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THE CONSISTENCY OF THE S^7 TRUNCATION IN $d = 11$ SUPERGRAVITY

B. de Wit

Institute for Theoretical Physics
Princetonplein 5, P.O. Box 80.006
3508 TA Utrecht, The Netherlands

and

H. Nicolai^{*)}

CERN - Geneva

A B S T R A C T

We give a complete proof that $d = 11$ supergravity compactified on S^7 admits a consistent truncation to its zero mass sector. The resulting theory is shown to coincide with gauged $N = 8$ supergravity to all orders.

^{*)} Address after April 1, 1986: Institut für Theoretische Physik, Universität Karlsruhe, P.O. Box 6380, D 7500 Karlsruhe 1, FR Germany.

1. - INTRODUCTION

The purpose of this paper is to resolve the longstanding problem of how gauged $N=8$ supergravity [1] is embedded in $N=1$ supergravity in eleven dimensions [2]. It has been known for some time that the $d=11$ theory possesses a solution where seven dimensions are compactified to the seven-sphere S^7 [3, 4]. This solution has $N=8$ supersymmetry and an internal $SO(8)$ symmetry and is therefore expected to correspond to gauged $N=8$ supergravity in four dimensions after a suitable truncation. At the linear level, the correspondence follows from the occurrence of one massless $N=8$ multiplet [4, 5] (accompanied by an infinite tower of massive $N=8$ multiplets [6]) in a small fluctuation analysis. Further evidence for the correctness of this hypothesis has been accumulated in refs. [7-19]. Here, we present a complete proof that $d=11$ supergravity compactified on S^7 admits a consistent truncation to gauged $N=8$ supergravity. We believe that this consistency proof constitutes the first example of a complete nonlinear analysis in the framework of Kaluza-Klein theories^{*)}.

The central problem in any Kaluza-Klein theory is how to make contact with our four-dimensional low-energy world. One usually starts by showing (or assuming) that the higher-dimensional theory spontaneously compactifies to four dimensions on an internal manifold whose size is small enough to prevent its immediate experimental discovery. The fields of the higher dimensional theory are then expanded about this background into certain harmonics on the internal manifold. Subsequently, only those

^{*)} Preliminary accounts of the work described here have appeared in [20, 21].

fields are kept which correspond to massless particles in four dimensions (in a first approximation). The massive states cannot be excited at low energies and are therefore discarded. The determination of both massless and massive modes involves an expansion of the higher dimensional field equations to linear order only. Having identified the zero modes, one would then like to calculate the residual interactions between the massless fields because those will ultimately lead to the final low-energy symmetry breaking. However, the evaluation of these interactions is a rather difficult problem. For its solution, the linear analysis, by which the zero modes were identified, is not sufficient. In particular, one does not obtain the correct couplings by substituting the linear modes back into the higher dimensional action and integrating over the internal manifold (the case of three-point couplings is an exception). The reason is that the correct ansätze for the massless fields involve nonlinear modifications beyond the possible nonlinear redefinitions of the $d=4$ fields. One might argue that such effects are irrelevant at low energy scales since one would expect them to be suppressed by inverse factors of the compactification scale (e.g., the Planck mass). Contrary to this naive expectation, a careful analysis indicates that this is not always the case and that there may arise certain renormalizable couplings of order unity through nonlinear modifications at the compactification scale^{*)}. The higher-order (nonrenormalizable) interactions of the massless fields are of course also sensitive to such effects, and their correct identification inevitably requires a complete nonlinear analysis of the type performed

*) To be sure, one must distinguish between the "effective low-energy theory" and the truncated theory. The latter may have solutions at which some of the scalar fields acquire vacuum expectation values of the order of the Planck mass. We are here concerned with the truncated theory.

here. In fact, our results exemplify how a sigma-model structure emerges from higher dimensions and may therefore be relevant in other contexts too.

In general, the nonlinear modifications are difficult to determine. A crucial ingredient turns out to be the requirement of consistency of the truncation to the zero-mass sector. Quite generally, this means that the states, which have been discarded in the truncation, are not reintroduced through the higher dimensional interactions or symmetry transformations after insertion of the truncated modes. In the case at hand, the consistency of the truncated supersymmetry transformations implies the consistency of the remaining bosonic transformations as well as of the truncated field equations. In the first step we therefore focus on the analysis of the supersymmetry transformations. From the consistency requirement one can determine the full embedding of gauged $N=8$ supergravity into $d=11$ supergravity and give a rigorous proof of the consistency of the S^7 truncation at the nonlinear level. A most important feature is that we base our proof on the recently constructed $SU(8)$ covariant version of $d=11$ supergravity [21, 22] rather than the original one of [2]. As we have shown there the field equations and constraints of the former are equivalent to the combined field equations and Bianchi identities of the latter. The nonlinear ansätze derived in this paper, which constitute gauged $N=8$ supergravity embedded into $d=11$ supergravity, satisfy some of these constraints, while the remaining conditions correspond to genuine $d=4$ field equations.

We now briefly review our notations and conventions^{*)} as well as some

^{*)} See also refs. [9, 13-15].

basic results concerning the S^7 compactification. The groundstate is assumed to be

$$M_{11} = (AdS)_4 \times S^7, \quad (1.1)$$

and the (finite) fluctuations will preserve the topology of this product manifold. The $d=11$ coordinates are split accordingly

$$z^M = (x^\mu, y^m), \quad (1.2)$$

where we distinguish between curved $d=4$ and $d=7$ indices $\mu, \nu, \dots$ and $m, n, \dots$. Flat $d=4$ indices are denoted by $\alpha, \beta, \dots$ whereas flat $d=7$ indices no longer appear in our treatment as they are replaced by $SU(8)$ indices $A, B, C, \dots$. Of fundamental importance are the eight Killing spinors $\eta^I(y)$ on S^7 , which satisfy

$$(\overset{\circ}{D}_m + \frac{1}{2} m_7 \overset{\circ}{\Gamma}_m) \eta^I(y) = 0 \quad (I = 1, \dots, 8) \quad (1.3)$$

and are normalized to

$$\overline{\eta}^I(y) \eta^J(y) = \delta^{IJ}. \quad (1.4)$$

Here, $|m_7|$ is the inverse S^7 radius, $\overset{\circ}{D}_m$ is the S^7 background covariant derivative and $\overset{\circ}{\Gamma}_m \equiv \overset{\circ}{e}_m^a \overset{\circ}{\Gamma}^a$ with $\overset{\circ}{e}_m^a(y)$ the (globally defined) siebenbein on S^7 . The 28 Killing vectors $K^{mIJ}(y)$ on S^7 can be expressed through the Killing spinors according to [4]

$$K^{mIJ}(y) = i \overset{\circ}{e}_a^m \overline{\eta}^I \overset{\circ}{\Gamma}^a \eta^J. \quad (1.5)$$

Their normalization, consistent with (1.4), is given by

$$K^{mIJ}(y) K^{nIJ}(y) = 8 g^{mn}(y) . \quad (1.6)$$

When lowering the index m on K^m it is understood that this is to be done with the round S^7 metric, i.e.

$$K_m^{IJ} \equiv g_{mn} K^{nIJ} . \quad (1.7)$$

We have adopted a representation in which the $d=7$ charge conjugation matrix equals the identity, so that the Killing spinors are real and the Γ^a are imaginary and antisymmetric. Throughout this paper we will use a set of orthonormal Killing spinors $\eta^i(y)$ to convert "curved" $SU(8)$ indices $A, B, C, \dots$ into "flat" $SU(8)$ indices $i, j, k, \dots$ (we introduce this terminology because in the truncation to $N=8$ supergravity the $SU(8)$ transformations acting on the "flat" indices $i, j, k, \dots$ are y -independent; hence the analogy with flat and curved indices in differential geometry should not be taken too literally). For instance, given an $SU(8)$ tensor $X^{ABC\dots}$, we define

$$X^{ijk\dots} = \eta_A^i \eta_B^j \eta_C^k \dots X^{ABC\dots} . \quad (1.8)$$

Because of the orthonormality and reality of the Killing spinors we can introduce transpose spinors η_i^A (such that $\eta_i^A \eta_A^j = \delta_i^j$, $\eta_i^A \eta_B^i = \delta_B^A$; see appendix) to invert (1.8):

$$X^{ABC\dots} = \eta_i^A \eta_j^B \eta_k^C \dots X^{ijk\dots} , \quad (1.9)$$

or to convert tensors with lower indices

$$Y_{ijk\dots} = \eta_i^A \eta_j^B \eta_{k\dots}^{C\dots} Y_{ABC\dots} \quad (1.10)$$

We also briefly remind the reader of the essential features of SU(8) invariant d=11 supergravity [21, 22]. In this formulation, the tangent space symmetry SO(1,10) of [2] is replaced by SO(1,3) x SU(8). Consequently, the basic fields are now SO(1,3) x SU(8) tensors. The fermionic sector is constituted by the fields ψ_μ^A and χ^{ABC} which transform as (s=3/2, 8) and s=1/2, 56) representations of SO(1,3) x SU(8). In the bosonic sector, we have the graviton field e_μ^α , a spin-1 field B_μ^m , a "56-bein" (e_{AB}^m , e^{mAB}) and two fields M and $\bar{M}$ which together form the adjoint representation of E_7 ($M = \mu, m$). Needless to say, all these fields still depend on all eleven coordinates and the physical degrees of freedom are still the same as before. The new fields are interrelated by certain equations which have been given in [22]. As there is no room here to review this construction in more detail, some familiarity with the preceding paper [22] will be assumed.

We conclude this introduction with an overview of the contents of this paper. In section 2, we study the bosonic transformation laws and their truncation. This leads to the identification the fields e_μ^α , B_μ^m , e_{AB}^m , ψ_μ^A and χ^{ABC} in terms of the fields of N=8 supergravity. In section 3, we solve the "generalized vielbein postulate" of [21, 22]; the solution is an indispensable prerequisite for the analysis of the fermionic transformation laws in section 4. Section 5 is the heart of this paper: it contains the most difficult part of the whole argument, namely the proof that the T-tensor as identified from d=11 supergravity also becomes y-independent (this tensor characterizes the extra terms induced by the gauging of SO(8) in N=8 supergravity [1]). At this point, the proof is already complete. Nonetheless, in section 6, we discuss the y-

independence of the field equations and deduce the N=8 potential [1] from d=11 supergravity. Finally, in section 7, we show how to obtain the full nonlinear ansätze for the original fields of d=11 supergravity [2] in terms of those of N=8 supergravity. The appendix contains some useful identities involving Killing spinors and vectors.

