A Note On Toroidal Compactification Of Heterotic String Theory

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Abstract
The connection of recently constructed lower dimensional heterotic strings with conventional toroidal compactification is clarified.
In a recent paper, one of us considered a more general scheme for toroidal compactification of superstrings than had been considered hitherto. The purpose of the present note is to clarify this scheme and consider some of the implications.

In ref [1] the following general expansion of left and right moving bosons $X^A$ and $\tilde{X}^I$ was considered:

\[
X^A = q^A + 2L^A u + \sum_{n \neq 0} \frac{1}{2n} a^A_n e^{-2i\nu n} \quad A = 1 \ldots q
\]

\[
\tilde{X}^I = q^I - 2L^I v + \sum_{n \neq 0} \frac{1}{2n} a^I_n e^{-2i\nu n} \quad I = 1 \ldots p
\]

(with $u = r+\sigma$, $v = r-\sigma$), and no a priori assumptions about the left and right moving "momenta" $(L^A, \tilde{L}^I)$ were made. It was then shown that modular invariance requires that two allowed momenta $(L^A, \tilde{L}^I)$ and $(L^A, \tilde{L}^I)$ should obey $L^A L^A - \tilde{L}^I \tilde{L}^I \in 2\mathbb{Z}$; thus $(L^A, \tilde{L}^I)$ and $(L^A, \tilde{L}^I)$ are points in an even lattice with a metric of signature $(q,p)$, i.e. $q$ positive and $p$ negative eigenvalues. It further emerged that this lattice must be unimodular or self-dual.

For $p,q > 0$, such lattices exist only for $q-p = 8n$ and moreover it is unique up to an $SO(q,p)$ transformation. However, it was pointed out that the contribution $L^A L^A$ or $\tilde{L}^I \tilde{L}^I$ of the momenta to the mass of a closed string does not have $SO(q,p)$ symmetry but only $SO(q) \times SO(p)$ symmetry. The possible compactifications achievable in this way thus depend on $\dim SO(q,p) - \dim SO(q) - \dim SO(p) = pq$ parameters. For instance, in the heterotic superstring theory, we take $p$ to be the number of compactified space-time dimension and $q = 16+p$, where 16 is the number of extra left-moving coordinates associated with the Cartan subalgebra of $SO(32)$ or $E_8 \times E_8$.

There are thus $p(16+p)$ parameters. What is their significance? Consider compactifying $p$ of the 10 space-time dimensions in this theory on a torus $T_p$. How many parameters would arise in specifying the vacuum state? $T_p$ must be flat if we are to make contact with Ref. [1]; if $T_p$ is not flat a simple expansion like (1) will not hold. The metric tensor $g_{ij}$, $i,j = 1 \ldots p$ of $T_p$ is a symmetric tensor with $\frac{1}{2} p(p+1)$ components. These must be constant (in some coordinate system with coordinates $X^i$) if $T_p$ is to be flat, but there are still $\frac{1}{2} p(p+1)$ freely adjustable parameters (with
discrete equivalences under "modular" transformations $X^i \rightarrow A^i_j X^j$, where $A$ is an element of $SL(p;\mathbb{Z})$. Also, if we are to find a vacuum solution, the Yang-Mills field strength $F^a_{ij}$ must vanish, but this still leaves the possibility that the Yang-Mills field on $T_p$ has a non-trivial global holonomy corresponding to non-trivial Wilson lines.

As $T_p$ is the product of $p$ circles $(S^1 \times S^1 \times \ldots \times S^1)_p$, there are $p$ independent Wilson lines $U_i$, $i=1 \ldots p$. These must commute, since the fundamental group of $T_p$ is abelian. The $p$ commuting elements $U_i$ of $G = E_8 \times E_8$ or $SO(32)$ can be simultaneously put in the Cartan subalgebra by conjugation by an element of $G$. As the Cartan subalgebra of $G$ is sixteen dimensional, the choice of $p$ elements of the Cartan subalgebra involves $16p$ parameters. (Again, there are discrete equivalences under elements of the Weyl group of $G$). The Wilson lines can be described by giving arbitrary constant values to $A^a_i$, the components in the Cartan subalgebra of the gauge field. So far we have $\frac{1}{2} p(p+1) + 16p$ parameters. However, the 10 dimensional supergravity multiplet also contains a two form $B_{MN}$. Its contribution to the vacuum energy vanishes if the field strength $H = dB + \Gamma$ vanishes. Here $\Gamma$ is the Chern-Simons three form, which vanishes with our ansatz, so $H = 0$ if $dB = 0$ or in other words if $B$ is a constant. A constant $B_{ij}$ would depend on $\frac{1}{2} p(p-1)$ parameters. Adding this to those we had previously, we have in all $p(p+16)$ parameters, the number found in Ref [1].

