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NONLINEAR BETATRON OSCILLATIONS

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ABSTRACT

Nonlinear fields are so important in modern accelerators or storage rings that they have to be included in the basic lattice design for both chromaticity correction and aperture determination. The power of prediction of a nonlinear theory is not yet firm enough and numerical simulations are necessary. These lectures deal with the principles of particle tracking, the equations of the betatron motion and the envelope distortions and tune shifts induced by multipole fields. The theory is applied to the correction of beam envelope distortions produced by chromaticity sextupoles in a large acceptance machine, the Antiproton Collector.

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Fig. 1. Swell on the sand (M. Wolfender, 1985).

1. INTRODUCTION

Three classes of problems can be distinguished in the study of the transverse nonlinear motion of a particle in a synchrotron. One concerns the properties of individual resonances in the context of beam extraction from an accelerator and reduction of background noise in colliders. The second one applies to the collective properties of a cluster of resonances; it is the realm of stochastic layers and chaotic situations; this subject is under thorough investigation in several domains of physics and in mathematics and it is of outstanding interest in accelerator physics in order to understand the *beam-beam effect* which is supposed to be the ultimate limit in collider performance. Third and last, which is the topic of this course: the determination of the change in beam shape and size induced by the nonlinear components of the magnetic field.

A general difficulty which is particularly acute in the treatment of nonlinear dynamical systems lies in the right balance to be maintained between rigor and completeness on one hand and easy access to the important results on the other hand. As proofs cannot be eliminated from a course, it is suggested to the visitor of the nonlinear wonderland to take or leave the magic train at one of the following stations:

Particle tracking (section 2) consists of calculating the transfer of a 4-vector whose components are the particle positions and momenta in the horizontal and vertical planes. It results from the integration of the equations of motion in each magnetic element of the synchrotron lattice. The three dimensional field configuration is that of the *hard edge model* which assumes a purely transverse magnetic field inside the magnet and an abrupt fall at the ends; the longitudinal component at the ends is expressed for tilted and curved faces up to the quadratic terms. The hard edge model provides a direct approach to the main characteristics of the particle-field coupling which operates via particle position inside

the magnet and via particle velocity at the ends; however, it cannot account for special effects linked to the non-negligible bending or focusing strengths which may appear in the transition region at the end of a real magnet.

The motion of a particle can be considered as an oscillation about a closed orbit (section 3). The oscillation is traditionally called the *betatron motion*. Its equations are derived using a variational calculus in the cartesian frame of a rectangular magnet. As with the classical linear theory, a privileged role is played by the coordinates of a particle which belongs to a plane normal to the closed orbit at the point where the variational treatment is applied. Among the many contributions to the betatron motion, it turns out that the field multipole components (sextupole, octupole, etc...) and, to a lesser extent, the solenoidal coupling are the most important ones.

Using adequate variables, the phase space trajectories of the linear betatron motion are windings of a 2-torus whose projections onto the horizontal and vertical sub-phase-spaces are circles. Polar instead of cartesian coordinates are then very attractive to describe the betatron motion since the radius is related to the beam envelope and the angle to the betatron phase. The distortions of the 2-torus are calculated in section 4 by defining the beam envelope as the limit of the change in radius at each turn over an infinite number of turns. In the same spirit, the betatron tune shift is the average value of the change in betatron phase at each turn. The formalism remains elementary and never resorts to Hamiltonian dynamics. The theory applies to one dimensional and coupled motions as well, it accepts any type of perturbation but it is restricted to a first approximation to the solutions of the betatron motion assuming that all the nonlinear deflections are superimposed without correlations between them. The formulae are illustrated by the correction of the beam envelope's non linear distortions created by chromaticity sextupoles in a large acceptance machine, the CERN Antiproton Collector.

2. PARTICLE TRACKING IN A THREE DIMENSION MAGNETIC FIELD

2.1 TOPOLOGY OF LATTICE REFERENCE SYSTEMS

The purpose of particle tracking is to define the position of a particle in a sequence of bending magnets, straight sections and focusing magnets, called the *lattice*. Each magnet is connected to the adjacent straight sections through transformations which are studied in this section.

2.1.1 REFERENCE SYSTEMS

The magnets are symmetric about a horizontal median plane. The geometry is defined by a reference line whose physical meaning will be explained later and called the *central orbit*. The central orbit lies in the median plane and traverses the magnet end faces in a_1 and a_2 (Fig. 2). There are three reference frames about the central orbit:

- (s_1): a_1, x_1, y_1, z_1 at the entry,
- (s): a, x, y, z inside,
- (s_2): a_2, x_2, y_2, z_2 at the exit.

Fig. 2. Reference frames in a magnet. The particle trajectory is defined with respect to the frames on the central orbit. The magnetic field at the magnet boundaries is calculated in the frames located at the intercept of the particle trajectory with the end faces.

The y-axes are vertical, a_1z_1 and a_2z_2 are tangent to the central orbit, az is aligned on a_1a_2 . The trajectory to be studied intercepts the $(a_1x_1y_1)$ plane in A_1 , the entry face in B_1 , the exit face in B_2 and the $(a_2x_2y_2)$ plane in A_2 . Particle tracking in a magnet consists of calculating the coordinates $(x_2, dx_2/dz_2, y_2, dy_2/dz_2)$ in A_2 as a function of the magnetic field and of the initial coordinates $(x_1, dx_1/dz_1, y_1, dy_1/dz_1)$ in A_1 .

2.1.2 POSITION TRANSFER

The end faces are cylinders (or planes) of equation:

$$z = c_1x + \frac{1}{2} c_2x^2 \quad (1)$$

expressed in (s_1) or (s_2) .

The coordinates of B_1 are solution of the system:

$$x = x_1 + x'_1 z$$

$$y = y_1 + y'_1 z$$

$$z = c_1x + \frac{1}{2} c_2x^2$$

where the symbol ' means d/dz_1 .

The abscissa is:

$$x = \frac{2x_1}{1 - c_1x'_1 + \text{sign}(1 - c_1x'_1) \sqrt{(1 - c_1x'_1)^2 - 2c_2x'_1x_1}} \quad (2)$$

The sign is chosen so that x becomes equal to $x_1/(1 - c_1x'_1)$ when the end face is a plane ($c_2 = 0$). In the magnet frame (s) , the coordinates of B_1 are:

$$\begin{bmatrix} x \\ y \\ z + \frac{\ell}{2} \end{bmatrix}_s = \begin{pmatrix} \cos\varphi_1 & 0 & \sin\varphi_1 \\ 0 & 0 & 0 \\ -\sin\varphi_1 & 0 & \cos\varphi_1 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{s_1} \quad (3)$$

where ℓ is the magnet length and φ_1 the angle:

$$\varphi_1 = (a_1z_1, az) \quad (4)$$

2.1.3 ANGLE TRANSFER

The same rotation has to be applied to the velocity vector $(dx/dt, dy/dt, dz/dt)$ and the quantities $dx/dz, dy/dz$ are then:

$$\frac{dx}{dz} = \frac{\tan\varphi_1 + (dx/dz)_1}{1 - \tan\varphi_1 (dx/dz)_1} \quad (5)$$

$$\frac{dy}{dz} = \frac{1}{\cos\varphi_1} \cdot \frac{(dy/dx)_1}{1 - \tan\varphi_1 (dx/dz)_1} \quad (6)$$

When the particle is submitted to the action of the magnetic field, the motion equations are integrated and this problem will be treated in the next sections. At the exit of the magnet, transformations reciprocal to the entry transformations are performed.