2. - THE GENERALIZED VIELBEIN AND THE BOSON TRANSFORMATION LAWS

In this section we express the fields e_μ^α , B_μ^m , e_{AB}^m , ϕ_μ^A and χ^{ABC} in terms of the fields of N=8 supergravity and examine the consistency of the supersymmetry transformations for the boson fields. These transformation rules are [21,22]

$$\delta e_\mu^\alpha = \frac{1}{2} \bar{\epsilon}^A \gamma_\mu^\alpha \phi_{\mu A} + \text{h.c.} , \quad (2.1)$$

$$\delta B_\mu^m = \frac{1}{8} \sqrt{2} e_{AB}^m (2\gamma_\mu^A B^B + \bar{\epsilon}_C \gamma_\mu^A \chi^{ABC}) + \text{h.c.} , \quad (2.2)$$

$$\delta e_{AB}^m = -\sqrt{2} \Sigma_{ABCD} e^{mCD} , \quad (2.3)$$

where

$$\Sigma_{ABCD} \equiv \epsilon_{[A} \chi_{BCD]} + \frac{1}{24} \epsilon_{ABCDEFGH} \bar{\epsilon}^E \chi^{FGH} . \quad (2.4)$$

As is well known the massless fermionic fluctuations about the S^7 background are proportional to Killing spinors [4,5], i.e.

$$\phi_{\mu A}(x,y) = \phi_{\mu i}(x) \eta_A^i(y) + \dots \quad (2.5)$$

$$\chi_{ABC}(x,y) = \chi_{ijk}(x) \eta_A^i(y) \eta_B^j(y) \eta_C^k(y) + \dots \quad (2.6)$$

where the dots indicate that this result only applies to infinitesimally small fluctuations. The fields ϕ_{μ}^i and χ_{ijk} are the four-dimensional spinor fields subject to the same chirality constraints as the fields $\phi_{\mu A}$ and χ_{ABC} . The supersymmetry parameter associated with $\phi_{\mu A}$ is decomposed similarly.

$$\epsilon_A(x,y) = \epsilon_i(x) \eta_A^i(y) + \dots \quad (2.7)$$

Decompositions such as (2.5-7) are not necessarily correct for finite deviations from the S^7 background. In the original version of the theory it was pointed out [9] that the supersymmetry transformations become inconsistent when all fields are naively restricted to the massless modes. It was then argued that a field-dependent chiral SU(8) transformation on the fermions was needed to obtain consistent transformation laws, and this conjecture was verified explicitly in a class of SO(7) invariant backgrounds [13,14]. However, in the present formulation of d=11 supergravity the theory is invariant under local (x- and y-dependent) SU(8) transformations, so that we can directly impose a gauge condition where (2.5-7) become exact. This requirement does not fix the SU(8) invariance completely, and one may still perform transformations of the form

$$U_B^A(x,y) = \eta_i^A(y) \eta_B^j(y) U_j^i(x) \quad (2.8)$$

where $U_j^i(x)$ is an arbitrary x-dependent SU(8) transformation, which will turn out to coincide with the SU(8) transformation of d=4, N=8 supergravity. Using the Killing spinors to convert "curved" SU(8) indices A, B, .. to "flat" SU(8) indices i, j, ..., the transformations (2.1-3) take the form

$$\delta e_{\mu}^{\alpha}(x,y) = \frac{1}{2} \bar{\epsilon}^i(x) \gamma^{\alpha} \psi_{\mu i}(x) + \text{h.c.} \quad (2.9)$$

$$\delta B_{\mu}^m(x,y) = \frac{1}{8} \sqrt{2} e_{ij}^m(x,y) (2\sqrt{2} \bar{\epsilon}^i(x) \psi_{\mu}^j(x) + \bar{\epsilon}_k(x) \gamma_{\mu} \chi^{ijk}(x)) + \text{h.c.} \quad (2.10)$$

$$\delta e_{ij}^m(x,y) = -\sqrt{2} \Sigma_{ijkl}(x) e^{mk\ell}(x,y) , \quad (2.11)$$

where

$$\Sigma_{ijkl}(x) = \bar{\epsilon}_{[i}(x) \chi_{jkl]}(x) + \frac{1}{24} \epsilon_{ijklmnpq} \bar{\epsilon}^m(x) \chi^{npq}(x). \quad (2.12)$$

It follows from general Kaluza-Klein theory that the correct ansatz for the massless SO(8) gauge fields is given to all orders by

$$B_{\mu}^m(x,y) = -\frac{\sqrt{2}}{4} A_{\mu}^{IJ}(x) K^{mIJ}(y) \quad (2.13)$$

where the proportionality constant is related to the normalization adopted for A_{μ}^{IJ} . According to (2.10) B_{μ}^m and e_{ij}^m have the same y -dependence so that we may assume the ansatz

$$e_{ij}^m(x,y) = w_{ij}^{IJ}(x) K^{mIJ}(y) . \quad (2.14)$$

Obviously, the transformations are now consistent for the ansätze (2.5-7), (2.13-14) provided we choose the vierbein field independent of y , i.e.

$$e_{\mu}^{\alpha}(x,y) = e_{\mu}^{\alpha}(x) , \quad (2.15)$$

in accordance with general Kaluza-Klein theory for the massless spin-2 modes.

What remains is to express the coefficients $w_{ij}^{IJ}(x)$ in terms of the spinless fields of N=8 supergravity. This follows from comparing the N=8

transformation rule^{*)}

$$\delta A_\mu^{IJ} = -\frac{i}{2}(u_{ij}^{IJ} + v_{ijIJ})(2\sqrt{2} \bar{\epsilon}_\mu^{ij} + \bar{\epsilon}_\mu^{jk} \chi^{ijk}) + \text{h.c.} \quad (2.16)$$

to the one that follows from combining (2.10), (2.13) and (2.14). This yields

$$w_{ij}^{IJ}(x) = u_{ij}^{IJ}(x) + v_{ijIJ}(x), \quad (2.17)$$

where u_{ij}^{IJ} and v_{ijIJ} are the 28×28 matrices that appear in the definition of the so-called 56-bein and its inverse,

$$v(x) = \begin{pmatrix} u_{ij}^{IJ}(x) & v_{ijKL}(x) \\ v_{klIJ}(x) & u_{KL}^{kl}(x) \end{pmatrix}, \quad (2.18a)$$

$$v^{-1}(x) = \begin{pmatrix} u_{ij}^{IJ}(x) & -v_{klIJ}(x) \\ -v_{ijIJ}(x) & u_{KL}^{kl}(x) \end{pmatrix}, \quad (2.18b)$$

which is an element of the coset space $E_7/SU(8)$ (for our notation, conventions and useful identities for u_{ij}^{IJ} and v_{ijIJ} we refer to [1]).

However, the consistency of (2.14) and (2.17) is not directly obvious;

e_{ij}^m has a special form in terms of $d=11$ quantities, namely

$$e_{ij}^m = i\Delta^{-\frac{1}{2}} e_a^m \eta_i^A \eta_j^B (\Phi^T \Gamma_a \Phi)_{AB} \quad (2.19)$$

with an as yet undetermined $SU(8)$ matrix $\Phi(x,y)$. The presence of Φ in (2.19) is absolutely crucial as (2.14) is complex whereas (2.19) is real for $\Phi = \mathbb{1}$ (note that the normalization of (2.14) and (2.19) is such that

^{*)} Note that ϵ differs by a factor $\frac{1}{2}$ from the ϵ used in [1].

the unit E_7 matrix corresponds to $\Phi = \mathbb{1}$, $e_a^m = \dot{e}_a^m$). An important consequence of (2.19) is that the generalized vielbein must satisfy the "Clifford property" [21].

$$e_{ij}^m e_{kl}^{nj} + e_{ij}^n e_{kl}^{mj} = \frac{1}{4} \delta_{ij}^{kl} e_{j\ell}^m e^{n\ell j} \quad (2.20)$$

This result indeed holds for the ansatz (2.14) by virtue of the E_7 properties of the matrices u_{ij}^{IJ} and v_{ijIJ} , as we will now show. Obviously (2.20) is true provided that the left-hand side vanishes when traced with an arbitrary traceless matrix Λ_k^i ; it is sufficient to assume Λ to be antihermitean, since any hermitean matrix can be rendered thus by multiplication with i . Inserting the solution (2.14) into the left-hand side of (2.20) we find

$$w_{ij}^{IJ} \Lambda_{kl}^{ij} w_{kl}^{KL} (K^{mIJ} K^{nKL} + K^{nIJ} K^{mKL}) \quad (2.21)$$

where $\Lambda_{kl}^{ij} = \delta_{[k}^{[i} \Lambda_{l]}^{j]}$. The matrix Λ_{kl}^{ij} characterizes the infinitesimal transformations of the $SU(8)$ subgroup of E_7 . This fact can be used to derive the E_7 Lie algebra relations [1]

$$(u\bar{u})_{KL}^{IJ} + (v\bar{v})_{KL}^{IJ} = \frac{2}{3} \delta_{[K}^{[I} \{ (u\bar{u})_{L]M}^{J]M} + (v\bar{v})_{L]M}^{J]M} \} \quad (2.22)$$

$$(u\bar{v})_{IJ, KL}^{IJ, KL} + (u\bar{v})_{KL, IJ}^{KL, IJ} = \text{antisymmetric, selfdual in } [IJKL] \quad *) \quad (2.23)$$

where $u, v, \bar{u}$ and $\bar{v}$ denote $u_{ij}^{IJ}, v_{ijIJ}, u^{ij}_{IJ}$ and v^{ijIJ} (hence $(u\bar{v})_{IJ, KL}^{IJ, KL} = u_{ij}^{IJ} \Lambda_{kl}^{ij} v^{klKL}$, etc.) . Substitution of (2.17) into (2.21) yields

*) By selfdual we always mean complex selfdual, i.e.
 $X_{IJKL} = \frac{1}{24} \epsilon_{IJKLMNPQ} X^{MNPQ}$, so the real part is selfdual and the imaginary part antiselfdual.

$$\begin{aligned} & \{ (u\Lambda\bar{u})_{KL}^{IJ} + (v\Lambda\bar{v})_{IJ}^{KL} + (u\Lambda\bar{v})^{IJ,KL} + (v\Lambda\bar{u})_{IJ,KL} \} \\ & \times (K^{mIJ}_K K^{nKL} + K^{nIJ}_K K^{mKL}) \end{aligned} \quad (2.24)$$

Using (2.22) for the first two terms in (2.24) leads to an expression containing

$$K^{mIJ}_K K^{nIK} + K^{nIJ}_K K^{mIK} = 2\delta^{JK} g^{mn}, \quad (2.25)$$

which multiplies the term (from (2.22))

$$(u\Lambda\bar{u})_{KM}^{JM} + (v\Lambda\bar{v})_{KM}^{JM}, \quad (2.26)$$

whose trace vanishes because (2.26) must be an element of the $SU(8)$ Lie algebra. In a similar fashion the last two terms cancel; because of the antihermiticity of Λ , the fourth term is just minus the complex conjugate of the third term, so that their sum constitutes an antiselfdual tensor. On the other hand $K^{m[IJ}_K K^{nKL]}$ is selfdual, so that their product vanishes because of opposite duality phases. Consequently (2.14) satisfies the Clifford property (2.20) provided we make the identification (2.17).

The above result is sufficient to show that the two expressions for the generalized vielbein, (2.14) and (2.19) are compatible. Let us first examine the case where all e^m_{ij} are real. From (2.14) and (2.17) one sees that in this case both u and v are real; more precisely, they parametrize the $SL(8, \mathbb{R})$ subgroup of E_7 . From the well-known uniqueness theorem for representations of the Clifford algebra (see, e.g. [23]), it then follows

that any e_{ij}^m satisfying (2.20) can be written in the form

$$e_{ij}^m = E^{ma} (S^{-1} \Gamma^a S)_{ij}, \quad (2.27)$$

with E^{ma} and S elements of $GL(7, R)$ and $SL(8, R)$, respectively. However, e_{ij}^m is manifestly antisymmetric in $[ij]$, which implies that S is actually an element of $SO(8)$ (i.e. $S^T = S^{-1}$), so that (2.27) is indeed of the form (2.19). Of course, S is only determined up to an $SO(7)$ rotation which may alternatively be absorbed into E . To discuss the general case with complex u and v , we apply an $SU(8)$ transformation to the real versions of (2.14) and (2.19) so that Φ in (2.19) becomes an $SU(8)$ transformation, while the matrices u and v constitute an element of E_7 which is the product of a real E_7 (i.e. $SL(8, R)$) element with an element of its $SU(8)$ subgroup. However, such products cover the whole E_7 group as there is no proper subgroup of E_7 which contains both $SL(8, R)$ and $SU(8)$. This proves our assertion that the e_{ij}^m defined by (2.14) can indeed be written in the form (2.19)*).