To show that quantization of the string in the presence of the general $g_{ij}$, $B_{ij}$, and $A^a_i$ fields gives the expansion considered in [1], consider first the role of $g_{ij}$, and $B_{ij}$. Propagation of a closed string on a torus with metric $g_{ij}$ and given expectation value of $B_{ij}$ is described by the action

$$I = \int_0^\pi \rho d\sigma \int d\tau \frac{1}{2\pi} (g_{ij} \partial_\alpha X^i \partial_\beta X^j + \epsilon^{a\beta}_{B_{ij}} \partial_\alpha X^i \partial_\beta X^j)$$  \hspace{1cm} (2)

Here we take the closed string to have circumference $\pi$, and the toroidal coordinates $X^i$ are normalized so that $\{X^i\}$ and $\{X^i + 2\pi k^i\}$ denote the same point in $T_p$ for any integers $\{k^i\}$ (The radius of $T_p$ is absorbed in its metric $g_{ij}$).
The first thing we note about (2) is that $g_{ij}$ and $B_{ij}$ will only affect the quantization of string zero modes. This is well known for $g_{ij}$, so we focus on $B_{ij}$. If $X^i$ were single-valued, the second term in (2) could be (as $B_{ij}$ is constant) the irrelevant total divergence $\partial_a \varepsilon^{\alpha\beta} B_{ij} X^i \partial_{\beta} X^j$. The second term hence plays a role only when the $X^i$ are not periodic. The non-periodic terms in (1) are the $L^A$ and $\tilde{L}^I$ terms, so it is only the determination of $L^A$ and $\tilde{L}^I$ that depends on $B_{ij}$.

To determine the allowed momenta, we write a zero mode ansatz

$$X^i = 2\sigma^i + q^i(\tau)$$

The $\sigma^i$'s label the homotopy class of the closed string in $\pi_1(T_p)$, while $q^i(\tau)$ the center of mass position of the string at time $\tau$. This ansatz in (2) gives

$$I = \int dt \left( \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j + 2B_{ij} \dot{q}^i \sigma^j - 2g_{ij} \dot{\sigma}^i \sigma^j \right)$$

(4)

The canonical momenta are hence

$$P_i = g_{ij} \dot{q}^j + 2B_{ij} \sigma^j$$

(5)

Since the $q^i$ are coordinates for a string moving on the torus $T_p$, the wave function $\psi(q_1, q_2, \ldots, q_p)$ must be periodic, $\psi(q_1, q_2, \ldots, q_p) = \psi(q_1 + 2\pi k_1, q_2 + 2\pi k_2, \ldots, q_p + 2\pi k_p)$, for any integers $k_i$. Hence the canonical momenta $P_i = -i \frac{\partial}{\partial q_i}$ are of the form

$$P_i = m_i$$

(6)

with integers $m_i$. Expressed back in (5), this gives

$$\dot{q}^i = g^{ik} m_k - 2B_{ik} \dot{\sigma}^k$$

with integers $m_k$, $\dot{\sigma}^k$.
In (7) \( q^i = 0 \) so the center of mass of the string travels on a straight line \( q^i(\tau) = q^i + \tau q^i \). Thus, (7) can be substituted in (3) to give

\[
\dot{x}^i = q^i + 2\sigma n^i + \tau (g^{ik} m_k - 2\beta n^k)
\]  

(8)