2.2 MAGNETIC FIELD

The magnetic field is expressed in the "hard edge" model. Inside the magnet, the field is derived from the two dimension magnetic potential:

$$V(\zeta) = a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + a_4\zeta^4 + \dots \quad (7)$$

where

$$\zeta = x + iy \quad (8)$$

The complex magnetic field is:

$$B(\zeta) = \frac{dV}{d\zeta} = B_y + iB_x \quad (9)$$

The coefficients $a_1, a_2, a_3, a_4, \dots$ are the dipole, quadrupole, sextupole, octupole, ... parts of the magnetic potential expansion. In the following, we shall use field and field gradient expansions. For this reason, it is preferable to rename these coefficients:

$$a_1 = B_0,$$

$$a_2 = \frac{1}{2} G_0,$$

$$a_3 = \frac{1}{3!} G',$$

$$a_4 = \frac{1}{4!} G'',$$

and the expressions of B_y and B_x can be written:

$$B_y = B_0 + Gx + \frac{1}{2} G'(x^2 - y^2) + \frac{1}{6} G''(x^3 - 3xy^2) + \dots \quad (10)$$

$$B_x = Gy + G'xy + \frac{1}{6} G''(3x^2y - y^3) + \dots \quad (11)$$

Outside the magnet, the field is zero. In other words, the expression of the field in the local system (B_1, X_1, Y_1, Z_1) is:

$$B(x_1, y_1, z_1) = B(x_1, y_1) \cdot E(z_1) , \quad (12)$$

$E(z)$ being a step function equal to zero for $z_1 < 0$ and equal to 1 for $z_1 > 0$. Maxwell's equations impose the conditions:

$$\begin{aligned} \frac{\partial B_{z_1}}{\partial y_1} &= \frac{\partial B_{y_1}}{\partial z_1} \\ \frac{\partial B_{z_1}}{\partial x_1} &= \frac{\partial B_{x_1}}{\partial z_1} . \end{aligned}$$

By integration of the above equations, B_{z_1} has the two symbolical forms:

$$\begin{aligned} B_{z_1} &= [\int B_{y_1}(x_1, y_1) dy_1 + f(x_1)] \delta(z_1) \\ &= [\int B_{x_1}(x_1, y_1) dx_1 + g(y_1)] \delta(z_1) \end{aligned}$$

The functions f and g are chosen so that B_{z_1} is uniquely determined. Furthermore, one can notice that the quantity $B_{x_1} dx_1 + B_{y_1} dy_1$ is the scalar product of the field $\vec{B}$ inside the magnet by the elementary length of trajectory $d\vec{s}$. Expressed in the reference frame (s), this scalar product is:

$$\vec{B} \cdot d\vec{s} = B_x dx + B_y dy .$$

Therefore, we have:

$$B_{x_1} dx_1 + B_{y_1} dy_1 = B_x dx + B_y dy .$$

As the vertical components are invariant in the transformation ($s \rightarrow s_1$),

$$B_{y_1} dy_1 = B_y dy ,$$

then

$$B_{x_1} dx_1 = B_x dx .$$

The integrals which come in the expression of B_{z_1} can thus be evaluated in the reference frame (s) and it is left as an exercise to prove that the field expansion of the axial component B_{z_1} is:

$$B_{z_1} = [B_0 + G_0 x + \frac{1}{2} G' (x^2 - \frac{y^2}{3}) + \dots] y \delta(z_1) \quad (13)$$

The degree of B_{z1} exceeds the degree of the central field by one unit: a quadrupole is not a linear element! The field which has been derived is not strictly Maxwellian because it does not satisfy the zero divergence condition:

$$\text{div } \mathbf{B} = \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} \right) E(z) + f(x,y) \delta'(z)$$

where $f(x,y)$ is the x and y dependence of B_z . The coefficient of $E(z)$ is zero since the two dimension field inside the magnet is Maxwellian. The divergence condition can be satisfied if terms in $\delta'(z)$ are added to the expansion of B_x and B_y . However, distributions like δ' make sense only inside an integral when they operate on some function $g(z)$:

$$\int_{-\infty}^{+\infty} g(z) \delta'(z) dz = -g'(0);$$

in our case the integral appears in the solution of the equations of motion and the functions g depend only on x and y and not on z : their derivatives with respect to z are zero. It is thus sufficient to limit the field expansions to terms in $E(z)$ and $\delta(z)$.

Exercise: Derive the expression of the field axial component at the downstream end B_2 .

2.3 EQUATIONS OF MOTION

The force exerted by a magnetic field on a charged particle is always perpendicular to the particle trajectory. Its work is zero and the energy, the mass and the length of the momentum vector $\vec{p}$ remain constant when the synchrotron radiation is neglected. The particle motion is defined by a position vector $\vec{q}(x,y,z)$ and a momentum vector $\vec{p}(p_x, p_y, p_z)$. It is the solution of a system of ordinary differential equations obtained from the definition of the velocity and from the application of Lorentz's force:

$$\frac{d\vec{q}}{dt} = \frac{\vec{p}}{m} \quad (14)$$

$$\frac{d\vec{p}}{dt} = e \vec{v} \times \vec{B} \quad (15)$$

It is convenient to take a space coordinate, say z , instead of time t as an independent coordinate and to define the reduced momentum:

$$\vec{\pi} = \frac{\vec{p}}{p} \quad (16)$$

The motion equations become:

$$\frac{d\vec{q}}{dz} = \frac{\vec{\pi}}{\pi_z} \quad (17)$$

$$\frac{d\vec{\pi}}{dz} = \frac{\vec{\pi}}{\pi_z} \times \frac{e}{p} \vec{B} \quad (18)$$

with

$$\pi_z = \sqrt{1 - \pi_x^2 - \pi_y^2} \quad (19)$$

In agreement with the field expansion, the following notations are introduced:

$$\frac{e}{p} B_0 = \frac{1}{\varrho} \quad (20)$$

$$\frac{e}{p} G_0 = K \quad (21)$$

$$\frac{e}{p} G' = K' \quad (22)$$

⋮

ϱ is the radius of curvature, K the focusing strength, K' its radial derivative, etc. The scalar form of the motion equations is:

$$\left| \begin{array}{l} \frac{dx}{dz} = \frac{\pi_x}{\pi_z} \\ \frac{dy}{dz} = \frac{\pi_y}{\pi_z} \\ \frac{d\pi_x}{dz} = \frac{e}{p} \left(\frac{\pi_y}{\pi_z} B_z - B_y \right) \\ \frac{d\pi_y}{dz} = \frac{e}{p} \left(B_x - \frac{\pi_x}{\pi_z} B_z \right) \end{array} \right. \quad (23)$$

The form of the equation seems to indicate a difference of nature between the actions of the transverse and longitudinal fields.

This is an effect of perspective due to the privileged role played by the longitudinal coordinate z .

2.4 INTEGRATION OF THE EQUATIONS OF MOTION

The motion equations can be integrated when the field is uniform, impulsive or in the linear approximation. Numerical techniques have to be used in the general case. The treatment of the linear approximation is, strictly speaking, outside the scope of particle tracking: however, it is basic in the description of the small amplitude betatron oscillations; it will therefore be summarized in a separate section.