Now that we have justified the ansatz for e_{ij}^m we turn to the supersymmetry transformation for w_{ij}^{IJ} . Combining w_{ij}^{IJ} with its complex conjugate w_{IJ}^{ij} , the transformation rule can be written in E_7 covariant form

$$\begin{pmatrix} \delta w^{IJ} \\ \delta w_{IJ} \end{pmatrix} = E \begin{pmatrix} w^{IJ} \\ w_{IJ} \end{pmatrix}, \quad (2.28)$$

*) For purely scalar fluctuations the matrix Φ has been computed in [17]. For $SO(7)^{\pm}$ invariant scalar and pseudoscalar fluctuations Φ has also been determined [13, 14].

where we have suppressed the SU(8) indices and E is an element of the E_7 Lie algebra equal to

$$E = \begin{pmatrix} 0 & -\sqrt{2} \Sigma_{ijkl} \\ -\sqrt{2} \Sigma_{mnpq} & 0 \end{pmatrix} \quad (2.29)$$

Equations such as (2.28) will also be encountered in section 3, so let us discuss them in full generality. From (2.28) expressions for δu and δv may be derived in the following fashion. First consider the contraction $u^{ij}_{IJ} \delta w^{KL}_{ij} - v_{ijIJ} \delta w^{ij}_{KL}$, which yields the equation

$$\Delta_{IJ}^{KL} + \Delta_{IJKL} = E_{IJ}^{KL} + E_{IJKL}, \quad (2.30)$$

where Δ_{IJ}^{KL} , Δ_{IJKL} and their complex conjugates are the 28×28 submatrices of $\psi^{-1} \delta \psi$, with

$$\delta \psi = \begin{pmatrix} \delta u & \delta v \\ \delta \bar{v} & \delta \bar{u} \end{pmatrix}, \quad (2.31)$$

while E_{IJ}^{KL} , E_{IJKL} and their complex conjugates are the 28×28 submatrices of $\psi^{-1} E \psi$. Because E belongs to the E_7 Lie algebra, $\psi^{-1} E \psi$ can also be decomposed according to the E_7 Lie algebra, so that E_{IJ}^{KL} characterizes the SU(8) components and E_{IJKL} is complex selfdual. In the case at hand, where $\delta \psi$ is the supersymmetry variation of ψ , $\psi^{-1} \delta \psi$ is also in the E_7 Lie algebra. Therefore (2.30) can be split into two separate equations

$$\Delta_{IJ}^{KL} = E_{IJ}^{KL}, \quad \Delta_{IJKL} = E_{IJKL} \quad (2.32)$$

or

$$(\delta \psi) \psi^{-1} = -\sqrt{2} \begin{pmatrix} 0 & \Sigma^{ijkl} \\ \Sigma^{mnpq} & 0 \end{pmatrix} \quad (2.33)$$

which is just the N=8 result. An alternative form of (2.33) is

$$\delta u_{ij}^{IJ} = -\sqrt{2} \Sigma_{ijkl} v^{klIJ}, \quad (2.34)$$

$$\delta v_{ijIJ} = -\sqrt{2} \Sigma_{ijkl} u^{kl}_{IJ}.$$

3. - SOLUTION OF THE GENERALIZED VIELBEIN POSTULATES

The generalized vielbeine satisfy two equations which take the form of a generalized vielbein postulate which extends the one of ordinary Riemannian geometry. They are [22]

$$\begin{aligned} (\partial_\mu - B_\mu^n D_n) e_{AB}^m + D_n B_\mu^m e_{AB}^n + \frac{1}{2} D_n B_\mu^n e_{AB}^m \\ + B_\mu^C [A^m_B]_C + A_{\mu ABCD} e^{mCD} = 0, \end{aligned} \quad (3.1)$$

$$D_m e_{AB}^n + B_m^C [A^n_B]_C + A_{m ABCD} e^{nCD} = 0. \quad (3.2)$$

As already explained these equations can be converted to "flat" SU(8) indices by employing the Killing spinors. Furthermore we can substitute the expressions (2.13) and (2.14) for B_μ^m and e_{ij}^m , so that (3.1) and (3.2) take the following form.

$$\begin{aligned} \partial_\mu w_{ij}^{IJ} K^{mIJ} + \frac{\sqrt{2}}{4} A_\mu^{KL} (K^{nKL} D_n K^{mIJ} - K^{nIJ} D_n K^{mKL}) w_{ij}^{IJ} \\ + (B_{\mu[i}^k w_{j]k}^{IJ} + A_{\mu ijkl} w^{kl}_{IJ}) K^{mIJ} = 0 \end{aligned} \quad (3.3)$$

$$w_{ij}^{IJ} D_m K^{nIJ} + (B_{m[i}^k w_{j]k}^{IJ} + A_{m ijkl} w^{kl}_{IJ}) K^{nIJ} = 0 \quad (3.4)$$

where $\hat{A}_{\mu ijkl}$ and $\hat{A}_{mijkl}$ are obtained from the tensors $A_{\mu ABCD}$ and A_{mABCD} by contracting with Killing spinors; $B_{\mu j}^i$ and B_{mj}^i contain an additional modification and are defined by

$$B_{\mu j}^i = \eta_A^i \eta_j^B (B_{\mu B}^A - i \frac{\sqrt{2}}{4} m_7 A_{\mu}^{KL} K^{nKL} \dot{e}_n^a (\Gamma_a)^A_B) \quad (3.5)$$

$$B_{mj}^i = \eta_A^i \eta_j^B (B_{mB}^A - i m_7 \dot{e}_m^a (\Gamma_a)^A_B) \quad (3.6)$$

Using the Lie algebra relation of the $SO(8)$ Killing vectors

$$K^{nIJ} \dot{D}_n K^{mKL} - K^{nKL} \dot{D}_n K^{mIJ} = 8 m_7 \delta^{[I[K} K^{mL]J]}, \quad (3.7)$$

we derive from (3.3)*)

$$\partial_{\mu} w_{ij}^{IJ} - 2g A_{\mu}^{K[I} w_{ij}^{J]K} + B_{\mu[i}^k w_{j]k}^{IJ} + A_{\mu ijkl} w^{kl}_{IJ} = 0 \quad (3.8)$$

where we have introduced the $SO(8)$ gauge coupling constant

$$g \equiv \sqrt{2} m_7. \quad (3.9)$$

Introducing the notation

$$B_{Mij}^{kl} \equiv \delta_{[i}^{[k} B_{Mj]}^{l]} \quad (M = \mu, m) \quad (3.10)$$

we can write (3.4) and (3.8) and their complex conjugates as

*) Here we have solved $O^{IJ}(x,y)K^{mIJ}(y) = 0$ by $O^{IJ}(x,y) = 0$; however, in general this equation only implies that $O^{IJ}(x,y)$ is proportional to $\dot{D}_n K^{nIJ}$. Our choice is based on the assumption that the y -dependence is entirely expressible in terms of Killing vectors, so that $O^{IJ}(x,y)$ is y -independent.

$$\overset{\circ}{D}_m K^{nIJ} \begin{pmatrix} w_{ij}^{IJ} \\ w_{kl}^{IJ} \end{pmatrix} + K^{nIJ} \begin{pmatrix} B_{mij}^{pq} & A_{mijuv} \\ A_m^{klpq} & B_{muv}^{kl} \end{pmatrix} \begin{pmatrix} w_{mn}^{IJ} \\ w_{uv}^{IJ} \end{pmatrix} = 0 \quad (3.11)$$

$$D_\mu^{SO(8)} \begin{pmatrix} w_{ij}^{IJ} \\ w_{IJ} \end{pmatrix} + \begin{pmatrix} B_{\mu ij}^{mn} & A_{\mu ij pq} \\ A_\mu^{klmn} & B_{\mu pq}^{kl} \end{pmatrix} \begin{pmatrix} w_{IJ}^{IJ} \\ w_{IJ} \end{pmatrix} = 0 \quad (3.12)$$

where the SO(8) covariant derivative $D_\mu^{SO(8)}$ is defined by the first two terms in (3.8).

Using the arguments preceding (2.32) with δw replaced by $D_\mu^{SO(8)} w$, where $D_\mu^{SO(8)}$ is the SO(8) covariant derivative, we conclude that (3.8) implies

$$D_\mu^{SO(8)} \varphi = - \begin{pmatrix} B_{\mu ij}^{mn} & A_{\mu ij pq} \\ A_\mu^{klmn} & B_{\mu pq}^{kl} \end{pmatrix} \varphi, \quad (3.13)$$

which shows that B_μ and A_μ have the same definition in terms of φ as in N=8 supergravity (modulo a different normalization factor for A_μ). The same arguments can also be applied to (3.11) but now with δw proportional to w . This yields

$$\overset{\circ}{D}_m K^{nIJ} + (B_{mIJ}^{KL} + A_{mIJKL}) K^{nKL} = 0 \quad (3.14)$$

where B_{mIJ}^{KL} , A_{mIJKL} and their complex conjugates are defined by

$$\begin{pmatrix} B_m & A_m \\ \bar{A}_m & \bar{B}_m \end{pmatrix} = \varphi^{-1} \begin{pmatrix} B_m & A_m \\ \bar{A}_m & \bar{B}_m \end{pmatrix} \varphi \quad (3.15)$$

We now construct an ansatz for B_{mIJ}^{KL} and A_{mIJKL} in terms of Killing vectors

$$B_{mIJ}^{KL} = -\alpha m_7 \delta_{[I}^{[K} K_{mJ]}^{L]} \quad (3.16)$$

$$A_{mIJKL} = -\beta \overset{\circ}{D}_m K_n^{[IJ} K^{nKL]}$$

where α and β are two real coefficients. Imposing (3.14) shows that α and β are related according to

$$\alpha + 4\beta = 1. \quad (3.17)$$

From (3.15) it is now easy to determine B_{mij}^{kl} and A_{mijkl} ,

$$\begin{aligned} B_{mij}^{kl} = & \alpha m_7 K_m^{IJ} (u_{ij}^{JK} u_{IK}^{kl} - v_{ijJK} v^{klIK}) \\ & + \beta \overset{\circ}{D}_m K_n^{[IJ} K^{nKL]} (u_{ij}^{[IJ} v^{klKL]} - v_{ij[IJ} u_{KL]}^{kl}) \end{aligned} \quad (3.18)$$

$$\begin{aligned} A_{mijkl} = & \alpha m_7 K_m^{IJ} (v_{ijJK} u_{kl}^{IK} - u_{ij}^{JK} v_{klIK}) \\ & - \beta \overset{\circ}{D}_m K_n^{[IJ} K^{nKL]} (u_{ij}^{IJ} u_{kl}^{KL} - v_{ijIJ} v_{klKL}) \end{aligned} \quad (3.19)$$

Both A_μ , B_μ and A_m , B_m can be written in a more suggestive form, namely

$$\begin{pmatrix} B_M & A_M \\ \bar{A}_M & \bar{B}_M \end{pmatrix} = v^{-1} O_M v \quad (M = \mu, m) \quad (3.20)$$

where

$$O_\mu = - D_\mu^{SO(8)}$$

$$O_m = \begin{pmatrix} -\alpha m_7 \delta_{[I}^{[K} \delta_{mJ]}^{L]} & -\beta \overset{\circ}{D}_m K_n^{[IJ} K^{nKL]} \\ -\beta \overset{\circ}{D}_m K_n^{[IJ} K^{nKL]} & -\alpha m_7 \delta_{[I}^{[K} \delta_{mJ]}^{L]} \end{pmatrix} \quad (3.21)$$

Note that A_μ and B_μ are y -independent, whereas A_m and B_m explicitly depend on y through the Killing vectors. The fermionic transformation rules that we shall discuss in the next section require that the T tensors, which can be written as products of e^m with B_m and A_m , are y -independent. In order to establish this property the ansätze for A_m and B_m , (3.17) and (3.18), are indispensable. Furthermore the coefficients α and β will be completely fixed by this analysis which we will give in section 5. Upon taking the limit $m_7 \rightarrow 0$ in (3.21), which leads to the torus reduction of [24], we recover the expression for ungauged $N=8$ supergravity, namely

$O_\mu = \partial_\mu$, $O_m = 0$. One may speculate that the general structure of (3.20) may be relevant for other compactifications as well.