To make contact now with formulas in ref [1], we must write \( x^i = \frac{1}{2} (x^i_L + x^i_R) \) where \( x^i_L \) and \( x^i_R \) (called \( x^i \) and \( x^i \) in (1)) are functions of \( u = \tau + \sigma \), \( v = \tau - \sigma \), respectively. We get

\[
x^i_L = q^i + u (2n^i + g^{ik} m_k - 2g^{ij} \beta_{jk} n^k)
\]

\[
x^i_R = q^i + v(-2n^i + g^{ik} m_k - 2g^{ij} \beta_{jk} n^k)
\]

(9)

Therefore, the allowed left and right moving momenta are

\[
\begin{align*}
L^i &= n^i + \frac{1}{2} g^{ik} m_k - g^{ij} \beta_{jk} n^k \\
\bar{L}^i &= -n^i + \frac{1}{2} g^{ik} m_k - g^{ij} \beta_{jk} n^k
\end{align*}
\]

(10)

for arbitrary integers \( n^i \), \( m_j \). Given two such points \( L = (L^i, \bar{L}^i) \) and \( L' = (L'^i, \bar{L}'^i) \) determined by integers \( (n^i, m_j) \) and \( (n'^i, m'_j) \) the inner product is

\[
L \cdot L' = \sum_{ij} (L^i L'^j - \bar{L}^i \bar{L}'^j) = \sum (n^i m'_j + n'^i m_j)
\]

(11)

Thus, the momenta lie in an integral lattice. Setting \( L = L' \), (11) shows that \( L \cdot L \) is even, so the lattice is even. Moreover, it is readily seen that this lattice is described by \( p \) copies of the unimodular matrix \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) (one copy for each of the \( p \) dimensions of \( T_p \)), so we have in fact constructed an even unimodular lattice of signature \( (p, p) \). If we include 16 left moving bosons propagating in the maximal torus of \( E_8 \times \mathbb{Z}_2 \) or \( \text{spin}(32) \) then the resulting \((16+p, p)\) lattice would still be unimodular.

We now discuss the role of the background gauge field \( A^i_I \). In the \( E_8 \times \mathbb{Z}_2 \) and \( \text{SO}(32) \) heterotic strings, the gauge group can be represented either by \( 32 \)
Weyl fermionic coordinates $\psi^{a}(\tau+\sigma)$ or by 16 left moving bosonic coordinates $X^{I}(\tau+\sigma)$. To parallel the discussion of $B_{ij}$ we will use the latter ones in which case the action is given by

$$I = \frac{1}{2\pi} \int d\tau d\sigma \left[ g_{ij} \partial_{\tau} x^{i} \partial_{\sigma} x^{j} + \partial_{\tau} x^{I} \partial_{\sigma} x^{I} + \varepsilon^{I}_{\alpha} A_{I}^{I} \partial_{\tau} x^{I} \partial_{\sigma} x^{I} \right]$$  \hspace{1cm} (12)

with the constraint $C^{I} \equiv (\partial_{\tau} - \partial_{\sigma}) x^{I} = 0$. The canonical momenta are

$$P_{i}(\sigma) = \frac{1}{\pi} \left( g_{ij} \partial_{\tau} x^{j} - \frac{1}{2} A_{I}^{I} \partial_{\sigma} x^{I} \right)$$  \hspace{1cm} (13)

and

$$P^{I}(\sigma) = \frac{1}{\pi} \left( \partial_{\tau} x^{I} + \frac{1}{2} A_{I}^{I} \partial_{\sigma} x^{I} \right).$$  \hspace{1cm} (14)

The constraints being of second class we need to introduce Dirac bracket as was also done in ref [2] in their discussion of ten dimensional heterotic string. It is easy to show that the momenta satisfy the following Dirac brackets:

$$\{P^{I}(\sigma), P^{J}(\sigma')\}_{DB} = \frac{1}{2\pi} \partial_{\sigma} \delta(\sigma - \sigma') \delta^{IJ}$$  \hspace{1cm} (15)

$$\{P_{i}(\sigma), P_{j}(\sigma')\}_{DB} = \frac{1}{8\pi} A_{I}^{I} A_{j}^{I} \partial_{\sigma} \delta(\sigma - \sigma')$$  \hspace{1cm} (16)

$$\{P^{I}(\sigma), P_{i}(\sigma')\}_{DB} = \frac{1}{4\pi} A_{I}^{I} \partial_{\sigma} \delta(\sigma - \sigma')$$  \hspace{1cm} (17)