2.4.1 UNIFORM FIELD

Let us prove that a particle trajectory inside a uniform field is a *helix*. For a field of components $(0, B_y, 0)$ the change of momentum is given by the equations:

$$\frac{d\pi_x}{dz} = -\frac{1}{\varrho} \quad (24)$$

$$\frac{d\pi_y}{dz} = 0 \quad (25)$$

whose integration is straightforward.

$$\pi_x = \pi_{x0} - \frac{z}{\varrho}$$

$$\pi_y = \pi_{y0}$$

The position transfer results from:

$$\frac{dx}{dz} = \frac{\pi_{x0} - z/\varrho}{\sqrt{1 - \pi_{y0}^2 - (\pi_{x0} - z/\varrho)^2}},$$

$$\frac{dy}{dx} = \frac{\pi_{y0}}{\sqrt{1 - \pi_{y0}^2 - (\pi_{x0} - z/\varrho)^2}}.$$

Although these equations can be integrated directly, the nature of the trajectory can be recognized more easily if the parameter u is introduced:

$$u = \text{Arcsin} \frac{\pi_{x0} - z/\varrho}{\sqrt{1 - \pi_{y0}^2}}, \quad (26)$$

with its initial value:

$$u_0 = \text{Arcsin} \frac{\pi x_0}{\sqrt{1 - \pi y_0^2}} \quad (27)$$

The integration with respect to u gives:

$$\left| \begin{aligned} x &= x_0 + \rho \sqrt{1 - \pi y_0^2} (\cos u - \cos u_0) \\ y &= y_0 - \rho \pi y_0 (u - u_0) \\ z &= -\rho \sqrt{1 - \pi y_0^2} (\sin u - \sin u_0) \end{aligned} \right. \quad (28)$$

This curve is precisely a helix whose axis is parallel to the field; the pitch is $2\pi\rho\pi y_0$ and the projection on the (x,y) plane is a circle of radius:

$$R = \rho \sqrt{1 - \pi y_0^2} ,$$

and of centre C:

$$x_C = x_0 - R \cos u_0 = x_0 - \rho \pi z_0$$

$$z_C = R \sin u_0 = \rho \pi x_0$$

Exercise: For what conditions does the particle stay trapped in the field?

Exercise: In order to be convinced that the shape of the trajectory is independent of the field direction, solve the motion equations when the field is directed along the z -axis.

IMPULSIVE FIELD

The field is impulsive when it varies in one of its coordinates, say z , like a delta-function. This is the case in the hard edge model for the stray field and in a thin lens where all the field is concentrated over a zero length. The integration of the momentum equations results from the very definition of the delta-function:

$$\text{stray field} \quad \left| \begin{aligned} \pi_x &= \pi_{x_0} + \frac{e}{p} \frac{\pi y_0}{\pi z_0} B_z(x_0, y_0) \\ \pi_y &= \pi_{y_0} - \frac{e}{p} \frac{\pi x_0}{\pi z_0} B_z(x_0, y_0) \end{aligned} \right. \quad (29)$$

$$\text{thin lens} \quad \left| \begin{aligned} \pi_x &= \pi_{x_0} - \frac{e}{p} B_y(x_0, y_0) \\ \pi_y &= \pi_{y_0} + \frac{e}{p} B_x(x_0, y_0) \end{aligned} \right. \quad (30)$$

The decomposition of a magnet into a chain of thin lenses separated by straight sections provides a model for numerical integration when the field contains nonlinear components.

3. THEORY OF NON-LINEAR BETATRON OSCILLATIONS

The simulation of the particle motion with a computer code is in the present state of our knowledge an indispensable tool to anticipate the real motion of a particle in a lattice of magnets. However, its virtue is essentially descriptive. If the amplitude of the motion grows beyond acceptable limits, one has to resort to physical insight or better to a theory to improve the lattice characteristics. If the theory of small amplitude oscillations in an accelerator is well established, the realm of large amplitude oscillations is still widely open to research.

In this section, the basic equations of the non-linear motion are established using a variational calculus about the closed orbit. The origin and the end of the betatron oscillations are defined in the planes normal to the closed orbit where the closed orbit crosses the magnet end faces. In the "hard edge" model, the betatron oscillation consists of two straight sections outside the magnet and of a curved trajectory inside the magnet. The oscillation could be correctly described by applying a variation to the closed orbit in the cartesian frame of the magnet for the curved part of the trajectory and by treating the straight sections separately. Unfortunately, the second order differential equations of motion would not be of Hill's type for the first order perturbation terms so that the connection with the classical Courant and Snyder (C-S) theory would be broken.

Some features of the linear theory must be maintained in a generalization to the non-linear case:

- a) the motion variable is the curvilinear coordinate of a reference point on the closed orbit,
- b) the coordinate system, called the *local system*, is a trihedron tangent to the closed orbit.

The new fact lies in the variational calculus which cannot be performed in a plane normal to the closed orbit because the field is not a continuous function at the ends of the magnet. This difficulty is basic even for small amplitude oscillations but, fortunately, it does not affect the results deduced from a first order calculation.

As for any motion, the betatron oscillation is completely defined by the transfer of the momentum and position vectors. The momentum transfer can be calculated inside the magnet only since it is constant along the outside straight sections; it is directly deduced from a cartesian variational calculus in the magnet system of coordinates. The position vector is this of the projection, $\mathbf{m}$, of a point, $\mathbf{M}$, of the betatron oscillation onto the plane normal to the closed orbit along the support of the momentum vector (Fig. 3); $\mathbf{m}$ is itself a point of the betatron oscillation in the entrance and exit planes; the one-to-one correspondence between $\mathbf{m}$ and $\mathbf{M}$ ensures that motion properties like stability established for $\mathbf{m}$ are valid for the betatron oscillation.

The closed orbit is defined in section 3.1 for any momentum compatible with the machine aperture; no privilege is given to the central orbit and the concepts of local curvature and focusing strength are introduced. After calculating the momentum components and position coordinates in the local system, the equations of the betatron motion limited to second order terms are established for rectangular

magnets and for special end shapes in the "impulse" approximation. The implications of the momentum conservation (Fig. 4) are explicitly derived and terms of kinematic and dynamic origin are systematically distinguished. Special sections (3.3.3, 3.4.4) are devoted to a qualitative analysis of the equations and to their experimental consequences.

3.1 CLOSED ORBIT

Assuming that the machine has a median plane of symmetry means that there is no horizontal field in that median plane. The closed orbit is thus that special trajectory in the horizontal median plane of symmetry which repeats itself at each turn and has the symmetry of the lattice.

Its differential equations are a special case of the general equations of motion:

$$\frac{dx}{dz} = \frac{\pi_x}{\pi_z} \quad (31)$$

$$\frac{d\pi_x}{dz} = - \frac{e}{p} [B_{y0} + G_0 x + \frac{1}{2} G'_0 (x^2 - y^2)] \quad (32)$$

$$\frac{d\pi_z}{dz} = \frac{e}{p} \frac{\pi_x}{\pi_z} [B_{y0} + G_0 x + \frac{1}{2} G'_0 (x^2 - y^2)] \quad (33)$$

Let us specify the coordinates of the closed orbit by the index "0" and define the *curvature*:

$$\frac{1}{\rho} = \frac{e}{p} (B_{y0} + G_0 x_0 + \frac{1}{2} G'_0 x_0^2) \quad (34)$$

and the *focusing strength*:

$$K = \frac{e}{p} (G_0 + G'_0 x_0) \quad (35)$$

The motion coordinates can be split into closed orbit and betatron coordinates (index "1"):

$$x = x_0 + x_1 \quad (36)$$

$$\pi_x = \pi_{x0} + \pi_{x1} \quad (37)$$

$$\pi_z = \pi_{z0} + \pi_{z1} \quad (38)$$

The equations of the closed orbit can then be written:

$$\frac{dx_0}{dz} = \frac{\pi_{x0}}{\pi_{z0}} \quad (39)$$

$$\frac{d\pi_{x0}}{dz} = -\frac{1}{\varrho} \quad (40)$$

$$\frac{d\pi_{z0}}{dz} = \frac{1}{\varrho} \frac{\pi_{x0}}{\pi_{z0}} \quad (41)$$

π_{x0} and π_{z0} are not independent; from their very definition they satisfy the condition:

$$\pi_{x0}^2 + \pi_{z0}^2 = 1 \quad (42)$$

and the third differential equation is redundant with the condition of momentum conservation. In the general case, the coordinates (x_0, π_{x0}) of the closed orbit are calculated numerically so that they remain unchanged after mapping through the lattice over one turn. In the horizontal phase plane (x, π_x) , the coordinates of a closed orbit are those of a *fixed point*. The balance between the centrifugal force pv/ϱ and the Lorentz force evB gives the basic relation:

$$p = e B \varrho \quad (43)$$

and shows that there is a one to one correspondence between particle momentum and closed orbit. If the orbits are always separated in bending magnets (*spectroscopic effect*), they may not be so in straight sections.