4. - FERMION TRANSFORMATION LAWS

The fermionic transformation rules are [21,22]

$$\begin{aligned} \delta \psi_\mu^A &= (\partial_\mu - B_\mu^m \overset{\circ}{D}_m - \frac{1}{4} \omega_\mu^{\alpha\beta} \gamma_{\alpha\beta} - \frac{1}{4} \gamma_\mu \gamma^\nu \overset{\circ}{D}_m B_{\nu}^m) \epsilon^A \\ &+ (\frac{1}{2} B_{\mu B}^A + \gamma^{\alpha\beta} \gamma_\mu \epsilon_{\alpha\beta}^{-AB}) \epsilon_B \\ &+ \frac{1}{2} e^{mAB} (\delta_B^C \overset{\circ}{D}_m + \frac{1}{2} B_{mB}^C) \gamma_\mu \epsilon_C \end{aligned}$$

$$- \frac{1}{4} e^m_{CD} A_m^{ABCD} \gamma_\mu \epsilon_B, \quad (4.1)$$

$$\begin{aligned} \delta\chi^{ABC} = & 3\sqrt{2} \gamma^{\alpha\beta} \epsilon_{\alpha\beta}^{-[AB} \epsilon^C] - \sqrt{2} \gamma^\mu A_\mu^{ABCD} \epsilon_D \\ & + \frac{3}{\sqrt{2}} e^m_{AB} (\delta^C_D \dot{D}_m + \frac{1}{2} B^C_{mD}) \epsilon^D \\ & - \frac{3\sqrt{2}}{4} e^m_{DE} A_m^{DE[AB} \epsilon^C] - \sqrt{2} e^m_{DE} A_m^{ABCD} \epsilon^E. \end{aligned} \quad (4.2)$$

Again we convert these equations to "flat" SU(8) indices. Substituting the ansätze for the various fields obtained in the previous section we find that the B^m_μ -dependent terms are again absorbed into $\frac{1}{\mu_j}$ (cf. (3.5)) and that also $\frac{1}{\mu_j}$ acquires the extra term shown in (3.6). Furthermore the spin connection $\hat{\omega}_\mu^{\alpha\beta}$ becomes the usual d=4 spin connection which we denote by $\omega_\mu^{\alpha\beta}$. The S^7 truncated version of (4.1) and (4.2) is then

$$\begin{aligned} \delta\psi_\mu^i = & (\partial_\mu - \frac{1}{4} \omega_\mu^{\alpha\beta} \gamma_{\alpha\beta}) \epsilon^i + \frac{1}{4} B^i_{\mu j} \epsilon^j \\ & + \frac{\sqrt{2}}{8} \gamma^{\alpha\beta} \gamma_\mu \bar{F}_{\alpha\beta}^{-ij} \epsilon_j \\ & + \frac{1}{4} \gamma_\mu (e^{mik} B_{mk}^j - A_m^{ijkl} e_{kl}^m) \epsilon_j \end{aligned} \quad (4.3)$$

$$\begin{aligned} \delta\chi^{ijk} = & \frac{3}{4} \gamma^{\alpha\beta} \bar{F}_{\alpha\beta}^{-[ij} \epsilon^{k]} - \sqrt{2} \gamma^\mu A_\mu^{ijkl} \epsilon_l \\ & + \frac{\sqrt{2}}{4} (3e^{m[ij} B_{m\lambda}^{k]} - 3e^m_{pq} A_m^{pq[ij} \delta_\lambda^{k]} - 4A_m^{ijkp} e_{pl}^m) \epsilon^l \end{aligned} \quad (4.4)$$

where we have introduced the definition

$$\overline{F}_{\alpha\beta}^{-ij} \equiv 4\sqrt{2} \eta_A^i \eta_B^j \mathcal{G}_{\alpha\beta}^{-AB}$$

$$(\text{thus } \overline{F}_{\alpha\beta ij}^+ = 4\sqrt{2} \eta_i^A \eta_j^B \mathcal{G}_{\alpha\beta AB}^+) \quad (4.5)$$

From its definition (cf. [22]) we know that $\mathcal{G}_{\alpha\beta}^{-AB}$ satisfies the identity (modulo terms proportional to $\partial_\mu e_\mu^\alpha$ which vanish in this truncation)

$$e_{AB}^m \mathcal{G}_{\alpha\beta}^{-AB} = - [e_\alpha^{[\mu} e_\beta^{\nu]}]_- (\partial_\mu - B_\mu^n \dot{D}_n) B_\nu^m, \quad (4.6)$$

where $[]_-$ indicates that we take the part antiselfdual in $[\mu\nu]$. Using the ansatz (2.13) and the $SO(8)$ Lie algebra property of the Killing vectors (3.7) one readily shows that

$$(\partial_{[\mu} - B_{[\mu}^n \dot{D}_n]}) B_{\nu]}^m = - \frac{\sqrt{2}}{8} F_{\mu\nu}^{IJ} K^{mIJ}, \quad (4.7)$$

where $F_{\mu\nu}^{IJ}$ is the $SO(8)$ field strength

$$F_{\mu\nu}^{IJ} = \partial_\mu A_\nu^{IJ} - \partial_\nu A_\mu^{IJ} - 2g A_\mu^{K[I} A_\nu^{J]K}, \quad (4.8)$$

so that (4.6) implies

$$e_{ij}^m \overline{F}_{\mu\nu}^{-ij} = [F_{\mu\nu}^{IJ}]_- K^{mIJ}. \quad (4.9)$$

The obvious solution of (4.9) is

$$(u_{ij}^{IJ} + v_{ijIJ}) \overline{F}_{\mu\nu}^{-ij} = [F_{\mu\nu}^{IJ}]_- \quad (4.10)$$

where we again make the assumption mentioned in the footnote preceding (3.8). We now observe that (4.10) is precisely the equation known from $N=8$

supergravity (cf. (2.26) of [1]), so that solving (4.10) leads to the same expression for $\bar{F}_{\mu\nu}^{-1j}$.

We have already established that $\omega_{\mu}^{\alpha\beta}$, $B_{\mu j}^i$, A_{μ}^{ijkl} , ϕ_{μ}^i , χ^{ijk} and ϵ^i are y -independent, and in direct correspondence with quantities that appear in N=8 supergravity. Now the same result also holds for $\bar{F}_{\mu\nu}^{-1j}$, and one can verify by direct comparison that the transformations (4.3) and (4.4) coincide with the corresponding ones of [1] as far as the terms proportional to $\partial_{\mu}\epsilon^i$, $\omega_{\mu}^{\alpha\beta}$, $B_{\mu j}^i$, A_{μ}^{ijkl} , are concerned. The remaining terms in (4.3) and (4.4), which involve e_{ij}^m , B_{mj}^i and A_m^{ijkl} , are evidently related to the extra terms that must be introduced in N=8 supergravity after the gauging of SO(8) to restore local supersymmetry. These extra variations are [1]

$$\delta\phi_{\mu}^i = \dots + \frac{1}{2}\sqrt{2} g A_1^{ij} \gamma_{\mu} \epsilon_j, \quad (4.11)$$

$$\delta\chi^{ijk} = \dots - g A_{2\ell}^{ijk} \epsilon^{\ell}, \quad (4.12)$$

where A_1^{ij} and $A_{2\ell}^{ijk}$ are (irreducible) SU(8)-tensors with the properties

$$A_1^{ij} = A_1^{ji}, \quad (4.13)$$

$$A_{2\ell}^{ijk} = A_{2\ell}^{[ijk]}, \quad A_{2k}^{ijk} = 0. \quad (4.14)$$

Together they form the "T-tensor"

$$\begin{aligned} T_i^{jkl} &= -\frac{3}{4} A_{2i}^{jkl} + \frac{3}{2} \delta_i^{[k} A_1^{\ell]j} \\ &= (u_{IJ}^{k\ell} + v^{k\ell IJ})(u_{im}^{JK} u_{KI}^{jm} - v_{imJK} v^{jmKI}) \end{aligned} \quad (4.15)$$

and it will be our task to rederive this directly from d=11 supergravity. We now proceed as before by comparing (4.11) and (4.12) with the associated terms in (4.3) and (4.4). This comparison will lead to expressions for A_1 and A_2 as functions of e_{ij}^m , A_{mijkl} and B_{mj}^i . Since for arbitrary values of α and β in (3.17) and (3.18), the new expressions will differ from the ones given above, we distinguish them by writing $A_1^{ij}(\alpha, \beta)$ and $A_{2\ell}^{ijk}(\alpha, \beta)$ as opposed to A_1^{ij} and $A_{2\ell}^{ijk}$. Hence we obtain from (4.3) and (4.4)

$$g A_1^{ij}(\alpha, \beta) = \frac{1}{4}\sqrt{2} (e^{m[k} B_{mk}^j - A_m^{ijkl} e_{kl}^m) \quad (4.16)$$

$$g A_{2\ell}^{ijk}(\alpha, \beta) = \frac{1}{4}\sqrt{2} (-3 e^{m[ij} B_{m\ell}^k] + 3 e_{pq}^m A_m^{pq[ij} \delta_{\ell}^k] + 4 A_m^{ijkp} e_{p\ell}^m) \quad (4.17)$$

We may now invoke the truncated vielbein postulate (3.4) which implies (because $\overset{\circ}{D}_m K^{mIJ} = 0$)

$$e^{mk[i} B_{mk}^j] - A_m^{ijkl} e_{kl}^m = 0 \quad (4.18)$$

to rewrite (4.16) in the form

$$g A_1^{ij}(\alpha, \beta) = \frac{1}{4}\sqrt{2} e^{mk(i} B_{mk}^{j)} \quad (4.19)$$

The symmetry property of (4.13) is now manifest. Similarly we can prove that (4.17) is traceless in accordance with (4.14),

$$g A_{2k}^{ijk}(\alpha, \beta) = -\frac{1}{4}\sqrt{2}(-2e^{mk[i} B_{m k}^{j]} + 6e_{pq}^m A_m^{pqij} - 4 A_m^{ijpq} e_{pq}^m) = 0 \quad (4.20)$$

where we used again (4.18) and $B_{mi}^1 = 0$.

Thus we have established the properties (4.13) and (4.14) from d=11 supergravity for arbitrary values of α and β subject to (3.17). The more difficult task is now to demonstrate that $A_1^{ij}(\alpha, \beta)$ and $A_{2\ell}^{ijk}(\alpha, \beta)$ can become y-independent and coincide with (4.11) and (4.12) as calculated from (4.15). As it turns out, this is only possible for one special value of α and β . This is the most subtle part of the whole consistency proof, and will be dealt with in the next section.

5. - MORE T-IDENTITIES

In the foregoing section we have arrived at expressions for $A_1^{ij}(\alpha, \beta)$ and $A_{2\ell}^{ijk}(\alpha, \beta)$ in terms of the SU(8) quantities A_m and B_m (cf. (4.19) and (4.17)). We can now substitute the solutions (3.18) and (3.19) into them and enquire under what conditions the result, which is now expressed through $u(x)$ and $v(x)$, becomes y-independent and coincides with (4.15). It turns out that the usual arguments based on the Γ -matrix completeness relation (A.5) are not sufficient to analyze these aspects and one must make use of further identities involving u and v which go beyond the "T-identities" given in [1]. Again the E_7 group, although manifestly not a symmetry of the theory, plays a crucial role.