The Dirac brackets involving the coordinates are the usual ones. As can be inferred from these, the momentum conjugate to $X^{I}(\sigma)$ is

$$P_{I}(\sigma) = P_{i}(\sigma) - \frac{1}{2} A_{I}^{I} P^{I}(\sigma)$$  \hspace{1cm} (18)

Therefore $q^{i}$ and $q^{I}$ in the expansion of $X^{i}$ and $X^{I}$ are given by the zero modes in (13) and (14):

$$q^{i} = g^{ij}(P_{j} + A_{j}^{I} (P^{I} - \frac{1}{2} A_{k}^{I} n^{k})))$$  \hspace{1cm} (19)

and

$$q^{I} = P^{I} - A_{I}^{I} n^{I}$$  \hspace{1cm} (20)

*Note that the last term is the bosonized version of $\psi_{\alpha} \psi_{\beta} x^{I}$. One could choose either $\alpha^{I}_{\alpha}$ or $\epsilon^{I}_{\alpha}$ in that term, however the difference is proportional to the constraint. Here we have chosen $\epsilon^{I}_{\alpha}$ for convenience.*
To make contact with ref. [1] one can again obtain the left moving and right moving momenta and show that the resulting lattice is self-dual with signature $(16+p,p)$. Note that in establishing this result the shift in momenta given in (19) was crucial. The discussion above makes it clear that this was due to the fact that the 16 dimensional bosonic coordinates $X^I$ were constrained to be left moving only. As we will see below, when we discuss the path integral approach using fermionic representation, this shift will manifest itself in the anomalies associated with chiral fermions.

In ref [1] it was shown how compactification to $\text{dim} < 10$ can result in the enlargement of the groups. It is quite interesting that the introduction of Wilson lines can achieve this enlargement. To illustrate this let us consider the simple case of the compactification of the 10-dimensional spin $(32)/Z_2$ heterotic string to $M^9\times S^1$, in which the $S^1$ has the periodicity $x+nx+n\pi$. The momentum is then some even integer $2m$. To get $SO(34)$ gauge group we take $A^I = (2,0,0,...,0)$. Using Eqs. (19) and (20) for the shifts in the momenta we can write the mass formula and the constraint for the massless states as:

$$n^2 = \frac{1}{2} (P^I - \frac{1}{2} nA^I)^2 + \frac{1}{4} k^2 + \frac{1}{4} n^2 - 1 = 0,$$

$$\frac{1}{2} kn + \frac{1}{2} (P^I - \frac{1}{2} n A^I)^2 = 1,$$  \hspace{1cm} (21)

where $k$ is the shifted momentum

$$k = 2m + A^I P^I = n.$$

$$\hspace{1cm} (22)$$

When $n = 0$ and $P$ is a root of $SO(32)$ one can set $k=0$ giving massless states corresponding to roots of $SO(32)$. When $n=1$, $k$ must be odd from Eq. (23). Therefore to get massless states we choose $k=1$. Moreover choosing $P$ to be in the scalar conjugacy class of $SO(32)$ we see that $P - \frac{1}{2} A$ gives the vector conjugacy class. Taking $P - \frac{1}{2} A$ to be the fundamental vectors of $SO(32)$ we again get massless states. Including also the case $n = -1$ we then get two copies of $SO(32)$ vectors that are massless. Adding the 17 massless states.
corresponding to the U(1) generators we obtain the full SO(34) adjoint representation.