In the context of the study of betatron oscillations, it is of interest of using the path length s on the closed orbit instead of the magnet axis coordinate z as an independent variable. The differential increments ds and dz are then related by the equation:

$$dz = \pi_{z0} ds \quad (44)$$

The equations of motion take the form:

$$\frac{dx_0}{ds} = \pi_{x0} \quad (45)$$

$$\frac{d\pi_{x0}}{ds} = -\frac{\pi_{z0}}{\varrho} \quad (46)$$

$$\frac{d\pi_{z0}}{ds} = \frac{\pi_{x0}}{\varrho} \quad (47)$$

Another relation on the closed orbit will be useful in the following sections:

$$\begin{aligned} \frac{d}{ds} \left(\frac{1}{\varrho} \right) &= \frac{d}{dx} \left(\frac{e}{p} B_y \right) \cdot \frac{dx}{ds} \\ &= K \pi_{x0} \end{aligned} \quad (48)$$

Fig. 3. Topology of the betatron oscillation. The plane $(M_0 M_h, M_0 y_0)$ is parallel to the end faces of the magnet. The plane $(M_0 x_0, M_0 y_0)$ is normal to the closed orbit in M_0 .

Fig. 4. Configuration of the closed orbit and betatron momentum vectors in the magnet frame. Due to the momentum conservation the extremities of the momentum vectors are located on a same sphere. For the clarity of the figure, perspective and scale are not the same as in Figure 3 and the magnitude of the perturbing vector $\bar{P}_1$ is exaggerated.

3.2 MOMENTUM COMPONENTS AND POSITION COORDINATES IN THE LOCAL FRAME

The betatron motion is defined in a local frame attached to the closed orbit. Let M_0 of curvilinear abscissa s be a point of the closed orbit. Its coordinates in the magnet frame are $(z, x_0, 0)$. The correspondence between a point M of the betatron oscillation and M_0 is defined by the translation vector $\overrightarrow{M_0M}$:

$$\overrightarrow{M_0M} = (0, x_1, y) \quad (49)$$

The local reference frame is a trihedron of origin M_0 and whose axes M_0z_ℓ , M_0x_ℓ , M_0y_ℓ are respectively tangent to the closed orbit, normal to the closed orbit and vertical. The local coordinates of M are $(\pi_{x_0}x_1, \pi_{z_0}x_1, y)$. The point m is the intersection of $\overrightarrow{MP}$, the momentum vector, tangent to the trajectory with the plane (M_0x_ℓ, M_0y_ℓ) . The origin of the betatron oscillation corresponds to a point M_0 located at the entry to the magnet. On the other hand, the momentum vector is $\overrightarrow{M_1P}$ and not the tangent to the trajectory of m which is not parallel to $\overrightarrow{MP}$.

Let $m_h M_h$ and $m_v M_v$ be the projections of the tangent to the trajectory mM onto the planes $(M_0z_\ell x_\ell)$ and $(M_0z_\ell y_\ell)$, respectively. The equations of the tangent are then:

$$x_\ell = \pi_{z_0}x_1 + (z_\ell - \pi_{x_0}x_1)\xi' \quad (50)$$

$$y_\ell = y + (z_\ell - \pi_{x_0}x_1)\eta' \quad (51)$$

The slopes ξ' and η' are calculated as a function of the components $(\pi_{z_\ell}, \pi_{x_\ell}, \pi_y)$ of the momentum $\overrightarrow{MP}$ in the local system:

$$\xi' = \pi_{x_\ell}/\pi_{z_\ell} \quad (52)$$

$$\eta' = \pi_{y_\ell}/\pi_{z_\ell} \quad (53)$$

Let us remark, in passing, that the closed orbit momentum is $(1,0,0)$ in the local system. The local system is deduced from the magnet system by a rotation of angle φ :

$$\varphi = (M_0z, M_0z_\ell) \quad (54)$$

As we know that

$$\cos\varphi = \pi_{z_0} \quad (55)$$

$$\sin\varphi = \pi_{x_0} \quad (56)$$

the change of coordinates can be written:

$$\begin{pmatrix} \pi_{z\ell} \\ \pi_{x\ell} \end{pmatrix} = \begin{pmatrix} \pi_{z0} & \pi_{x0} \\ -\pi_{x0} & \pi_{z0} \end{pmatrix} \begin{pmatrix} \pi_{z0} + \pi_{z1} \\ \pi_{x0} + \pi_{x1} \end{pmatrix} \quad (57)$$

π_y remains unchanged in the rotation and the slopes are:

$$\xi' = \frac{\pi_{z0}\pi_{x1} - \pi_{x0}\pi_{z1}}{1 + \pi_{x0}\pi_{x1} + \pi_{z0}\pi_{z1}} \quad (58)$$

$$\eta' = \frac{\pi_y}{1 + \pi_{x0}\pi_{x1} + \pi_{z0}\pi_{z1}} \quad (59)$$

The modulus of the momentum is the same for the betatron oscillation and for the closed orbit:

$$(\pi_{x0} + \pi_{x1})^2 + \pi_y^2 + (\pi_{z0} + \pi_{z1})^2 = \pi_{x0}^2 + \pi_{z0}^2$$

or,

$$\pi_{x0}\pi_{x1} + \pi_{z0}\pi_{z1} = -\frac{1}{2} (\pi_{x1}^2 + \pi_y^2 + \pi_{z1}^2) \quad (60)$$

In the case of small amplitude oscillations (first order approximation) the perturbing momentum

$$\vec{P}_1 = \vec{P} - \vec{P}_0 \quad (61)$$

is orthogonal to $\vec{P}_0$ since:

$$\pi_{x0}\pi_{x1} + 0 \cdot \pi_y + \pi_{z0}\pi_{z1} = 0 \quad (62)$$

As $\pi_{x0}\pi_{x1} + \pi_{z0}\pi_{z1}$ is a second order quantity, the slopes ξ' and η' can be written to the second order:

$$\xi' = \pi_{z0}\pi_{x1} - \pi_{x0}\pi_{z1} \quad (63)$$

$$\eta' = \pi_y \quad (64)$$

After solving (60) exactly with respect to π_{z1} :

$$\pi_{z1} = \pi_{z0} \left(-1 + \sqrt{1 - \frac{2\pi_{x0}\pi_{x1} + \pi_{x1}^2 + \pi_y^2}{\pi_{z0}^2}} \right) \quad (65)$$

and expanding the right hand side to second order:

$$\pi_{z1} = - \frac{\pi_{x0}}{\pi_{z0}} \pi_{x1} - \frac{1}{2\pi_{z0}} \left(\frac{\pi_{x1}^2}{\pi_{z0}^2} + \pi_y^2 \right) , \quad (66)$$

the expression of ξ' becomes:

$$\xi' = \frac{\pi_{x1}}{\pi_{z0}} + \frac{\pi_{x0}}{2\pi_{z0}} \left(\frac{\pi_{x1}^2}{\pi_{z0}^2} + \pi_y^2 \right) . \quad (67)$$

To determine the coordinates of m , ξ' and η' have to be known to first order only:

$$z_l = 0 \quad (68)$$

$$\xi = \pi_z x_1 \quad (69)$$

$$\eta = y - \pi_{x0} \pi_y x_1 . \quad (70)$$

3.3 BETATRON MOTION IN A PURE TRANSVERSE FIELD

Inside a magnet, the field is completely defined by its two transverse components B_y and B_x .