The substitution of (3.18) and (3.19) into (4.17) and (4.19) leads to

products of two and three Killing vectors, namely $K^m K_m$ and $\dot{D}_m K_n K^n K^m$.

These expressions can be simplified by using the representation (1.5) and properties of Γ -matrices. In this way one proves from the Γ -matrix completeness relation (A.5),

$$K^{mIJ} K_m^{KL} = 2 \delta_{KL}^{IJ} + K^{IJKL} \quad (5.1)$$

where K^{IJKL} is the antisymmetric selfdual tensor

$$K^{IJKL} \equiv K^m [IJ K_m^{KL}] \quad (5.2)$$

Furthermore, using

$$\dot{D}_m K_n^{IJ} = -m_7 \eta^I \Gamma_{mn} \eta^J \quad (5.3)$$

and the identity

$$\Gamma_{IJ}^n \Gamma_{KL}^{mn} = \Gamma_{IJ}^{mn} \Gamma_{KL}^n - 8 \delta_{[I[K} \Gamma_{L]J]}^m \quad (5.4)$$

one finds

$$\dot{D}_m K_n^{[IJ} K^{nKL]} K^{mMN} = 8 m_7 \delta^{[I[M} K^{N]JKL]} \quad (5.5)$$

With (5.1) and (5.5) it is straightforward to write $A_1^{ij}(\alpha, \beta)$ and $A_{2l}^{ijk}(\alpha, \beta)$ in the following form

$$\begin{aligned} A_1^{ij}(\alpha, \beta) &= \frac{1}{3} \alpha T_k^{jki} \\ &+ \frac{1}{6} K^{IJKL} \{ \alpha (u_{IJ}^{kl} + v^{klIJ}) (u_{MK}^{jm} u_{km}^{LM} - v^{jmMK} v_{kmLM}) \} \end{aligned}$$

$$- 8 \beta (u_{IM}^{ki} + v^{kiIM}) (u_{[MJ}^{jm} v_{kmKL]} - v^{jm[MJ} u_{km}^{KL]})^{(ij)} \quad (5.6)$$

$$\begin{aligned} A_{2\ell}^{ijk}(\alpha, \beta) = & -\frac{7}{3} \alpha T_{\ell}^{[ijk]} \\ & - \frac{1}{2} K^{IJKL} \{ \alpha (u_{IJ}^{ij} + v^{ijIJ}) (u_{MK}^{km} u_{\ell m}^{LM} - v^{kmMK} v_{\ell mLM}) \\ & - 2 \alpha (u_{\ell m}^{IJ} + v_{\ell mIJ}) (u_{MK}^{ij} v^{kmLM} - v^{ijMK} u_{LM}^{km}) \\ & - 8 \beta (u_{IM}^{ij} + v^{ijIM}) (u_{[MJ}^{km} v_{\ell mKL]} - v^{km[MJ} u_{\ell m}^{KL]}) \\ & + 16 \beta (u_{\ell m}^{IM} + v_{\ell mIM}) (u_{[MJ}^{ij} u_{KL}^{km} - v^{ij[MJ} v_{\ell mKL]}) \}^{[ijk]} \end{aligned} \quad (5.7)$$

where the terms in parenthesis are symmetrized in (ij) and antisymmetrized in $[ijk]$, respectively. In deriving (5.7) we dropped terms containing Kronecker symbols, because we have already shown that $A_{2\ell}^{ijk}(\alpha, \beta)$ is traceless (cf. (4.20)). Furthermore we used the relation [1]

$$(v^{klIJ} u_{KJ}^{ij} - u_{IJ}^{kl} v^{ijKJ}) (u_{mn}^{IK} + v_{mnIK}) = -\frac{4}{3} \delta_{[m}^{[i} T_{n]}^{jkl]} \quad (5.8)$$

According to (5.6) and (5.7) both A_1 and A_2 decompose into two terms: the first one is just proportional to the T-tensor, and remarkably enough, the proportionality constants appear in the same ratio as in gauged N=8 supergravity. Precise agreement is obtained for

$$\alpha = \frac{4}{7}, \quad \beta = \frac{3}{28}, \quad (5.9)$$

where β follows from α through (3.17). The second terms are proportional to K^{IJKL} which is y -dependent. Therefore it remains to show that these

terms cancel for α and β given by (5.9). to prove this one must exploit the fact that u and v constitute an element of E_7 according to (2.18). First consider the identities

$$\begin{aligned} & u^{ik}_{IJ} u^{KL}_{jk} + v^{ikKL}_{jkIJ} - \frac{1}{8} \delta^i_j (u^{kl}_{IJ} u^{KL}_{kl} + v^{klKL}_{klIJ}) \\ &= \frac{2}{3} \delta^i_j [K(u^{ik}_{JM} u^{LJ}_{jk} + v^{ikLJ}_{jkJM}) \\ &\quad - \frac{1}{12} \delta^i_j \delta^k_l [K(u^{kl}_{JM} u^{LJ}_{kl} + v^{klLJ}_{klJM})] , \end{aligned} \quad (5.10)$$

$$\begin{aligned} & u^{ik}_{IJ} v^{jkKL} + u^{ik}_{KL} v^{jkIJ} - \frac{1}{8} \delta^i_j (u^{kl}_{IJ} v^{klKL} + u^{kl}_{KL} v^{klIJ}) \\ &= -\frac{1}{24} \epsilon_{IJKLMNPQ} (2u^{MN}_{jk} v^{ikPQ} - \frac{1}{4} \delta^i_j u^{MN}_{kl} v^{klPQ}) . \end{aligned} \quad (5.11)$$

Although these relations look similar to the ones given in section 4 of [1], they are in fact different in that the role of $SU(8)$ and E_7 (or rather $SO(8)$) indices has been interchanged. Using (5.10) one easily derives

$$\begin{aligned} & u^{\lambda i}_{MN} (u^{IJ}_{\lambda k} v^{jkKL}) + u^{\lambda j}_{MN} (u^{KL}_{\lambda k} v^{ikIJ}) \\ &= - (u^{IJ}_{\lambda k} u^{ik}_{MN} + v_{\lambda kMN} v^{ikIJ}) v^{\lambda jKL} \\ &\quad - (u^{KL}_{\lambda k} u^{jk}_{MN} + v^{jkKL}_{\lambda kMN} v^{\lambda iIJ}) \\ &= -\frac{2}{3} \delta^i_j [I(u^{J]P}_{\lambda k} u^{ik}_{N]P} + v_{\lambda kMN} v^{ikJ]P}) v^{\lambda jKL} \\ &\quad -\frac{2}{3} \delta^i_j [K(u^{L]P}_{\lambda k} u^{jk}_{N]P} + v^{jkL]P}_{\lambda kMN} v^{\lambda iIJ}) \end{aligned}$$

$$\begin{aligned}
 & - v^{ijKL} \left(\frac{1}{8} (u_{kl}^{IJ} u_{MN}^{kl} + v_{klMN} v^{klIJ}) - \frac{1}{12} \delta_{[M}^{[I} (u_{kl}^{J]P} u_{N]P}^{kl} + v_{klN]P} v^{klJ]P}) \right) \\
 & + v^{ijIJ} \left(\frac{1}{8} (u_{kl}^{KL} u_{MN}^{kl} + v_{klMN} v^{klKL}) - \frac{1}{12} \delta_{[M}^{[K} (u_{kl}^{L]P} u_{N]P}^{kl} + v_{klN]P} v^{klL]P}) \right) , \\
 \end{aligned} \tag{5.12}$$

$$\begin{aligned}
 & v^{\lambda i MN} (v_{\lambda k IJ} u_{KL}^{jk}) + v^{\lambda j MN} (v_{\lambda k KL} u_{IJ}^{ik}) \\
 & = -(u_{\lambda k}^{MN} u_{IJ}^{ik} + v_{\lambda k IJ} v^{ikMN}) u_{KL}^{\lambda j} \\
 & \quad - (u_{\lambda k}^{MN} u_{KL}^{jk} + v_{\lambda k KL} v^{jkMN}) u_{IJ}^{\lambda i} \\
 & = -\frac{2}{3} \delta_{[I}^{[M} (u_{\lambda k}^{N]P} u_{J]P}^{ik} + v_{\lambda k J]P} v^{ikN]P}) u_{KL}^{\lambda j} \\
 & \quad - \frac{2}{3} \delta_{[K}^{[M} (u_{\lambda k}^{N]P} u_{L]P}^{jk} + v_{\lambda k L]P} v^{ikN]P}) u_{IJ}^{\lambda i} \\
 & - u_{KL}^{ij} \left(\frac{1}{8} (u_{kl}^{MN} u_{IJ}^{kl} + v_{kl} v^{klMN}) - \frac{1}{12} \delta_{[I}^{[M} (u_{kl}^{N]P} u_{J]P}^{kl} + v_{kl J]P} v^{klN]P}) \right) \\
 & + u_{IJ}^{ij} \left(\frac{1}{8} (u_{kl}^{MN} u_{KL}^{kl} + v_{kl KL} v^{klMN}) - \frac{1}{12} \delta_{[K}^{[M} (u_{kl}^{N]P} u_{L]P}^{kl} + v_{kl L]P} v^{klN]P}) \right) \\
 \end{aligned} \tag{5.13}$$

Symmetrizing (5.12) and (5.13) over (ij), contracting over I and M, and antisymmetrizing in JNKL leads to two simple identities

$$\begin{aligned}
 & \{ u_{MN}^{\lambda i} (u_{\lambda k}^{MJ} v^{jkKL} + u_{\lambda k}^{KL} v^{jkMJ}) \\
 & \quad + \frac{2}{3} v^{\lambda i KL} (u_{\lambda k}^{JM} u_{NM}^{ik} + v_{\lambda k NM} v^{ikJM}) \} (ij)_{[NJKL]} = 0 ,
 \end{aligned}$$

$$\begin{aligned} & \{v^{\lambda i M N} (v_{\lambda k M J} u^{j k}_{KL} + v_{\lambda k K L} u^{j k}_{M J}) \\ & + \frac{2}{3} u^{\lambda j}_{KL} (u^{\lambda k}_{NM} u^{i k}_{J M} + v_{\lambda k J M} v^{i k N M})\} \frac{(i j)}{[N J K L]} = 0 . \end{aligned} \quad (5.15)$$

Substitution of these identities into (5.6) gives

$$\begin{aligned} A_1^{i j}(\alpha, \beta) &= \frac{1}{3} \alpha T_k^{j k i} \\ &+ \frac{1}{6} K^{I J K L} \left\{ \left(\frac{3}{2} \alpha - 8 \beta \right) (u^{k i}_{I M} + v^{k i I M}) (u^{j m}_{[K L} v_{k m M J]} - v^{j m [K L} u_{k m}^{M J]}) \right. \\ &\quad \left. - \frac{3}{2} \alpha (u^{k i}_{I M} - v^{k i I M}) (u^{j m}_{[K L} v_{k m M J]} + v^{j m [K L} u_{k m}^{M J]}) \right\} \frac{(i j)}{[N J K L]} \end{aligned} \quad (5.16)$$

According to (5.11) the combination $\bar{u}v + \bar{v}u$ that appears in the last term is antiselfdual in indices $[KLMJ]$ (modulo terms that vanish by virtue of the (ij) symmetrization). Because K^{IJKL} is a selfdual tensor, one can prove that their contraction over JKL is symmetric in IM . Consequently the contribution of the last term in (5.16) vanishes as it is multiplied by a tensor $u-v$ which is antisymmetric in IM . Choosing $\frac{3}{2} \alpha - 8 \beta = 0$ cancels the other y -dependent term, and this yields precisely the values for α and β given in (5.9). Hence with these values we have shown that

$$A_1^{i j}(\alpha, \beta) = \frac{4}{21} T_k^{j k i} \quad (5.17)$$

is y -independent, and coincides with the result for gauged $N=8$ supergravity [1].

