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REMARKS ON FERMION PROPAGATION IN RANDOM LATTICES

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ABSTRACT

We consider free naive Dirac fermions on a hypercubic lattice with random hopping parameters and show, by analytic mean-field methods, that the replica fermions required by the Nielsen-Ninomiya theorem do not contribute to the large-distance behaviour of the fermion propagator.

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Following an early proposal by Jacobs¹⁾, Weingarten and Velikson have made in a recent paper²⁾ an interesting suggestion concerning the well-known problem of fermion species multiplication in lattice gauge theories. Carrying out a computer simulation in two dimensions, they show that - for a particular random lattice to be described below - replica fermions implied by the Nielsen-Ninomiya theorem³⁾ become localized states and do not contribute to the fermion propagator at large distances [for related work on this subject, see Refs. 4) and 5)]. Given the importance of the problem, it is desirable to obtain further theoretical understanding of this phenomenon. We shall present in this paper an analytic approach to this problem, based on mean-field methods for fermions. Our solution becomes exact in the (formal) limit in which the dimensionality d of the Euclidean lattice tends to infinity, while the dimensionality $\mathbb{C}$ of spinor space is kept fixed.

Following Refs. 1) and 2), we consider free naive fermions interacting with a random field $z_\mu(x)$ with Gaussian distribution. Thus the action S is

$$\bar{a}^d S = m \sum_x \bar{\Psi}(x) \Psi(x) + \sum_{x,y} \bar{\Psi}(x) Q(x,y) \Psi(y) - \frac{1}{2\Delta^2} \sum_{x,\mu} [z_\mu(x)]^2 \quad (1)$$

where the matrix Q is given by

$$Q(x,y) = \sum_\mu \left\{ \left[\frac{1}{2a} + z_\mu(x) \right] \gamma^\mu (\delta_{x+\hat{\mu}a,y} - \delta_{x,y}) + \left[\frac{1}{2a} - z_\mu(x) \right] \gamma^\mu (\delta_{x,y} - \delta_{x-\hat{\mu}a,y}) \right\} \quad (2)$$

which, for a given $z_\mu(x)$, represents a finite-difference approximation to first derivatives accurate to order a . Therefore, the interaction terms induced by the random field $z_\mu(x)$ will be irrelevant operators. The fermionic quadratic form (2) lacks reflection positivity⁶⁾ for a given z_μ configuration, and this is why the Nielsen-Ninomiya theorem can be avoided²⁾.

Integrating the field $z_\mu(x)$, one obtains an effective action

$$a^d S_{eff} = m \sum_x \bar{\Psi}(x) \Psi(x) + \frac{1}{2a} \sum_{x,\mu} [\bar{\Psi}(x) \gamma^\mu \Psi(x+\hat{\mu}a) - \bar{\Psi}(x) \gamma^\mu \Psi(x-\hat{\mu}a)] \\ + \frac{\Delta^2}{2} \sum_{x,\mu} [\bar{\Psi}(x) \gamma^\mu \Psi(x+\hat{\mu}a) + \bar{\Psi}(x) \gamma^\mu \Psi(x-\hat{\mu}a) - 2 \bar{\Psi}(x) \gamma^\mu \Psi(x)]^2 \quad (3)$$

where the irrelevant nature of the interaction is obvious from the fact that Δ^2 has dimensions of $(\text{length})^{d-2}$. Notice that the interaction vanishes like a^4 for long-wavelength configurations, while it takes its maximum value for staggered configurations of wavelength $2a$, where replica fermions sit.