3.3.1 HORIZONTAL PLANE

The evolution of the motion is described by the derivatives of position ξ and slope ξ' with respect to the curvilinear coordinate s . Let us first study the variation of ξ :

$$\begin{aligned} \frac{d\xi}{ds} &= \frac{d}{dz} (\pi_z x_1) \cdot \frac{dz}{ds} \\ &= \pi_{z0} \left(\pi_z \frac{dx_1}{dz} + x_1 \frac{d\pi_z}{dz} \right) . \end{aligned} \quad (71)$$

The first term is of *kinematic* origin since it is independent of any force whereas the second one is *field* dependent.

The derivative of x_1 is determined by the momentum components:

$$\begin{aligned} \frac{dx_1}{dz} &= \frac{dx}{dz} - \frac{dx_0}{dz} \\ &= \frac{\pi_x}{\pi_z} - \frac{\pi_{x0}}{\pi_{z0}} , \end{aligned} \quad (72)$$

and

$$\pi_{z0} \pi_z \frac{dx_1}{dz} = \pi_{x1} \pi_{z0} - \pi_{x0} \pi_{z1} \quad (73)$$

From (63), the right hand side is precisely ξ' . In other words, the kinematic contribution to the change of horizontal position is the slope of the momentum vector.

The field term is deduced from the projection of Lorentz's force onto the z-axis:

$$\frac{d\pi_z}{dz} = \frac{e}{p} \left(\frac{\pi_x}{\pi_z} B_y - \frac{\pi_y}{\pi_z} B_x \right) \quad (74)$$

As it is sufficient to know $d\pi_z/dz$ to the first order, the second term in the right hand side can be neglected. From the momentum conservation (66) and vertical field (10) expansions:

$$\frac{\pi_x}{\pi_z} = \frac{\pi_{x0}}{\pi_{z0}} + \frac{\pi_{x1}}{\pi_{z0}^2} \quad (75)$$

$$\frac{e}{p} B_y = \frac{1}{\varrho} + Kx_1 \quad (76)$$

the field term is:

$$\pi_{z0} x_1 \frac{d\pi_z}{dz} = \frac{\pi_{x0} x_1}{\varrho} + \frac{\pi_{x1} x_1}{\varrho \pi_{z0}^2} + K \pi_{x0} x_1^2 \quad (77)$$

The expressions of the cartesian coordinates as a function of the local coordinates are drawn from (69) and (67):

$$x_1 = \left(1 + \frac{\pi_{x0}}{\pi_{z0}} \xi' \right) \frac{\xi}{\pi_{z0}} \quad (78)$$

$$\pi_{x1} = \pi_{z0} \xi' - \frac{\pi_{x0}}{2} (\xi'^2 + \eta'^2) \quad (79)$$

so that the first equation of motion is:

$$\frac{d\xi}{ds} = \xi' + \frac{\pi_{x0}}{\pi_{z0}} \frac{\xi}{\varrho} + \frac{1 + \pi_{x0}^2}{\pi_{z0}^2} \frac{\xi \xi'}{\varrho} + \frac{K \pi_{x0}}{\pi_{z0}^2} \xi^2 \quad (80)$$

The second equation of motion is derived from the variation of the slope of the horizontal momentum vector (63):

$$\frac{d\xi'}{ds} = \frac{d}{ds} (\pi_{z0}\pi_{x1} - \pi_{x0}\pi_{z1}) \quad (81)$$

$$= \frac{d\pi_{z0}}{ds} \pi_{x1} - \frac{d\pi_{x0}}{ds} \pi_{z1} + \pi_{z0} \frac{d\pi_{x1}}{ds} - \pi_{x0} \frac{d\pi_{z1}}{ds}$$

The first two terms are easily calculated using the closed orbit formulae (46) and (47) and the momentum conservation expanded to the second order (66):

$$\frac{d\pi_{z0}}{ds} \cdot \pi_{x1} - \frac{d\pi_{x0}}{ds} \cdot \pi_{z1} = - \frac{1}{2\varrho} \left(\frac{\pi_{x1}^2}{\pi_{z0}^2} + \pi_y^2 \right) \quad (82)$$

For the last two terms, the components of Lorentz's force have to be expanded to the second order:

$$\frac{d\pi_{x1}}{dz} = -Kx_1 - \frac{K'}{2} (x_1^2 - y^2) \quad (83)$$

$$\frac{d\pi_z}{dz} = \frac{e}{p} \left(\frac{\pi_x}{\pi_z} B_y - \frac{\pi_y}{\pi_z} B_x \right) \quad (84)$$

The calculation of the right hand side of the above equation can be decomposed into four steps:

$$\frac{\pi_x}{\pi_z} = \frac{\pi_{x0}}{\pi_{z0}} + \frac{\pi_{x1}}{\pi_{z0}^3} + \frac{\pi_{x0}}{2\pi_{z0}^3} \left(3 \frac{\pi_{x1}^2}{\pi_{z0}^2} + \pi_y^2 \right) \quad (85)$$

$$\frac{e}{p} B_y = \frac{1}{\varrho} + Kx_1 + \frac{1}{2} K' (x_1^2 - y^2) \quad (86)$$

$$\frac{\pi_y}{\pi_z} = \frac{\pi_y}{\pi_{z0}} + \frac{\pi_{x0}}{\pi_{z0}^3} \pi_{x1} \pi_y \quad (87)$$

$$\frac{e}{p} B_x = Ky \quad (88)$$

so that $d\pi_{z1}/dz$ is given by:

$$\begin{aligned} \frac{d\pi_{z1}}{dz} = & - \frac{1}{\varrho} \frac{\pi_{x1}}{\pi_{z0}^3} + \frac{\pi_{x0}}{\pi_{z0}} Kx_1 + \frac{\pi_{x0}}{2\varrho\pi_{z0}^3} \left(3 \frac{\pi_{x1}^2}{\pi_{z0}^2} + \pi_y^2 \right) + \frac{K}{\pi_{z0}^3} \pi_{x1} x_1 \\ & + \frac{\pi_{x0}}{2\pi_{z0}^3} K' (x_1^2 - y^2) - K \frac{\pi_y \pi_y}{\pi_{z0}} \end{aligned} \quad (89)$$

Summing all the terms in the right hand side of $d\xi'/ds$ gives:

$$\begin{aligned} \frac{d\xi'}{ds} = & -Kx_1 - \frac{\pi_{x0}\pi_{x1}}{\rho\pi_{z0}^2} - \frac{1}{2\rho\pi_{z0}^2} [(2\pi_{x0}^2 + 1) \frac{\pi_{x1}^2}{\pi_{z0}^2} + \pi_y^2] \\ & - K\pi_{x0} \left(\frac{\pi_{x1}x_1}{\pi_{z0}^2} - \pi_y y \right) - \frac{K'}{2} (x_1^2 - y^2) \end{aligned}$$

or, in terms of the local coordinates,

$$\begin{aligned} \frac{d\xi'}{ds} = & -K \frac{\xi}{\pi_{z0}} - \frac{\pi_{x0}}{\pi_{z0}} \frac{\xi'}{\rho} - \frac{K'}{2} \left(\frac{\xi^2}{\pi_{z0}^2} - \eta^2 \right) + K\pi_{x0} (\eta\eta' - 2 \frac{\xi\xi'}{\pi_{z0}^2}) \\ & - \frac{1}{2\rho} \left(\frac{1 + \pi_{x0}^2}{\pi_{z0}^2} \xi'^2 + \eta'^2 \right) \end{aligned} \quad (90)$$