We now turn to the expression (5.7) for $A_{2\lambda}^{i j k}(\alpha, \beta)$. For reasons already explained we may consistently ignore all terms containing Kronecker symbols with $SU(8)$ indices in the subsequent manipulations.

Using (5.10) one easily derives the following two identities

$$\begin{aligned}
 & (u^{kp}_{IJ} + v^{kpIJ})(u^{[KL}_{\lambda p} v^{ijMN]} + (u^{IJ}_{\lambda p} + v_{\lambda pIJ})(v^{kp[KL}_{\lambda p} v^{ijMN]}) \\
 & = \frac{2}{3} v^{ij[KL} \delta^M_{[I} (u^{N]P}_{\lambda p} u^{kp}_{J]P} + v^{kpN]P} v_{\lambda pJ]P}) \\
 & + v^{ij[KL} (u^{MN}_{\lambda p} v^{kpIJ} + v^{kpMN}] u^{IJ}_{\lambda p}) \quad (5.18)
 \end{aligned}$$

$$\begin{aligned}
 & (u^{kp}_{IJ} + v^{kpIJ})(v_{\lambda p[KL} u^{ij}_{MN]}) + (u^{IJ}_{\lambda p} + v_{\lambda pIJ})(u^{ij}_{[KL} u^{kp}_{MN]}) \\
 & = \frac{2}{3} u^{ij}_{[KL} \delta^I_M (u^{kp}_{N]P} u^{J]P}_{\lambda p} + v_{\lambda pN]P} v^{kpJ]P}) \\
 & + u^{ij}_{[KL} (u^{kp}_{MN}] v_{\lambda pIJ} + v_{\lambda pMN}] u^{kp}_{IJ}) \quad (5.19)
 \end{aligned}$$

The difference of these equations, contracted over I and K and antisymmetrized over JLMN leads to

$$\begin{aligned}
 & (u^{kp}_{IJ} + v^{kpIJ})(v_{\lambda p[IL} u^{ij}_{MN]} - u^{[IL}_{\lambda p} v^{ijMN]}) \\
 & + (u^{IJ}_{\lambda p} + v_{\lambda pIJ})(u^{ij}_{[IL} u^{kp}_{MN]} - v^{kp[IL}_{\lambda p} v^{ijMN]}) \\
 & + \frac{1}{3} (u^{ij}_{MN} + v^{ijMN})(u^{kp}_{JP} u^{LP}_{\lambda p} + v^{kpLP}_{\lambda p} v_{\lambda pJP}) \\
 & + \frac{1}{2} (u^{ij}_{IJ} + v^{ijIJ})(u^{kp}_{[IL} v_{\lambda pMN]} - v^{kp[IL}_{\lambda p} u^{MN]}) \\
 & + \frac{1}{2} (u^{ij}_{IJ} - v^{ijIJ})(u^{kp}_{[IL} v_{\lambda pMN]} + v^{kp[IL}_{\lambda p} u^{MN]}) = 0 \quad (5.20)
 \end{aligned}$$

When contracting this result with K^{JLMN} the last term vanishes because of antiselfduality (cf. the argument following(5.16)). For the same reason

the first term in (5.20) contains only the selfdual part of $\bar{u}v - \bar{v}u$, for which we can use another identity that follows from the E_7 Lie algebra:

$$\begin{aligned} & \{ (v_{\ell p IL} u^{ij}_{MN} - u_{\ell p}^{IL} v^{ijMN}) \\ & - \frac{2}{3} \delta_{[\ell}^{[i} (v_{p]m IL} u^{j]m}_{MN} - u_{p]m}^{IL} v^{j]mMN}) \\ & + \frac{1}{12} \delta_{\ell p}^{ij} (v_{mn IL} u^{mn}_{MN} - u_{mn}^{IL} v^{mnMN}) \} [ILMN]^+ = 0 \end{aligned} \quad (5.21)$$

where $[ILMN]^+$ indicates that we consider only the selfdual components.

Combining (5.20) and (5.21) we thus find

$$\begin{aligned} & K^{JLMN} \{ (u_{\ell p}^{IJ} + v_{\ell p IJ}) (u^{ij}_{[IL} u^{kp}_{MN]} - v^{kp[IL} v^{ijMN]}) \\ & + \frac{1}{6} (u^{ij}_{IJ} + v^{ijIJ}) (u^{kp}_{[IL} v_{\ell p MN]} - v^{kp[IL} v^{ijMN]}) \\ & + \frac{1}{3} (u^{ij}_{MN} + v^{ijMN}) (u^{kp}_{JP} u_{\ell p}^{LP} + v^{kpLP} v_{\ell p JP}) \} [ijk] = 0 \end{aligned} \quad (5.22)$$

Substituting this result into (5.7) leads to the following expression for $A_{2\ell}^{ijk}(\alpha, \beta)$:

$$\begin{aligned} A_{2\ell}^{ijk}(\alpha, \beta) = & -\frac{7}{3} \alpha T_{\ell} [ijk] \\ & - \frac{1}{2} K^{IJKL} \{ (-\alpha + \frac{16}{3}\beta) (u^{ij}_{IJ} + v^{ijIJ}) (u^{km}_{KM} u_{\ell m}^{LM} + v^{kmLM} v_{\ell m KM}) \\ & + 2 \alpha (u_{\ell m}^{IJ} + v_{\ell m IJ}) (u^{ij}_{KM} v^{kmLM} - v^{ijKM} u_{LM}^{km}) \\ & - \frac{32}{3} \beta (u^{ij}_{IM} + v^{ijIM}) (u^{km}_{[MJ} v_{\ell m KL]} - v^{km[MJ} u_{\ell m}^{KL]}) \} \end{aligned} \quad (5.23)$$

For the remaining part of the proof we need two more relations which follow from the E_7 Lie algebra

$$\begin{aligned} u^{[ij}_{IJ} v^{kl]KL} - \frac{1}{24} \epsilon^{ijklmnpq} v_{mnIJ} u^{KL}_{pq} = \\ = \frac{2}{3} \delta^{[K}_{[I} (u^{ij}_{J]M} v^{kl]L]M} - \frac{1}{24} \epsilon^{ijklmnpq} v_{mnJ]M} u^{L]M}_{pq}) , \end{aligned} \quad (5.24)$$

$$\begin{aligned} u^{[ij}_{IJ} u^{kl]}_{KL} - \frac{1}{24} \epsilon^{ijklmnpq} v_{mnIJ} v_{pqKL} = \\ = \frac{1}{24} \epsilon_{IJKLMNPQ} \left\{ \frac{1}{24} \epsilon^{ijklmnpq} u^{MN}_{mn} u^{PQ}_{pq} - v^{[ijMN} v^{kl]PQ} \right\} , \end{aligned} \quad (5.25)$$

Proceeding in analogy with (5.12) and (5.13) we exploit (5.10), (5.11), (5.18) and (5.19) to obtain

$$\begin{aligned} & \left\{ \frac{1}{2} v^{kpMN} (u^{KL}_{\lambda p} u^{ij}_{IJ} + v_{\lambda pIJ} v^{ijKL}) \right. \\ & \quad \left. + \frac{1}{24} \epsilon^{ijkprstu} v_{rsIJ} u^{KL}_{tu} u^{MN}_{\lambda p} \right\} [ijk] \\ & = \left\{ \frac{1}{2} (u^{MN}_{\lambda p} u^{kp}_{IJ} + v_{\lambda pIJ} v^{kpMN}) v^{ijKL} \right. \\ & \quad \left. + \frac{1}{2} (u^{MN}_{\lambda p} v^{kpKL} + u^{KL}_{\lambda p} v^{kpMN}) u^{ij}_{IJ} \right\} [ijk] \\ & \quad - u^{MN}_{\lambda p} (u^{[ij}_{IJ} v^{kp]KL} - \frac{1}{24} \epsilon^{ijkprstu} v_{rsIJ} u^{KL}_{tu}) \\ & = \left\{ \frac{1}{3} \delta^{[M}_{[I} (u^{N]P}_{\lambda p} u^{kp}_{J]P} + v_{\lambda pJ]P} v^{kpN]P}) v^{ijKL} \right. \\ & \quad \left. + u^{[MN}_{\lambda p} v^{kpKL]} u^{ij}_{IJ} \right\} [ijk] \end{aligned}$$

$$- \frac{2}{3} \delta_{[I}^{[K} (u^{ij}_{J]P} v^{kpL]P} - \frac{1}{24} \epsilon^{ijkprstu} v_{rsJ]P} u_{tu}^{L]P} u_{\lambda p}^{MN} , \quad (5.26)$$

$$\begin{aligned} & \left\{ \frac{1}{2} u_{MN}^{kp} (v_{\lambda p IJ} v^{ijKL} + u_{\lambda p}^{KL} u_{IJ}^{ij}) \right. \\ & \quad \left. + \frac{1}{24} \epsilon^{ijkprstu} v_{rsIJ} u_{tu}^{KL} v_{\lambda pMN} \right\} [ijk] \\ & = \left\{ \frac{1}{2} (v_{\lambda pMN} u_{IJ}^{kp} + v_{\lambda p IJ} u_{MN}^{kp}) v^{ijKL} \right. \\ & \quad \left. + \frac{1}{2} (v_{\lambda pMN} v^{kpKL} + u_{\lambda p}^{KL} u_{MN}^{kp}) u_{IJ}^{ij} \right\} [ijk] \\ & \quad - v_{\lambda pMN} (u_{IJ}^{[ij} v^{kp]KL} - \frac{1}{24} \epsilon^{ijkprstu} v_{rsIJ} u_{tu}^{KL}) \\ & = \left\{ \frac{1}{3} \delta_{[M}^{[K} (v_{\lambda pN]P} v^{kpL]P} + u_{\lambda p}^{L]P} u_{N]P}^{kp} \right\} u_{IJ}^{ij} \\ & \quad + v_{\lambda p[MN} u_{IJ}^{kp} v^{ijKL} \} [ijk] \\ & \quad - \frac{2}{3} \delta_{[I}^{[K} (u^{ij}_{J]P} v^{kpL]P} - \frac{1}{24} \epsilon^{ijkprstu} v_{rsJ]P} u_{tu}^{L]P}) v_{\lambda pMN} , \quad (5.27) \end{aligned}$$

where we have suppressed Kronecker symbols with index λ . Upon contraction with δ_K^I and antisymmetrization over the remaining indices one can show that the first terms on both sides of (5.26) and (5.27) become identical. To see this one needs (4.7) of the second work of [1]. Using (4.8) of the same reference, one subsequently removes the ϵ -tensors, and is left with two identities

$$\begin{aligned} & \{ u_{\lambda p}^{IL} v^{kpMN} u_{IJ}^{ij} - u_{IJ}^{kp} v^{ijIL} u_{\lambda p}^{MN} \} [JLMN] = 0 \\ & \{ v_{\lambda p IJ} u_{MN}^{kp} v^{ijIL} - u_{IJ}^{ij} v^{kpIL} v_{\lambda pMN} \} [LJMN] = 0 \quad (5.28) \end{aligned}$$

Using these identities it is straightforward to show that

$$\begin{aligned}
 & \{ (u_{IM}^{ij} + v^{ijIM}) (u_{[MJ}^{km} v_{\ell mKL]} - v^{km[MJ} u_{\ell m}^{KL]}) \\
 & - (u_{\ell m}^{IJ} + v_{\ell mIJ}) (u_{KM}^{ij} v^{kmLM} - v^{ijKM} u_{LM}^{km}) \\
 & - (u_{IM}^{ij} - v^{ijIM}) (u_{[MJ}^{km} v_{\ell mKL]} + v^{km[MJ} u_{\ell m}^{KL]}) \} \frac{[ijk]}{[IJKL]} = 0
 \end{aligned} \tag{5.29}$$

Substituting this result into (5.23) the last term in (5.29) does not contribute because of anti-selfduality, so that all terms in (5.23) proportional to K^{IJKL} cancel provided that $3\alpha = 16\beta$. This yields the values given in (5.9), so that we have established that also $A_{2\ell}^{ijk}$ is y -dependent for this choice of parameters and equal to

$$A_{2\ell}^{ijk}(\alpha, \beta) = -\frac{4}{3} T_{\ell}^{[ijk]} \tag{5.30}$$

and coincides with the result for gauged N=8 supergravity [1].

