu\nu} + \dots \quad (33)$$

where dots stand for terms involving derivatives of $\bar{\beta}_G$ and (or)

$\bar{\beta}_{\phi}$ (these terms are probably scheme-dependent).

$$\Sigma^{\mu\nu} = G^{\mu\nu} + d_1 \lambda R^{\mu\nu} + d_2 \lambda^2 R^{\mu}_{\alpha\beta\gamma} R^{\nu\alpha\beta\gamma} + \dots \quad (\text{compare with } \Omega_2^{\mu\nu}$$

in (9)). The coefficients d_n are scheme-dependent; redefining

$G^{\mu\nu}$ we can always make $\Sigma^{\mu\nu} = G^{\mu\nu}$. It is easy to see that $\tilde{\beta}_G$

depends on ϕ only through $\bar{\beta}_G$ (30) and that $\tilde{\beta}_{\phi}$ is at most

quadratic in $\partial_{\mu} \phi$ (the only quadratic term is $(\partial_{\mu} \phi)^2$ in $\bar{\beta}_{\phi}$ (31)).

In the 3-loop approximation we find using (9), (23), (31), (33)

[4,9, 2]

$$\tilde{\beta}_\phi = \frac{D-26}{6} - \frac{1}{4} \left\{ R + 4 \partial^2 \phi - 4 (\partial_\mu \phi)^2 + \frac{1}{4} \lambda R_{\mu\nu}^2 + d_1' \lambda R_{\mu\nu}^2 \right\} + O(\lambda^3) \quad (34)$$

where $d_1' = 2 - \frac{3}{2} (\chi - \ln 4\pi)$ is related to d_1 (we included the contribution of ghosts). It is likely that the conditions [9] $\tilde{\beta}_G = 0, \tilde{\beta}_\phi = 0$ are equivalent to the WI conditions $\bar{\beta}_G = 0, \bar{\beta}_\phi = 0$. A conjecture (which is true in the 2-loop approximation [9,13]) is that in some particular scheme $\tilde{\beta}_G = 0, \tilde{\beta}_\phi = 0$ follow from the effective action (see also [2]) $\Gamma = \int d^D x \sqrt{G} e^{-2\phi} \tilde{\beta}_\phi$. To prove it one is to show that the variation of $\tilde{\beta}_\phi$ under variations of $G_{\mu\nu}$ and ϕ has the form: $\delta \tilde{\beta}_\phi = -\frac{1}{4} \tilde{\beta}_\phi \delta G^{\mu\nu} + (\partial_\mu - 2 \partial_\mu \phi) B^\mu(\delta G, \delta \phi)$. Eqs. (30), (31), (33) should provide a basis for such a proof. Since $\tilde{\beta}_G = 0, \tilde{\beta}_\phi = 0$ imply $\bar{\beta}_G = 0, \bar{\beta}_\phi = 0$ we may consider $\bar{\beta}_G = 0, \bar{\beta}_\phi = 0$ as an alternative form of the effective equations,

$$\beta_{G\mu\nu} + 2 \partial_\mu W_\nu + 2 \lambda \partial_\mu \partial_\nu \phi = 0 \quad , \quad (34)$$

$$-(\chi'' - W'' - \lambda \partial^2 \phi) \partial_\mu \phi + \omega = 0 \quad . \quad (35)$$

The important advantage of (34), (35) is that they simply depend on the dilaton ϕ and are directly expressed in terms the RG functions. Further analysis of (34), (35) may be simplified by choosing a scheme in which $\chi'' = \frac{1}{2} \partial^2 +$ (terms with higher covariant derivatives) (see (9)) and a parameterisation in which $W'' = 0$. Let us stress once more that without the W'' -dependent terms eqs.(34), (35) could hardly be considered as effective equations for the metric and the dilaton. The point is that the β -functions (β_G and $\beta_\phi = -\delta\phi/\omega$) depend on a parametrization chosen while effective equations should certainly be independent of a parametrization of a G^2 -model field space (though their form may of course depend on a parametrization in the coupling constant space (G, ϕ) , i.e. on a 2-d renormalization scheme).

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