For the consequences that we shall draw from equation (80) and (90) it is of primary importance to check that the treatment that we have adopted coincides with Courant and Snyder (C-S) theory for small amplitude horizontal oscillations. The equations (80) and (90) reduced to their first order terms, are:

$$\frac{d\xi}{ds} = \xi' + \frac{\pi_{x0}}{\pi_{z0}} \frac{\xi}{\rho} \quad (91)$$

$$\frac{d\xi'}{ds} = -K \frac{\xi}{\pi_{z0}} - \frac{\pi_{x0}}{\pi_{z0}} \frac{\xi'}{\rho} \quad (92)$$

After deriving the first equation with respect to s and using (92), (46), (47), (48) and (91), one obtains the second order differential equation:

$$\frac{d^2\xi}{ds^2} + (K\pi_{z0} + \frac{1}{\rho^2}) \xi = 0 \quad (93)$$

This equation is indeed the basis for CS theory. It has nevertheless to be carefully interpreted as describing the motion of m . The distinction between $d\xi/ds$, the slope of the trajectory of m , and ξ' , the slope of the betatron oscillation has vanished; all the formulae related to angles in CS theory concern $d\xi/ds$, they can be related to the actual slope of the betatron oscillation by the transformation:

$$\xi' = \frac{d\xi}{ds} - \frac{\pi_{x0}}{\pi_{z0}} \frac{\xi}{\rho} \quad (94)$$

3.3.2 VERTICAL PLANE

The vertical variation of m results from the derivation of (70) with respect to s :

$$\frac{d\eta}{ds} = \frac{d}{ds} (y - \pi_{x0}\pi_y x_1) \quad . \quad (95)$$

Still, kinematic and field contributions can be distinguished. The kinematic part is:

$$\begin{aligned} \frac{dy}{ds} - \pi_{x0}\pi_y \frac{dx_1}{ds} &= \pi_{z0} \left(\frac{dy}{dz} - \pi_{x0}\pi_y \frac{dx_1}{dz} \right) \\ &= \pi_y (1 - \pi_{x0}\pi_{x1} - \pi_{z0}\pi_{z1}) \quad . \end{aligned} \quad (96)$$

Since $\pi_{x0}\pi_{x1} + \pi_{z0}\pi_{z1}$ is second order, the right hand side reduces itself to π_y or η' and the statement made for the horizontal motion can be generalized:

The kinematic contribution to the momentum of m is the betatron momentum.

The field term results from Lorentz's force application:

$$- x_1 \frac{d}{ds} (\pi_{x0}\pi_y) = \pi_{z0} \left(\pi_y \frac{x_1}{\rho} - K\pi_{x0} x_1 y \right) \quad . \quad (97)$$

By adding the two contributions, the change of vertical position in local coordinates is given by:

$$\frac{d\eta}{ds} = \eta' + \eta' \frac{\xi}{\rho} - K\pi_{x0} \xi \eta \quad . \quad (98)$$

The change of slope of the vertical betatron oscillation is given by:

$$\begin{aligned} \frac{d\eta'}{ds} &= \frac{d\pi_y}{ds} \\ &= \pi_{z0} \frac{d\pi_y}{dz} \quad . \end{aligned} \quad (99)$$

The second order expansion of the vertical component of Lorentz's force is:

$$\begin{aligned} \frac{d\pi_y}{dz} &= K y + K' x_1 y \\ &= K\eta + K\pi_{x0} \frac{\xi}{\pi_{z0}} \eta' + K' \frac{\xi}{\pi_{z0}} \eta \end{aligned} \quad (100)$$

so that the second equation of the vertical motion is:

$$\frac{d\eta'}{ds} = K\pi_{z0}\eta + K\pi_{x0}\xi\eta' + K'\xi\eta \quad (101)$$

In the linear approximation, the system of first order equations is simply:

$$\left| \begin{array}{l} \frac{d\eta}{ds} = \eta' \\ \frac{d\eta'}{ds} = K\pi_{z0}\eta \end{array} \right. \quad (102)$$

and the passage to the second order differential equation is straightforward

$$\frac{d^2\eta}{ds^2} - K\pi_{z0}\eta = 0 \quad (103)$$

In contrast with the case of the horizontal motion, there is an identity between the slope $d\eta/ds$ of the trajectory of m and the slope η' of the betatron oscillation, an identity which is no longer true at the second order approximation.

3.3.3 SUMMARY AND DISCUSSION OF THE EQUATIONS OF THE BETATRON MOTION INSIDE A MAGNET

The equations of the betatron motion inside a magnet will be written in the form of a system of four first order differential equations whose left and right hand sides contain the first and second order terms respectively:

$$\frac{d\xi}{ds} - \xi' - \frac{\pi_{x0}}{\pi_{z0}} \frac{\xi}{\varrho} = \frac{1 + \pi_{x0}^2}{\pi_{z0}^2} \frac{\xi\xi'}{\varrho} + \frac{K\pi_{x0}}{\pi_{z0}^2} \xi^2 \quad (104)$$

$$\begin{aligned} \frac{d\xi'}{ds} + \frac{K\xi}{\pi_{z0}} + \frac{\pi_{x0}}{\pi_{z0}} \frac{\xi'}{\varrho} = & - \frac{K'}{2} \frac{\xi^2}{\pi_{z0}^2} - \eta^2 + K\pi_{x0} (\eta\eta' - \frac{2\xi\xi'}{\pi_{z0}^2}) \\ & - \frac{1}{2\varrho} \left(\frac{1 + \pi_{x0}^2}{\pi_{z0}^2} \xi'^2 + \eta'^2 \right) \end{aligned} \quad (105)$$

$$\frac{d\eta}{ds} - \eta' = - K\pi_{x0}\xi\eta + \frac{\xi}{\varrho} \eta' \quad (106)$$

$$\frac{d\eta'}{ds} - K\pi_{z0}\eta = K\pi_{x0}\xi\eta' + K'\xi\eta \quad (107)$$

Let us recall the meaning of the various variables:

- i) s is the curvilinear coordinate of the origin of the local frame (Fig. 3).
- ii) ξ and η are respectively the abscissa and ordinate of the intersection m of the tangent to the betatron oscillation with the plane normal to the closed orbit.
- iii) ξ' and η' are respectively the horizontal and vertical slopes of the tangent to the betatron oscillation. ξ , ξ' , η and η' are expressed in the local frame.
- iv) w_{x0} and w_{z0} are respectively the horizontal and longitudinal components of the reduced closed orbit momentum vector P_0/p in the magnet cartesian frame (Fig. 4). In a bending magnet of deflection angle θ , one has:

$$w_{x0} = \sin(\theta/2), \quad w_{z0} = \cos(\theta/2)$$

at the upstream end, and

$$w_{x0} = -\sin(\theta/2), \quad w_{z0} = \cos(\theta/2)$$

at the downstream end. In a quadrupole on the central orbit, w_{x0} and w_{z0} are given by:

$$w_{x0} = 0, \quad w_{z0} = 1$$

all along the magnet.

- v) $1/\rho$ is the closed orbit curvature. K is the focusing strength related to the first derivative of the field on the closed orbit with respect to the radial coordinate x in the magnet cartesian frame. K' is the radial derivative of K .

It is totally outside the scope of this paper to perform a general integration of the equations of the nonlinear betatron motion. We shall discuss the system in a qualitative way and try to establish a hierarchy in the contribution of the various non-linear terms.