6. - FIELD EQUATIONS

With the proof that there exist values of α and β which render the tensors $A_1^{ij}(\alpha, \beta)$ and $A_{2\ell}^{ijk}(\alpha, \beta)$ y -independent, we have established that all supersymmetry transformations can be truncated consistently to a massless supermultiplet. As the commutator of two supersymmetry transformations yields the (full) field equations, it is clear that also the field equations must be consistent in the sense that the field equations of the truncated theory coincide with the original $d=11$

equations, without further constraints. The latter implies that the truncated equations must either be y-independent, or the y-dependence must factorize into a common factor. In this section we will explicitly demonstrate in a few examples, how the field equations of d=11 supergravity as derived in [22] become entirely equivalent to those of gauged N=8 supergravity after the truncation.

The first example is the Einstein equation, which, in the formulation of [22], reads

$$\begin{aligned}
 R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = & -\frac{1}{6} A_{\mu}^{ABCD} A_{\nu ABCD} + \\
 & + \frac{1}{12} g_{\mu\nu} A_{\rho}^{ABCD} A_{ABCD}^{\rho} - 32 G_{\mu\rho}^{-AB} G_{\nu}^{+\rho}{}_{AB} \\
 & - \frac{1}{8} g_{\mu\nu} [e^{mAB} e^{nCD} D_m A_{nABCD} - \\
 & - e^{mAB} e_{CD}^n A_m^{CDEF} A_{nEFAB} + \frac{1}{12} e^{mAB} e_{AB}^n A_m^{CDEF} A_{nCDEF}] \quad (6.1)
 \end{aligned}$$

where we have already omitted all terms that trivially vanish in the truncation on account of the Killing condition $\overset{\circ}{D}_m B_{\mu}^m = 0$ or $\partial_m e_{\mu}^{\alpha} = 0$. The term in brackets in (6.1) is the only one whose y-independence is not obvious. Comparison with gauged N=8 supergravity suggests that this term is nothing but the scalar field potential [1]

$$P(v) = g^2 \left\{ \frac{3}{4} |A_1^{ij}|^2 - \frac{1}{24} |A_2^{ijkl}|^2 \right\} \quad (6.2)$$

To verify this assertion, we proceed "backwards" by substituting (4.17) and (4.19) into (6.2). This yields

$$\frac{3}{4} g^2 |A_1^{ij}|^2 = \frac{3}{64} e^{mij} B_{mj}^k B_{nk}^\lambda e_{\lambda j}^n + \frac{3}{64} e_{\lambda i}^n B_{mj}^i e^{mjk} B_{nk}^\lambda \quad (6.3)$$

and

$$\begin{aligned} & - \frac{1}{24} g^2 |A_{2jkl}^i|^2 \\ & = - \frac{3}{64} \{ e^{m[ij} B_{m\lambda}^{k]} - e_{pq}^m A_m^{pq[ij} \delta_{\lambda}^{k]} + \frac{4}{3} A_m^{ijkp} e_{\lambda p}^m \} \\ & \quad \times \{ e_{ij}^n B_{nk}^\lambda - e^{nrs} A_{nrsij} \delta_k^\lambda + \frac{4}{3} A_{nijkr} e^{n\lambda r} \} \end{aligned} \quad (6.4)$$

Combining (6.3) and (6.4), using (4.16) and the selfduality of A_m^{ijkl} , we obtain, after a little rearrangement,

$$\begin{aligned} P(\psi) = & - \frac{1}{8} \{ 2e_{[ij}^m e_{kl]}^n B_{mp}^i A_n^{p jkl} + \\ & - e^{mij} e_{kl}^n A_m^{klpq} A_{npqij} + \frac{1}{12} e^{mpq} e_{pq}^n A_m^{ijkl} A_{nijkl} \} \end{aligned} \quad (6.5)$$

The first term in parentheses is just the remnant of the full SU(8) covariant derivative which appears in the brackets in (6.1) after the truncation. In this way, we see that the term in brackets in (6.1) reduces indeed to the scalar potential (6.2) in the truncation. Observe that the y-independence of A_1^{ij} and A_{2jkl}^i is a crucial ingredient in this argument and that we get the correct normalization.

As a second example, consider (7.7) of [22]. In the truncation, it becomes

$$\begin{aligned} \hat{F}_{\mu\nu i}^j + \frac{2}{3} A_{[\mu i k \ell m} A_{\nu]}^{j k \ell m} = \\ = \sqrt{2} e_{ik}^m B_m^{[k} \bar{F}_{\mu\nu}^{-j] \ell} - \sqrt{2} e_{ik}^m A_m^{j k p q} \bar{F}_{\mu\nu p q} + \text{(h.c.; trace)} \end{aligned} \quad (6.6)$$

where we have again neglected terms that manifestly vanish in the S^7 truncation. The caret in (6.6) is to indicate that the $SU(8)$ field strength in eleven dimensions contains an extra term, namely

$$\hat{F}_{\mu\nu i}^j = F_{\mu\nu i}^j + 2D_{[\mu} B_{\nu]}^m B_{mi}^j \quad (6.7)$$

where

$$F_{\mu\nu i}^j \equiv \partial_\mu B_{\nu i}^j - \partial_\nu B_{\mu i}^j + \frac{1}{2} [B_\mu, B_\nu]_i^j \quad (6.8)$$

is evidently y -independent. The additional term in (6.7) is needed for $SU(8)$ covariance (see [22]). In (6.6), we now write

$$\frac{1}{2} e_{ik}^m B_{m\ell}^k = \sqrt{2} g A_{1i\ell} + \frac{1}{2} e_{k[i}^m B_{m\ell]}^k \quad (6.9)$$

$$e_{ik}^m A_m^{j k p q} = -\frac{1}{2} \sqrt{2} g A_{2i}^{j p q} - \frac{3}{4} e^{m[pq} B_{mi}^j - \frac{3}{4} e^{mk[p} B_{mk}^q \delta_{i}^j] \quad (6.10)$$

After a little algebra, the right hand side of (6.6) can thus be reexpressed as

$$\begin{aligned} (6.6) = \frac{4}{3} g (T_{ik\ell}^j \bar{F}_{\mu\nu}^{-k\ell} - T_i^{j k \ell} \bar{F}_{\mu\nu k \ell} + \\ - \frac{\sqrt{2}}{4} (e_{k\ell}^m \bar{F}_{\mu\nu}^{-k\ell} + e^{mk\ell} \bar{F}_{\mu\nu k \ell}) B_{mi}^j \end{aligned} \quad (6.11)$$

Recalling the relation (4.9), we recognize that the term with m in (6.11) coincides with the extra term in (6.7), and we thus arrive at

$$\begin{aligned}
 F_{\mu\nu i}^j + \frac{2}{3} A_{[\mu i k l m} A_{\nu]}^{j k l m} &= \\
 = \frac{4}{3} g \left(T_{i k l}^j \bar{F}_{\mu\nu}^{-k l} - T_i^{j k l} \bar{F}_{\mu\nu k l}^+ \right) &= \\
 = \frac{4}{3} \left(u_{i k}^{IK} u_{JK}^{j k} - v_{i k I K} v^{j k J K} \right) F_{\mu\nu}^{IJ} & \quad (6.12)
 \end{aligned}$$

But this is just the integrability equation that follows from our solution for A_μ and B_μ in (3.13); (6.12) also coincides with (5.6) of [1]. Once more we draw the reader's attention to the fact that the numerical factors come out correctly, too.

The final example to be discussed in detail is the spin-1 field equation. It corresponds to (7.8) of [22], and repeating the by now familiar steps, we obtain in the S^7 truncation

$$\begin{aligned}
 D_\nu \bar{F}^{+\mu\nu}_{ij} + A_{\nu i j k l} \bar{F}^{-\mu\nu k l} &= \\
 = -\frac{\sqrt{2}}{2} e^{m k l} B_{m[i}^p A_{j k l]p}^\mu + \frac{\sqrt{2}}{3} e_{k[i}^m A_{j]l p q}^\mu A_m^{k l p q} & \quad (6.13)
 \end{aligned}$$

Invoking (6.9) and (6.10) once more, it is straightforward to show that all y -dependence cancels and we are left with

$$D_\nu \bar{F}^{+\mu\nu}_{ij} + A_{\nu i j k l} \bar{F}^{-\mu\nu k l} = -\frac{1}{3} g A_2^{p q r} A_{[i}^\mu A_{j] p q r}^\mu \quad (6.14)$$

This equation contains both the field equation and the Bianchi identity for the field strength $F_{\mu\nu}^{IJ}$ as its real and imaginary parts. We emphasize

again that the consistency of these equations crucially relies on the y -independence of A_1^{ij} and A_{2i}^{jkl} .

We could continue with this exercise by analyzing the other equations given in section 7 of [22] but we refrain from doing so because their consistency follows by independent arguments just as for the equations discussed above. Besides it is almost selfevident that (7.6) and (7.9) of [22] are either trivially satisfied or correspond to some of the T-identities derived in [1]. We therefore leave it to the reader to verify some of these assertions.

7. - THE NONLINEAR EMBEDDING

So far, we have only investigated the relation between the new formulation of $d=11$ supergravity [21,22] and gauged $N=8$ supergravity. However, it is also possible to express the fields occurring in the original version [2] in terms of those of $N=8$ supergravity and to derive the full nonlinear embedding. In the linearized approximation the results will coincide with those of [4,5], but we can now also explicitly display the deviations from the linearized behavior for finite fluctuations. In this section, we will briefly describe how to obtain the nonlinear ansätze without aiming at an exhaustive discussion. As it turns out the final results are rather unwieldy expressions (such as (7.6) below), whose usefulness is limited; the diligent reader should be able to complete the arguments wherever he wishes to do so.

The basic idea is best illustrated with the truncated metric which was already given in [15]. To derive it, one simply combines (2.14) with

(2.19) such that

$$8\Delta^{-1} g^{mn} = e^m_{ij} e^{nj} = K^{mIJ} K^{nKL} (u_{ij}^{IJ} + v_{ijIJ}) (u_{KL}^{ij} + v^{ijKL}) \quad (7.1)$$

This expression is already symmetric in m and n owing to the E_7 properties of u and v . From (7.1), the original siebenbein $e_a^m(x,y)$ may be determined up to an $SO(7)$ rotation. In [15], we have "tested" (7.1) by inserting the vacuum expectation values corresponding to $SO(7)^+$, G_2 and $SU(4)^-$ invariant stationary points of the $N=8$ potential [25,9] and verified that the resulting metric indeed coincides with the respective metrics required by the $d=11$ field equations. It is also not difficult to show that (7.1), when expanded to lowest nontrivial order, reproduces the linearized metric ansatz of [4,5].