In order to study the interacting fermion theory (3), we shall apply the mean-field techniques developed in Refs. 7) and 8). These techniques rely on a specific bosonization procedure⁷⁾ which allows saddle-point methods to be applied. In the effective action (3), each fermion bilinear is replaced by a bosonic field as follows (α, β are Dirac indices):

$$\Psi_\alpha(x) \bar{\Psi}_\beta(y) \rightarrow \sigma_{\alpha\beta}(x, y) \quad (4)$$

and the Grassmannian integration in the original partition function is turned into an integration over the matrix elements of σ and its conjugate matrix field λ :

$$Z = \int d\sigma d\lambda \exp V \quad (5)$$

where

$$V = \text{Tr} [i \lambda \sigma + \ln \lambda] + S_{eff}(\sigma) \quad (6)$$

Matrix multiplication in space-time and spinor space is understood. The trace acts on space-time and Dirac indices. Notice that the integration over the matrix elements of σ which do not explicitly occur in $S_{eff}(\sigma)$ - or in the integrand of the path integral when one computes vacuum expectation values - can be explicitly done and sets the corresponding matrix elements of λ equal to zero.

The saddle-point equations are:

$$i \lambda_{\alpha\beta}(x,y) + \frac{\delta S_{\text{eff}}}{\delta \sigma_{\alpha\beta}(x,y)} = 0 \quad (7a)$$

$$i \sigma_{\alpha\beta}(x,y) + (\lambda^{-1})_{\alpha\beta}(x,y) = 0 \quad (7b)$$

The first equation determines the matrix λ in terms of σ , while the second equation, after matrix inversion, yields self-consistent equations for σ . From the form of $S_{\text{eff}}[\sigma]$, it is clear that in Eqs. (7) y is restricted to $y = x$ or $y = x \pm \hat{\mu}a$, all other matrix elements of λ being zero. When one computes the fermion propagator, it is easy to see that it is given by $\sigma_{\alpha\beta}(x,y)$ for arbitrary (x,y) as determined from Eq. (7b), with the matrix λ given by its saddle-point value.

When the saddle-point equations are explicitly written, it turns out that the only relevant variational parameter that occurs in this problem is

$$A^\mu(x) = t_\nu \left\{ \tau^\mu [\sigma(x+\hat{\mu}a, x) + \sigma(x-\hat{\mu}a, x) - 2\sigma(x, x)] \right\} \quad (8)$$

and that there is no way in which chiral symmetry can be spontaneously broken. The order parameter (8) is of vector nature, but it vanishes like a^2 in the continuum limit. We consider only saddle points that are translational invariant, of the form $A^\mu(x) = \eta^\mu A$ with $\eta^\mu = \pm 1$. This set of two-dimensional saddle points preserve the discrete rotational symmetry as well as the reflection symmetry of the lattice provided they are degenerate, as will be the case. Going to momentum space, we find after some algebra

$$-i\lambda(p) = mI + i \sum_{\mu} \tau^\mu a_\mu \quad (9)$$

where

$$a_\mu = \frac{1}{a} \sin p_\mu a + i \eta_\mu \cdot 2\Delta^2 A (1 - \cos p_\mu a) \quad (10)$$

Furthermore, the matrix elements of σ are given by

$$\sigma(x,y) = \int_{-\pi}^{\pi} \frac{d^d p}{(2\pi)^d} \left[mI + i \sum_{\mu} \gamma^{\mu} a_{\mu} \right]^{-1} \exp i p \cdot (x-y) \quad (11)$$

Inserting this result into Eq. (8) to compute A^{μ} , we obtain for any η^{μ} configuration a self-consistent equation for $B = 2\Delta^2 A$ (notice that B has dimension of mass):

$$B = \frac{4\Delta^2 c}{d} f(B) \quad (12)$$

$$f(B) = \int_{-\pi}^{\pi} \frac{d^d p}{(2\pi)^d} \frac{\sum_{\nu} i a_{\nu} (1 - \cos p_{\nu} a)}{m^2 + \sum_{\mu} a_{\mu}^2} \quad (13)$$