The very first observation which can be made is the coupling between horizontal and vertical motions. To analyze the strength of each term, we shall simplify the problem by considering a particle oscillating about the central orbit. In that case, the quadrupole term comes always under the form Kw_{x0} and is thus zero at least in machines whose bending and focusing functions are separated. The importance of the dipole contribution is best appreciated in the second equation where terms like $K'\xi^2$ and ξ'^2/ρ can be directly compared. As a very general rule, the dipole term is much smaller for two reasons:

- i) $1/\rho \ll K'$,
the difference between dipole curvature and sextupole component is huge in large machines; it may, however, not be as significant in small rings like CERN Antiproton Accumulator.
- ii) $\xi' \sim \xi/\beta$,
for a given emittance, angle and position are in the ratio of the β function and β is at least equal to several metres in dipoles so that in practice the sextupole term exceeds the dipole term by more than an order of magnitude.

This analysis must be confirmed by a careful particle tracking in a lattice made of bending and focusing magnets only. To my

knowledge, there are no practical examples of strong non-linearities induced by dipoles or quadrupoles*. For these reasons, the system of non-linear equations will be limited to the sextupole terms; moreover, the assumption of betatron oscillations about the central orbit in a separated function lattice will be maintained:

$$\begin{aligned}
 & \frac{d\xi}{ds} - \xi' - \frac{w_{x0}}{w_{z0}} \frac{\xi}{\varrho} = 0 \\
 & \frac{d\xi'}{ds} + K\xi + \frac{w_{x0}}{w_{z0}} \frac{\xi'}{\varrho} = - \frac{K'}{2} (\xi^2 - \eta^2) \\
 & \frac{d\eta}{ds} - \eta' = 0 \\
 & \frac{d\eta'}{ds} - K\eta = K'\xi\eta
 \end{aligned} \tag{108}$$

3.4 BETATRON MOTION IN THE NEIGHBOURHOOD OF TILTED AND OF CURVED MAGNET BOUNDARIES

The previous section was dedicated to the calculation of the betatron oscillation inside a magnet where the magnetic field has only transverse components. The treatment was composed of two parts:

- i) Motion of the point M deduced from the point M_0 on the closed orbit by variation of the equations of motion in the cartesian frame of the magnet.
- ii) Projection m of M onto the normal plane to the closed orbit in M_0 along the direction of the momentum vector $\vec{MP}$.

The reason for the second stage was to concile the boundary conditions of the betatron oscillation with the region of validity of a variational calculus. At the end of a magnet, where one assumes an abrupt transition between a zero field region (outside the magnet) and a finite field region (inside the magnet), a variational calculus is impossible because of the field discontinuity. However, one can obviate this difficulty and restore the validity of a variational calculus by using the *hard edge* model in which all the components of the end field are assumed to vary impulsively over an infinitely small length.

3.4.1 END FIELD EXPANSION

The description of the field at the end of a magnet is complicated by the presence of the longitudinal component. The *hard edge* model which is developed in this section is the simplest approach to the problem and puts the main physical features of a magnet end in evidence although some special but relatively rare effects related to

* Such lattices are often referred to as "linear" lattices. This is quite an improper definition of linearity.

Fig. 5. Hard edge model.

the finite extension of the fringe field are outside the scope of this treatment.

At the end of a magnet, one can distinguish the *stray field* whose lines are closed outside the magnet in a plane normal to the end face and the *wedge field* which occurs in that part of the magnet which deviates from the rectangular geometry (Fig. 5). In the hard edge model, both stray and wedge fields act like *thin lenses*: their strength is confined in slabs of infinitely small thickness stuck to the end faces of the reference rectangular magnet and is represented by a field like vector:

$$\vec{b}(x, y, z) = \vec{B}(x, y) \ell \delta(z) \quad (109)$$

where ℓ is a fictitious length which is present for the sake of homogeneity of the expression and δ is the delta-function.

As the equation of the end face is:

$$z = c_1 x + \frac{1}{2} c_2 x^2 \quad (110)$$

the radial dependence of the vertical component is expected to be:

$$B_y(x, 0) \cdot \ell = -(B_0 + G_0 x) \left(c_1 x + \frac{1}{2} c_2 x^2 \right) . \quad (111)$$

The sign "--" means that there is a lack of field with respect to a rectangular magnet when c_1 or c_2 are positive. The above quantity consists of a quadrupole-like term:

$$-B_0 c_1 x ,$$

a sextupole-like term:

$$- (G_0 c_1 + \frac{1}{2} B_0 c_2) x^2 ,$$

and an octupole-like term that we shall not study. Maxwell's equations impose that the two-dimensional expansion of the vertical and radial components of the equivalent field be:

$$B_y(x,y) \cdot \ell = -B_0 c_1 x - \frac{1}{2} (B_0 c_2 + 2G_0 c_1)(x^2 - y^2) \quad (112)$$

$$B_x(x,y) \cdot \ell = -B_0 c_1 y - (B_0 c_2 + 2G_0 c_1)xy . \quad (113)$$

The radial component which has been found in an abstract way can be interpreted as the sum of a wedge contribution

$$-G_0 c_1 xy ,$$

and of a stray field contribution

$$- B_0 c_1 y - (B_0 c_2 + G_0 c_1)xy .$$

It has been shown indeed (13) that the expression of the stray field in a plane normal to the end face is:

$$B_{z_1} = (B_0 + G_0 x)y \delta(z_1) .$$

In the magnet system of coordinates, the stray field is decomposed into a radial component:

$$B_x = B_{z_1} \sin\psi$$

and a longitudinal component

$$B_z = B_{z_1} \cos\psi$$

where ψ , the angle of the normal to the end face with the z-axis, is given by:

$$\begin{aligned} \tan\psi &= -dz/dx \\ &= -(c_1 + c_2 x) \end{aligned}$$

The assumption of an abrupt variation of the field in the wedge implies that ψ is small (an assumption which is not necessary in Helm's treatment):

$$\sin\psi \approx \tan\psi \approx \psi$$

$$\cos\psi \approx 1 .$$

We are thus led to adopt for the expression of the longitudinal component of the field:

$$B_z \ell = (B_0 + G_0 x) y \quad . \quad (114)$$

Knowing the three components of the field at the end of a magnet, we can determine the variations of the closed orbit and of the betatron oscillations.

3.4.2 CLOSED ORBIT

By isolating the coordinate x_0 in x , it turns out that the change of closed orbit momentum results from the integration of the equations:

$$\frac{d\pi_{x_0}}{dz} = - \frac{e}{p} B_y(x_0, 0) \cdot \ell \delta(z)$$

$$\frac{d\pi_{z_0}}{dz} = \frac{e}{p} \frac{\pi_{x_0}}{\pi_{z_0}} B_y(x_0, 0) \cdot \ell \delta(z) \quad ,$$

which gives:

$$\Delta\pi_{x_0} = \frac{c_1}{Q_0} x_0 + \frac{1}{2} \left(\frac{c_2}{Q_0} + 2K_0 c_1 \right) x_0^2 \quad (115)$$

$$\Delta\pi_{z_0} = - \frac{\pi_{x_0}}{\pi_{z_0}} \Delta\pi_{x_0} \quad . \quad (116)$$

3.4.3 CHANGE OF BETATRON POSITION

When there is a deflection of the closed orbit, the plane of reference of the betatron oscillation which is normal to the closed orbit is different at the entrance and at the exit of the thin lens. The kinematic terms have no contributions in a thin lens since they are finite and the equation of the horizontal change of position is deduced from (69):

$$\begin{aligned} \frac{d\xi}{dz} &= x_1 \frac{d\pi_z}{dz} \\ &= \frac{e}{p} x_1 \frac{\pi_x}{\pi_z} (B_y \ell) \delta(z) \quad . \end{aligned} \quad (117)$$