In [15], it was also shown that the "internal" component F_{mnpq} of the four-index field strength F_{MNPQ} acquires a nonzero value for the G_2 -invariant solution in analogy with Englert's solution [26]. However, we did not give the full nonlinear ansatz for F_{mnpq} there. This expression can now be determined from the results of this paper, as well as similar expressions for the remaining components of F_{MNPQ} . Let us demonstrate this for F_{abcd} and $f \equiv (24i)^{-1} \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta\gamma\delta}$, where F_{abcd} and $F_{\alpha\beta\gamma\delta}$ are the $d=11$ field strength components with flat indices a,b,c,d and $\alpha,\beta,\gamma,\delta$ taking values in the 7- and 4-dimensional subspace, respectively. These components occur in both B_m and A_m [21,22]; because B_m is not $SU(8)$ covariant the determination of F_{abcd} and f from B_m would require knowledge of the $SU(8)$ rotation $\Phi = \Phi(u,v)$. ^{*)} Hence it is more convenient to look at A_m , which, in the gauge $\Phi = 1$, contains the terms [21,23]

^{*)} We recall that $\Phi(u,v)$ is implicitly determined through (2.14), (2.17) and (2.27, up to an $SO(7)$ rotation.

$$A_m^{ABCD} = \dots + \frac{i\sqrt{2}}{56} e_{ma} f \Gamma_{[AB}^a \Gamma_{CD]}^b - \frac{\sqrt{2}}{32} e_m^a F_{abcd} \Gamma_{[AB}^b \Gamma_{CD]}^{cd} \quad (7.2)$$

As A_m is SU(8) covariant we can project out f and F_{abcd} in an SU(8)-invariant manner, just as in (7.1); by SU(8) invariance we are then free to adopt the gauge $\Phi = 1$ as in (7.2). The projection is again accomplished by means of the generalized vielbein. So we consider the SU(8) invariant expression

$$\begin{aligned} e^m_{ij} (e^{[n} e^p e^q e^r e^s])_{kl} A^{ijkl}_v &= \\ &= \frac{\sqrt{2}}{8} \Delta^{-3} e_v^a \left\{ -\frac{1}{7} i f \Gamma_{[AB}^a \Gamma_{CD]}^b + \frac{1}{4} F_{abcd} \Gamma_{[AB}^b \Gamma_{CD]}^{cd} \right\} \Gamma_{AB}^m \Gamma_{CD}^{npqrs} \\ &= \frac{\sqrt{2}}{16} i \Delta^{-3} e_v^a \frac{\epsilon^{npqrstuv}}{\sqrt{g}} \left\{ -\frac{1}{7} i f \Gamma_{[AB}^a \Gamma_{CD]}^b + \frac{1}{4} F_{abcd} \Gamma_{[AB}^b \Gamma_{CD]}^{cd} \right\} \Gamma_{AB}^m (\Gamma_{tn})_{CD} \end{aligned} \quad (7.3)$$

where in the first step we have substituted (2.19) with $\Phi = 1$. It is important here that SU(8) covariance forces us to use a five-fold product of the vielbeins e^m_{ij} , rather than a two-fold one, as the latter does not have SU(8) indices in the required position. Furthermore, note that $d=7$ world indices in (7.3) are related to flat indices by use of the full sevenbein. Using $\sqrt{g} = \Delta \sqrt{\hat{g}}$, where $\hat{g}$ is the determinant of the S^7 metric, a little calculation yields

$$(7.3) = i\sqrt{2} \Delta^{-4} \frac{\epsilon^{npqrstuv}}{\sqrt{\hat{g}}} \left\{ \frac{4}{7} i f g_{v[t} \delta_{u]}^m + \frac{1}{2} F^m_{vtu} \right\} \quad (7.4)$$

Inverting (7.4) leads to

$$\begin{aligned} \frac{4}{7} g_{n[p} \delta_{q]}^m + \frac{1}{2} F^m_{npq} &= \\ &= -\frac{i\sqrt{2}}{480} \Delta^4 \frac{\epsilon_{pqrstuv}}{\sqrt{\hat{g}}} e^m_{ij} (e^r e^s e^t e^u e^v)_{kl} A^{ijkl}_n \end{aligned} \quad (7.5)$$

To obtain the full nonlinear expressions for F_{mnpq} and f one simply exploits the fact that all quantities on the right-hand side can be expressed directly in terms of the 56-bein $\vartheta(x)$ and the S^7 Killing vectors $K^m(y)$: Δ can be computed from (7.1), and e^m and A_m are known from (2.14), (2.17), (3.19) and (5.7). Direct substitution gives

$$\begin{aligned}
 & \frac{4}{7} if(x,y) g_{n[p}(x,y) \delta_{q]}^m + \frac{1}{2} F_{npq}^m(x,y) = \\
 & = - \frac{i\sqrt{2}}{480} \Delta^4(x,y) \frac{\epsilon_{pqrstuv}}{\sqrt{g}} K^{mIJ}(y) K^{rI_1 J_1}(y) K^{sI_2 J_2}(y) \dots K^{vI_5 J_5}(y) \\
 & \quad \times \{ w_{ij}^{IJ}(x) w_{kk_1}^{I_1 J_1}(x) w_{I_2 J_2}^{k_1 k_2}(x) \dots w_{k_4 l}^{I_5 J_5}(x) \} \\
 & \quad \times \{ \frac{4}{7} m_7 K_n^{KL}(y) (v^{ijLM}(x) u^{kl}_{KM}(x) - u^{ij}_{LM}(x) v^{klKM}(x)) \\
 & \quad - \frac{3}{28} D_n^{n'} [K_L(y) K_{n'}^{MN}(y) (u^{ij}_{KL}(x) u^{kl}_{MN}(x) - v^{ijKL}(x) v^{klMN}(x)) \} \\
 & \hspace{25em} (7.6)
 \end{aligned}$$

where $w = u+v$ was defined in (2.17).

It is not obvious that the right-hand side of (7.6) will decompose into the two tensors on the left-hand side, which are real and have a certain symmetry. Nonetheless, our results ensure that this must be the case. In addition f and F_{abcd} must also satisfy the Bianchi identities for the 11-dimensional field strength F_{MNPQ} . However, as explained in [22], the combined Bianchi identities and field equations correspond to a number of $SU(8)$ covariant equations whose validity can be verified directly for the $SU(8)$ covariant fields defined in this paper. We have not made exhaustive attempts in that direction, but we have verified (7.6) for the $SO(8)$ invariant solution of $N=8$ supergravity. In that case

$$u^{ij}_{IJ} = \delta^{ij}_{IJ}, v_{ijIJ} = 0; \text{ a straightforward calculation then leads to}$$

$$\begin{aligned}
 f &= 3\sqrt{2} m_7 \\
 F_{mnpq} &= 0 \\
 g_{mn} &= g_{mn}
 \end{aligned} \tag{7.7}$$

which is the expected result for the sphere S^7 . To further analyse (7.6) it may be convenient to rewrite it by means of the generalized vielbein postulate (3.4) into the form

$$\begin{aligned}
 \frac{4}{7} \text{if } g_{n[p} \delta_{q]}^m + \frac{1}{2} F_{npq}^m &= \\
 &= \frac{i\sqrt{2}}{480} \Delta^4 \frac{\epsilon_{pqrstuv}}{\sqrt{g}} e_{ik}^r e_{skk'}^s e_{k'l}^t e_{ull'}^u e_{l'j}^v D_n e^{mij}
 \end{aligned} \tag{7.8}$$

where D_n is an $SU(8)$ covariant derivative with $SU(8)$ connection $B_{m j}^i$.

Finally the full nonlinear expressions for the remaining field strength components $F_{\alpha\beta\gamma}$, $F_{\alpha abc}$ and $F_{\alpha\beta ab}$ can be obtained in a completely analogous fashion by projecting out the appropriate components in $A_\mu^{ijk\lambda}$ and $\bar{F}_{\alpha\beta}^{-ij}$, using the $d=4$ results for these quantities.

APPENDIX

In this appendix we collect several formulae involving the Killing spinors and vectors defined in (1.3) and (1.4). A useful explicit representation for the Killing spinors on S^7 is provided by the formula [27]

$$\eta_I^I(y) = [\exp(iy_a \Gamma^a)]_I^I \quad (A.1)$$

where y_a are local coordinates on S^7 (in the neighborhood of the northpole, say), and the 8x8 matrices Γ^a generate the d=7 Clifford algebra. From (A.1) it is evident that $\eta(y)$ is an element of $SO(8)$ (or, more precisely, of the coset space $SO(8)/SO(7)$) and thus $\eta(y) \eta^T(y) = \mathbf{1}$ as matrices. This implies the orthonormality relations

$$\begin{aligned} \eta_I^I(y) \eta_J^I(y) &= \delta_{IJ} , \\ \eta_I^I(y) \eta_J^J(y) &= \delta^{IJ} , \end{aligned} \quad (A.2)$$

used in the text (in particular, see (1.8-10)). The fundamental property (1.3) follows directly from (A.1) together with explicit parametrization for the siebenbein $e_m^a(y)$ and spin connection $\omega_m^{ab}(y)$. Furthermore, from the definition (1.5) one verifies for the 28 Killing vectors on S^7 ,

$$\begin{aligned} D_m K_n^{IJ} &= - m_7 \frac{1}{\eta^I} \Gamma_{mn}^I \eta^J \\ &= m_7 K_m^{K[I} K_n^{J]K} , \end{aligned} \quad (A.3)$$

from which the Killing property $D_{(m} K_{n)}^{IJ} = 0$ is manifest. Using the completeness relation for d=7 Γ -matrices [24]

$$- 2\Gamma_{AB}^a \Gamma_{aCD} + \Gamma_{AB}^{ab} \Gamma_{abCD} = 16 \delta_{A[C} \delta_{D]B} \quad (A.4)$$

one can then derive the $SO(8)$ Lie algebra relation (3.7). In terms of Killing vectors (A.4) takes the form

$$2K_m^{IJ} K^{mKL} + m_7^{-2} \overset{\circ}{D}_m K_n^{IJ} \overset{\circ}{D}^m K^{nKL} = 16 \delta^{I[K} \delta^{L]J} \quad (A.5)$$

From (A.4-5) one can then prove (5.1).

The following combinations, together with their duality properties play an important role, as in the linearized analysis of [4,5]

$$\begin{aligned} K_m^{[IJ} K_n^{KL]} &= \frac{1}{24} \varepsilon^{IJKLMNPQ} K_m^{MN} K_n^{PQ}, \\ \overset{\circ}{D}_m K_n^{[IJ} K^{nKL]} &= \frac{1}{24} \varepsilon^{IJKLMNPQ} \overset{\circ}{D}_m K_n^{MN} K^{nPQ}, \\ \overset{\circ}{D}_{[m} K_{n}^{[IJ} K_{p]}^{KL]} &= -\frac{1}{24} \varepsilon^{IJKLMNPQ} \overset{\circ}{D}_{[m} K_{n}^{MN} K_{p]}^{PQ}. \end{aligned} \quad (A.6)$$

The tensor K^{IJKL} defined in (5.2) is thus selfdual, and is invariant under the $SO(7)^+$ subgroup of $SO(8)$. It obeys the relation [9]

$$K^{IJKP} K_{LMNP} = -6\delta_{LMN}^{IJK} - 9\delta_{[L}^{[I} K^{JK]_{MN}} \quad (A.7)$$

which may be derived either by certain Fierz rearrangements or by directly substituting the explicit formula (A.1).

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