One can immediately check that $f(B)$ is an odd function of B , so that Eq. (12) always admits the trivial solution $B = 0$, in which case the matrix σ given by Eq. (11) becomes the free propagator of naive fermions. The function $f(B)$ can easily be computed for large d . We find (for $m = 0$)

$$f(B) = \frac{3B}{3B^2 - 1} \quad (14)$$

This exhibits a singularity at $B^2 = 1/3$ and a $1/B$ behaviour at large B , which can also be directly read from Eq. (13). Of course, the singularity is not present at finite d . Indeed, we have checked numerically that for values of d as small as $d = 2$ or 4 , $f(B)$ starts with a negative slope at the origin and the singularity is replaced by a sudden rise from negative to positive values near $B \sim 1/\sqrt{3}$. This shape of $f(B)$ guarantees the existence of two further solutions of the self-consistent equation (12), provided Δ^2 is above some critical value Δ_c^2 , which means that the disorder is important enough. At large d , one of these solutions is close to $B^2 = 1/3$, and the other is

$$B^2 = \frac{1}{3} + \frac{4\Delta^2 c}{d} \quad (15)$$

We have calculated the free energy per site $F(B)$ associated with the solution (15) and compared to the free energy of the trivial solution, $F(0)$. We obtain

$$F(B) = F(0) - \frac{c}{2} \left[\ln(3B^2a^2 - 1) - \frac{3B^2a^2}{3B^2a^2 - 1} \right] \quad (16)$$

for any η^μ configuration. This result shows that all saddle points are degenerate (this is a general feature independent of the $1/d$ expansion), and that for $B > 1.237a^{-1}$ the non-trivial solution gives the absolute minimum of the free energy. In the phase we are interested in, B is of the order of the cut-off mass, and from now on we can safely take $m \ll B$ and $B \approx a^{-1}$.

The existence of degenerate saddle points means that all vacuum expectation values, and in particular the fermion propagator (11), have to be averaged over the set of η^μ . Notice that even without averaging $\sigma(x,y)$ is real. However, it is only after averaging that $\sigma(x,y)$ acquires all the symmetries of the lattice. The presence of complex energy denominators before averaging is, of course, a trace of the lack of reflection positivity of the action (1). Since B is of order a^{-1} , it is quite easy to see that in the naive continuum limit $a \rightarrow 0$ the only contribution to the low-energy spectrum arises from states in which all components of momenta are near zero. Replica fermions develop infinite complex energies, and there is no species multiplication. We have verified by numerical calculation using Fast Fourier Transform methods, for $d = 2$ and 3 , that the propagator (11) (for $an = 0.05$, $aB = 2$) follows within numerical errors the behaviour of the continuum Dirac propagator between 3 and 30 lattice spacings.

Let us next take a closer look at the asymptotic behaviour of $\text{tr } \sigma$ in the x -direction, for example. Calling $x-x' = na$ and integrating the transverse momentum components, we can write

$$\text{tr } \sigma(n) = m c \int_{-\pi}^{\pi} \frac{dp}{2\pi} e^{ipna} \frac{1}{2} [g_+(p,B) + g_-(p,B)] \quad (17)$$

where $g_{\pm}(p,B)$ depends on p through $M = [m^2 + ((1/a)\sin p \pm iB(1-\cos p))^2]^{\frac{1}{2}}$, and is given by a $(d-1)$ -dimensional integral of an energy denominator D . The definition of $g_{\pm}(p,B)$ includes the average over the sign of B in the transverse direction. The conditions $D = 0$ and $\partial D / \partial p_i = 0$ ($i = 2, \dots, d$) yield the singularities in M , which are located at $M^2 = n_1[B - \sqrt{B^2 - a^{-1}}]^2 + n_2[B + \sqrt{B^2 - a^{-1}}]^2$ with n_1, n_2 being zero or integers such that $n_1 + n_2 \leq (d-1)$. We next consider Eq. (17) as a contour integral along the unit circle C in the $z = \exp(ip)$ plane. Clearly, the asymptotic behaviour of $\text{tr } \sigma(n)$ will be controlled by the singularity closest to $|z| = 1$ inside C . All singularities at $M^2 \neq 0$ map in the z plane to an even number of branch points