It is instructive to see how the above results in the canonical approach can be reproduced in the path integral formulation. First consider the motion of a string on $\tau^P = \frac{R^P}{2\pi}$ in the absence of $A_1$ and $B_{i\bar{j}}$. Let $ds^2 = e^\phi|d\sigma + \tau dt|^2$ be the metric of the world sheet torus where $\sigma, t \in [0, 1]$ and $\tau$ is a complex parameter with $\text{Im}\tau > 0$. We are interested in the contribution of the zero mode $X^i_0 = 2\pi \delta^i_0 + 2\pi \delta^i_0 \tau (\bar{W}, \bar{W}' \in \Lambda)$ to the path integral. It is given by

$$
\sum_{\bar{W}, \bar{W}' \in \Lambda} e^{-I_0} = \sum_{\bar{W}, \bar{W}' \in \Lambda} \exp\left\{ -\frac{1}{2\pi} \int d^2 \sigma \ g^{\alpha\bar{\beta}} \partial_\sigma X^\alpha \partial_\bar{\sigma} X^\bar{\beta} \right\}
$$

$$
= \sum_{\bar{W}, \bar{W}' \in \Lambda} \exp\left\{ -2\pi (\tau_2 \bar{W}^2 + \frac{1}{\tau_2} (\bar{W}' - \bar{W})^2) \right\}
$$

$$
(24)
$$

where $\tau_1 = \text{Re}\tau$ and $\tau_2 = \text{Im}\tau$. We can now use the Poisson summation formula to replace the lattice sum $\sum_{\bar{W}' \in \Lambda}$ by its dual $\sum_{\bar{P} \in \Lambda^*}$. The result contains the familiar form:

$$
\sum_{\bar{W}_L, \bar{W}_R} \exp\left\{ i\pi (\tau_2 \bar{W}_L^2 - \bar{W}_R^2) \right\}
$$

$$
(25)
$$

where $\bar{W}_L = +\frac{1}{2} g^{ij} p_{ij} + \bar{W}$ and $\bar{W}_R = -\frac{1}{2} g^{ij} p_{ij} + \bar{W}$. As was shown above the lattice $(\bar{W}_L, \bar{W}_R)$ with $\bar{W}_L - \bar{W}_R \in \Lambda^*$ is a self dual lattice with signature $(p, p)$. When we turn on the $B_{i\bar{j}}$ field there is the additional contribution $-4\pi i B_{i\bar{j}} \bar{W}^i \bar{W}^\bar{j}$ to the zero mode action $I_0$. Carrying out the Poisson summation formula with this term present we arrive at (25) but with $\bar{W}_L = \frac{1}{2} g^{ij} p_{ij} + \bar{W}^i + \bar{W}^i$ and $\bar{W}_R = -\frac{1}{2} g^{ij} p_{ij} + B_{i\bar{j}} \bar{W}^i + \bar{W}$. Therefore the momentum $p_{i\bar{j}}$ is shifted by $B_{i\bar{j}} \bar{W}^i$. This is the same shift that we found earlier in (9) by the canonical approach.

Now we consider the partition function in the presence of Wilson lines. It is more convenient to use the fermionic representation for the spin $\text{(32)/}\mathbb{Z}_2$ (or $E_8 \times E_8$) gauge group of the heterotic string theory. For definiteness we
shall consider spin (32)/Z_2. In this case we have 32 real left moving 2-dimensionsal fermions and we have to sum over all the spin structures. In other words for every non trivial generator of \( \pi_1 \) of the string world sheet we have to sum over periodic and antiperiodic boundary conditions for the fermions. The effect of switching on constant background gauge fields is basically to change these boundary conditions.

Since the gauge field \( A \) lies in the Cartan subalgebra of SO(32), it is convenient to group the fermions into 16 complex fermions \( (\psi_a, \bar{\psi}_a) \) \( (a=1, \ldots, 16) \), so that \( A^a_i \) are represented by diagonal elements \( A^a_i (a=1, \ldots, 16) \).

If the string world sheet is a torus \( 0 < \sigma < 1, 0 < t < 1 \), and \( \chi^i \) have winding numbers \( L_1 \) and \( L_2 \) in \( \sigma \) and \( t \) directions, then the boundary conditions of the fermions are

\[
\psi_a (\sigma + n, t + m) = (\pm)^n (\pm)^m e^{2i\pi (nA^a_1 L_1 + mA^a_2 L_2)} \psi_a (\sigma, t) \quad (26)
\]

where the \( \pm \) signs indicate the four spin structures. Let

\[
A^a \cdot L_1 = \chi^a \text{ mod integer, } 0 < \chi^a < 1
\]

\[
A^a \cdot L_1 = v^a \text{ mod integer, } 0 < v^a < 1
\]

Path integral over \( \psi^a \) and \( \bar{\psi}^a \) gives the determinant of free Dirac operator \( \gamma_L \) subject to the above boundary conditions. As \( \psi^a \) are chiral fermions then the determinants are determined only modulo an overall phase which can depend on \( \chi \) and \( v \). It is easy to show that the determinants (modulo phases) are, in each of the four sectors respectively

\[
Z(\pm,+) = q^{\frac{1}{12}} (1-6\chi^a(1-\chi^a)) \prod_{n=1}^{\infty} \left( 1 + n^{-\chi^a} e^{-2i\pi v^a} \right)
\]

\[
Z(\pm,-) = q^{\frac{1}{12}} (1-6(\chi^a + \frac{1}{2})(\frac{1}{2} - \chi^a)) \prod_{n=1}^{\infty} \left( 1 + \frac{1}{2} n^{-\chi^a} e^{-2i\pi v^a} \right)
\]

\[
Z(\mp,+) = q^{\frac{1}{12}} (1-6\chi^a(1-\chi^a)) \prod_{n=1}^{\infty} \left( 1 + n^{-\chi^a} e^{-2i\pi v^a} \right)
\]

\[
Z(\mp,-) = q^{\frac{1}{12}} (1-6(\chi^a + \frac{1}{2})(\frac{1}{2} - \chi^a)) \prod_{n=1}^{\infty} \left( 1 + \frac{1}{2} n^{-\chi^a} e^{-2i\pi v^a} \right)
\]

(27)
where $Z(t, \tau)$ correspond to the four sectors with the first entry referring to $t$ and the second to $\sigma$ directions. In the second equation we have assumed that $0 < \chi^a < \frac{1}{2}$, (if $\chi^a > \frac{1}{2}$ then in the second equation $\chi^a$ has to be replaced by $\chi^a - 1$). These equations can also be easily obtained within the operator formalism. Indeed path integral over $\psi^a$ and $\bar{\psi}^a$ is just

$$\text{Tr}((-1)^F H g)$$

where $g$ is the operator that rotates $\psi^a$ by $e^{2i\pi v^a}$ and $\bar{\psi}^a$ by $e^{-2i\pi v^a}$. The powers of $q$ in eq (27) are shifted by $\pm \chi^a$ as the oscillator modes of $\psi^a$ and $\bar{\psi}^a$ are shifted by precisely $\pm \chi^a$. The factor involving $q$ in front of the products is just the energy of the ground state in the twisted sectors. In the operator formalism one would also get overall phases corresponding to the eigenvalues of $g$ of the ground state in each sector respectively, however we have omitted these phases as equation (27) is meant to be true only up to phases (independent of $\tau$). In the following we will determine these phases by the requirement that firstly it should reproduce the correct anomalies and secondly that it should give modular invariant result.

To determine the anomalies, we consider the change in the determinants of the chiral Dirac operator as we shift $\chi^a + \chi^a + 1$ (or $\psi^a + \psi^a + 1$), that leaves the holonomy unchanged. Consider a one parameter family of $\chi$'s depending on a parameter $u \in [0,1]$ such that $\chi^a(0) = \chi^a$ and $\chi^a(1) = \chi^a + 1$. Since the holonomy at $u=0$ and $u=1$ is the same, we can identify the world sheet $S^1 \times S^1$ at $u=0$ and $u=1$, yielding a closed three surface $M = (S^1 \times S^1 \times S^1)$, parametrised by $(t, \sigma, u)$. Next, we define a gauge field on $M$ such that $\int_{A_t} dt = 2\pi v$ and

$$\int_{A_\sigma} d\sigma = 2\pi \chi^a$$

at $u=0$

$$\int_{A_\sigma} d\sigma = 2\pi (\chi^a + 1)$$

at $u=1$

We cannot, of course globally define $A$ on $M$ (as it would develop singularity at $u=0$), however we can still define gauge field strength $F_{\mu \nu}$ on $M$. $F_{\mu \nu}$ must have the property that $F_{\mu \nu} = 0$ at $u=0$ and $u=1$, $\int_{A_t} dt F_{\sigma \tau} = 0$, $\int_{A_\sigma} du F_{u \tau} = 0$ and $\int_{A_\sigma} du F_{\sigma \tau} = 2\pi$. A convenient choice is $F = du A_d$ and $A_\tau$ constant. Now we wish to calculate the change in the chiral fermion determinant between $u=0$ and $u=1$. In fact we will calculate the change in