inside C sufficiently far from $|z| = 1$, provided $B \sim a^{-1}$. However, the singularity at $M^2 = 0$ maps to three branch points inside (outside) C and one branch point outside (inside) C for B positive (negative). Then the function $g_{\pm}(p, B)$ does not appear single-valued on C , and the transformation of Eq. (17) into a contour integral requires further clarification. Nevertheless, one can still check that one of the branch points arising from the $M^2 = 0$ singularity is always at $z = 1 - ma + O(a^2)$, while the one that would be at $z = -1 + ma + O(a^2)$ for $B = 0$ is shifted well inside the unit circle. This fact, as well as the fact that the strength of the $M^2 = 0$ singularity of $g_{\pm}(p, B)$ is $2^{-(d-1)}$ times the strength of the corresponding singularity of $g(p, 0)$, makes the normalization of the asymptotic behaviour of $\text{tr } \sigma(n)$ the one corresponding to the propagation of a single Dirac fermion.

A more refined analysis can be carried out in two dimensions, where $g_{\pm}(p, B)$ can be computed by contour integration. We find:

$$g_{\pm}(p, B) = [f(M) + f(-M)] - \theta(p_0 - |p|) f(M) ; \quad f(M) = \frac{1}{2M \sqrt{1 - 2BM + M^2}} \quad (18)$$

where $p_0 = \cos^{-1}[(B^2 - 1)/(B^2 + 1)] + O(m^2)$. Then, $g_{\pm}(p, B)$ is the sum of two terms. The first one, $[f(M) + f(-M)]$, is not singular at $M^2 = 0$, and is therefore a single-valued function of p . It can be integrated on the unit circle, and it does not contribute to the leading behaviour as $n \rightarrow \infty$. The remaining term in $g_{\pm}(p, B)$ is not single-valued, but it is integrated only in $(-p_0, p_0)$. These end points are potentially dangerous because they could give rise to asymptotic expansions in inverse powers of n ⁹⁾. However, after averaging over $\pm B$, one picks $\text{Re}f(M)$, which is symmetric under $p \rightarrow -p$, and the two asymptotic expansions arising from $p = \pm p_0$ cancel each other⁹⁾. Therefore, the asymptotic behaviour is controlled by the nearest singularity inside the unit circle, which is located at $z = 1 - ma + O(a^2)$. The terms of $O(a^2)$ depend, however, on the sign of B . In fact, up to obvious powers of n , the behaviour of $\text{tr } \sigma(n)$ is:

$$\begin{aligned} \text{tr } \sigma(n) &\simeq \frac{1}{2} \left[\frac{B - m + (\bar{a}^2 - 2Bm + m^2)^{1/2}}{B + \bar{a}^1} \right]^n + \frac{1}{2} \left[\frac{-B - m + (\bar{a}^2 + 2Bm + m^2)^{1/2}}{\bar{a}^1 - B} \right]^n \\ &\simeq \exp \left[-m|x - x'| + O(a^2) \right] \cosh \left[|x - x'| \frac{Bm^2 a^2}{2} \right] \end{aligned} \quad (19)$$

which is the correct one in the continuum limit $ma \rightarrow 0$. However, away from the

continuum limit we see a new mass scale emerging, $\Delta m = BM^2 a^2 \ll m$, which corresponds to a mass difference between right- and left-moving fermions, induced by the breaking of reflection symmetry. Averaging over $\pm B$ gives the correct behaviour at length scales bigger than m^{-1} but smaller than $(\Delta m)^{-1}$. Ultimately, at bigger length scales, the lightest state takes over and the normalization of the propagator is not the correct one. We have been unable to see this effect (which is clearly a lattice artifact) in our numerical calculations, due to the smallness of the term $\exp -m|x-x'|$ in the relevant region.

We have considered in this paper the case of massive Dirac fermions. One can easily check that the whole calculation applies to massless Weyl fermions: the presence of helicity projection operators does not modify the results. Moreover, all the lattice artifacts we have just described disappear in this case. However, avoiding the Nielsen-Ninomiya theorem is not enough to define a correct lattice regularization of free chiral fermions; one still has to check that the peculiar form of the dispersion relation at energies of the order of the cut-off does not lead to residual effects which break Lorentz invariance in the continuum limit, in the one-loop calculation of current-current correlation functions. This problem clearly deserves further investigation.

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