*The phase ambiguity associated with chiral determinants has recently been discussed in [3].
\[ \ln \phi = \ln \det_\beta(x^a, v^a) - \ln \det_\beta(x', 0) \]  

(28)

going from \( u = 0 \) to \( u = 1 \). Here the subscript \( \beta \) denotes the spin structure \((\pm, \pm)\) in \( t \) and \( \sigma \) directions. The second determinant is for arbitrary \( x' \) and \( v = 0 \). We use the well known formula relating the change in determinant to the \( n \)-invariants of \( M \) (see [4]):

\[
\Delta \ln \phi = \frac{i\pi}{2} \left( \eta_\beta(A) - \eta_\beta(A') \right) \tag{29}
\]

where \( A \) and \( A' \) are the gauge fields on \( M \) for the two cases. To calculate \( n \)-invariants on \( M \) we consider the open four space \( B \) whose boundary is \( M \) and extend the gauge fields to \( B \). The ball \( B \) is conveniently constructed as \( (S^1 \times S^1 \times \mathbb{R}^2) \) where the first two \( S^1 \)'s are \( \sigma \) and \( u \) circles and \( \mathbb{R}^2 \) is a disc parametrised by the angle \( t \) and the radial coordinate \( \rho \in [0, 1] \). \( \rho = 1 \) gives the boundary \( M \) of \( B \). One can easily extend \( F_{\mu \nu} \) to \( B \). A convenient choice is \( F = d\alpha d\sigma + v^a d\alpha dt \) and \( F' = d\alpha d\sigma \). Now using the Atiyah-Singer-Patodi theorem we can calculate \( n \)-invariants. The result is

\[
\Delta_x \ln \phi = -\frac{2\pi i}{2\pi} \left[ \int_B FAF - \int_B F' A' F' \right] 
= -\pi i v^a \mod 2\pi
\]  

(30)

Similarly we can calculate the change in the determinant due to a change \( v^a + v^a + 1 \). The result is

\[
\Delta_v (\ln \det_\beta(x^a, v^a) - \ln \det_\beta(0, v')) = \pi i x^a \mod 2\pi
\]  

(31)

Now the \( Z(\pm, \pm) \) in eq (27) are related to the Jacobi \( \Theta \) functions. One can use the infinite sum form of \( \Theta \) functions to rewrite eq (27) (modulo phases) as

\[
Z(\pm, +) = \prod_{a} \left( \frac{1}{24} f(q) \right)^{-1} e^{-i\pi x^a v^a} \sum_{n=-\infty}^{\infty} \left( \frac{1}{2\pi} \left( n + \frac{1}{2} + x^a \right)^2 \right) e^{2i\pi(n + \frac{1}{2} + x^a)v^a}
\]

\[
Z(\pm, -) = \prod_{a} \left( \frac{1}{24} f(q) \right)^{-1} e^{-i\pi x^a v^a} \sum_{n=-\infty}^{\infty} \left( \frac{1}{2\pi} \left( n + x^a \right)^2 \right) e^{2i\pi(n + x^a)v^a}
\]

(32)
where $\chi^a$ and $\nu^a$ can be any real numbers (i.e. not restricted to $[0,1]$) and we have chosen phases so that they satisfy the anomaly conditions (30) and (31). Unfortunately the latter does not fix the phase ambiguity completely. For instance one can add phases linear in $\chi$ and $\nu$, or indeed relative constant phases between different $Z$'s. Can one fix this ambiguity by requiring modular invariance? First we note that the form of the above partition functions (ignoring $q^{-\frac{2}{3}} f(q)^{-\frac{16}{3}}$) is modular invariant. For example for $\tau + \tau + 1$: $(\chi^a, \nu^a) + (\chi^a, \nu^a + \chi^a)$ and $Z(\tau, -) \rightarrow Z(\tau, -)$ and for $\tau + -\frac{1}{\tau}$ : $(\chi^a, \nu^a) + (\nu^a, -\chi^a)$ and $Z(+,-) \leftrightarrow Z(-,+)$ as expected. If we add phases that are continuous functions of $\chi^a$ and $\nu^a$, then in general the anomaly condition and modular invariance conditions would be destroyed, unless the phases are constants. Moreover, since modular transformations mix $Z(\tau, -)$ and $Z(-,+)$ the relative phases between these three sectors must be zero. Thus it appears that one has only the freedom to add a constant relative phase (say $\alpha$) between $Z(+, +)$ and the other three sectors. Combining all the sectors, one obtains

$$Z_{\text{fermions}} = q^{-\frac{2}{3}} (f(q))^{-\frac{16}{3}} \left[ \sum_{\text{We scalar}} + \frac{1}{2} (1 + e^{i\alpha}) \sum_{\text{We spinor}} \right]$$

$$+ \frac{1}{2} (1 - e^{i\alpha}) \sum_{\text{We spinor}} \frac{1}{q^2} (W + A \cdot L_1)^2 e^{2\pi i (W + A \cdot L_1) A \cdot L_2} \quad (33)$$

where the sums are over the scalar, spinor and spinor' conjugacy classes of spin (32). The choice $e^{i\alpha} = 1$, then gives the partition function of the shifted lattices of spin (32)/$Z_2$. Including the partition function corresponding to $T^p$ and converting the $L_2$-sum over $\Lambda$ to that of a sum over the momentum lattice $\Lambda^*$ by means of Poisson summation formula, we arrive at

$$Z_{\text{lattice}} = \sum_{\text{LeA}} \sum_{\text{We spin(32)/Z_2}} \sum_{\text{We spin(32)/Z_2}} \frac{1}{q^2} (\frac{1}{2} k + L)^2 \frac{1}{q^2} (\frac{1}{2} k - L)^2 \frac{1}{q^2} (W + A \cdot L)^2 \quad (34)$$

where
\[ k^i = \varepsilon^{ij} (p_j - \omega^{a_i}_a - \frac{1}{2} A^{a_i}_A A^a_L k) \]  \hspace{1cm} (35)

Once again we have reproduced the shift in \( k^i \) found in eq (19), which gave rise to the self dual lattice of signature \((16+p,p)\).

Some interesting conclusions can be drawn from this investigation. First of all, the construction in ref [1], gives not new string theories but more general toroidal compactifications of the already known heterotic superstring. \( p(p+16) \) parameters of ref [1] are expectation values of physical fields (\( g_{ij} \), \( B_{ij} \), and \( A^a_i \)) that play a role in the compactification. However the treatment in ref [1], certainly gives a unified treatment of these parameters.

The interesting new physics that emerges from this construction is first of all the observation that compactification to \( d < 10 \) gives possibilities for enlargement of the gauge group not hitherto recognized. Second, this construction gives a new indication that the heterotic \( SO(32) \) and \( E_8 \times E_8 \) superstrings are two different vacuum states in the same theory, something that has been argued before on various grounds \([5,6,7]\). Before compactification the \( SO(32) \) and \( E_8 \times E_8 \) theories can be described in terms of 16 dimensional even self-dual lattices \( \Gamma_{16} \) or \( \Gamma_8 \times \Gamma_8 \), respectively. After compactification to, say 9 dimensions, we would encounter an even self-dual lattice of signature \((17,1)\). The \( SO(32) \) or \( E_8 \times E_8 \) theories would give rise, respectively, in the absence of Wilson lines, to the \((17,1)\) self-dual lattice \( \Gamma_{16} \times P \) or \( \Gamma_8 \times \Gamma_8 \times \Gamma_8 \), where \( P \) denotes the even self-dual lattice of signature \((1,1)\). The uniqueness theorem for even self-dual lattice of signature \((17,1)\) ensures that it is possible to continuously interpolate between these theories after compactification (by suitable introduction of Wilson lines), and this strongly suggests that it is also possible to interpolate between them (perhaps in a much more difficult way) before compactification.

References