The calculation is driven as in section 3.3.1:

$$- \frac{e}{p} \frac{\pi_x}{\pi_z} (B_y \ell) = - \frac{e}{p} \left(\frac{\pi_{x_0}}{\pi_{z_0}} + \frac{\pi_{x_1}}{\pi_{z_0}^3} \right) B_y(x_0, 0) \cdot \ell - [B_0 c_1 + (B_0 c_2 + 2G_0 c_1) x_0] x_1$$

$$= \Delta \pi_{z0} - \frac{\Delta \pi_{x0}}{\pi_{z0}^3} \pi_{x1} - \frac{\pi_{x0}}{\pi_{z0}} \left[\frac{c_1}{\varrho_0} + \left(\frac{c_2}{\varrho_0} + 2K_0 c_1 \right) x_0 \right] x_1$$

and, finally,

$$\Delta \xi = \frac{\Delta \pi_{z0}}{\pi_{z0}} \xi - \frac{1 + \pi_{x0}^2}{\pi_{z0}^3} \Delta x_0 \xi \xi' - \frac{\pi_{x0}}{\pi_{z0}^3} \left[\frac{c_1}{\varrho_0} + \left(\frac{c_2}{\varrho_0} + 2K_0 c_1 \right) x_0 \right] \xi^2 \quad (118)$$

One can verify, at least for the first order term, that this expression is equivalent to:

$$x_1 = \frac{\xi}{\pi_{z0}} = \text{cste}$$

as it must be in a thin lens.

In the vertical plane, we have to perform a calculation similar to that of section 3.3.2:

$$\begin{aligned} \frac{d\eta}{dz} &= -x_1 \frac{d}{dz} (\pi_{x0} \pi_y) \\ &= -x_1 \left\{ \pi_y \frac{d\pi_{x0}}{dz} + \pi_{x0} \frac{e}{p} [(B_x \ell) - \frac{\pi_{x0}}{\pi_{z0}} (B_z \ell)] \delta(z) \right\} \quad (119) \end{aligned}$$

After replacement of $(B_x \ell)$ and $(B_z \ell)$ by their detailed expression (113) and (114) limited to the first order and conversion of the cartesian coordinates of the magnet system into local coordinates, one gets:

$$\Delta \eta = \frac{\pi_{x0}}{\pi_{z0}} \left(c_1 + \frac{\pi_{x0}}{\pi_{z0}} \right) \frac{\xi \eta}{\varrho} - \frac{\Delta \pi_{x0}}{\pi_{z0}} \xi \eta' \quad (120)$$

3.4.4 EDGE FOCUSING

If the orbit deflection in a magnet end has been studied and used only recently for the control of off-momentum orbit length and orbit dispersion, especially in antiproton rings whose momentum acceptance is very large, the edge focusing has always been embodied in the design of an accelerator lattice. The calculation methods are the same as those used in sections 3.3.1 and 3.3.2 but, beyond the well known linear focusing, we shall emphasize the chromatic properties of a magnet end and its non-linear characteristics.

The horizontal deflection results from the integration of the differential equation (81):

$$\begin{aligned}
\frac{d\xi'}{dz} &= \frac{d}{dz} (\pi_{z0} \pi_{x1} - \pi_{x0} \pi_{z1}) \\
&= \pi_{z0} \frac{d\pi_{x1}}{dz} - \pi_{x0} \frac{d\pi_{z1}}{dz} .
\end{aligned} \tag{121}$$

since the kinematic terms can be forgotten in a thin lens. As all the field terms contain a δ -function, one can integrate right away and work on the total variation:

$$\begin{aligned}
\Delta\xi' &= \pi_{z0} \cdot \frac{e}{p} \left[\frac{\pi_y}{\pi_z} (B_z \ell) - (B_y \ell) + (B_y(x_0, 0) \ell) \right] \\
&\quad - \pi_{x0} \cdot \frac{e}{p} \left[\frac{\pi_x}{\pi_z} (B_y \ell) - \frac{\pi_{x0}}{\pi_{z0}} (B_y(x_0, 0) \ell) - \frac{\pi_y}{\pi_z} (B_x \ell) \right] .
\end{aligned}$$

Following the same steps as in section 3.3.1 but with the new field components, one gets:

$$\begin{aligned}
\Delta\xi' &= - \frac{\Delta\pi_{z0}}{\pi_{z0}} \left(\xi' + \frac{\pi_{x0}}{\pi_{z0}} \xi'^2 \right) \\
&\quad + \frac{1}{\pi_{z0}^2} \left(\frac{c_1 + c_2 x_0}{\varrho_0} + 2K_0 c_1 x_0 \right) \left(\xi + 2 \frac{\pi_{x0}}{\pi_{z0}} \xi \xi' \right) \\
&\quad + \frac{1}{2\pi_{z0}} \left(\frac{c_2}{\varrho_0} + 2K_0 c_1 \right) \left(\frac{\xi^2}{\pi_{z0}^2} - \eta^2 \right) \\
&\quad + \left[\left(1 - \frac{\pi_{x0}}{\pi_{z0}} c_1 \right) \frac{1}{\varrho_0} + K_0 x_0 \right] \eta \eta' .
\end{aligned} \tag{122}$$

In the vertical plane, the equation (99) still applies:

$$\begin{aligned}
\frac{d\eta'}{dz} &= \frac{d\pi_y}{dz} \\
&= \frac{e}{p} \left[(B_x \ell) - \frac{\pi_x}{\pi_z} (B_z \ell) \right] \delta(z)
\end{aligned}$$

or, after integration,

$$\Delta\eta' = \frac{e}{p} \left[(B_x \ell) - \frac{\pi_x}{\pi_z} (B_z \ell) \right] . \tag{123}$$

The innovation here lies in the contribution of the longitudinal component which contains a first order term. After calculation, the detailed form of $\Delta\eta'$ is:

$$\begin{aligned}\Delta\eta' = & - \left[\frac{c_1 + c_2 x_0}{\rho_0} + 2K_0 c_1 x_0 + \left(\frac{1}{\rho_0} + K_0 x_0 \right) \frac{x_0}{w_{z0}} \right] \left(\eta + \frac{x_0}{w_{z0}} \xi \eta' \right) \\ & - \left(\frac{c_2}{\rho_0} + 2K_0 c_1 + \frac{x_0}{w_{z0}} K_0 \right) \frac{\xi \eta}{w_{z0}} \\ & - \left(\frac{1}{\rho_0} + K_0 x_0 \right) \frac{\xi' \eta}{w_{z0}^2} .\end{aligned}\quad (124)$$

By collecting the formulae (118), (120), (122) and (124), and limiting them to their first order terms, the transfer matrices of a magnet end can be constructed:

$$E_h = \begin{pmatrix} 1 + \frac{\Delta w_{z0}}{w_{z0}} & 0 \\ \frac{1}{w_{z0}^2} \left(\frac{c_1 + c_2 x_0}{\rho_0} + 2K_0 c_1 x_0 \right) & 1 - \frac{\Delta w_{z0}}{w_{z0}} \end{pmatrix} \quad (125)$$

$$E_v = \begin{pmatrix} 1 & 0 \\ - \left[\frac{c_1 + c_2 x_0}{\rho_0} + 2K_0 c_1 x_0 + \frac{x_0}{w_{z0}} \left(\frac{1}{\rho_0} + K_0 x_0 \right) \right] & 1 \end{pmatrix} \quad (126)$$

The determinant of the horizontal motion matrix is not strictly equal to 1 but it is a well known fact that perturbative treatments cannot respect conservation theorems to all orders. Positive coefficients c_1 or c_2 correspond to defocusing and focusing actions in the horizontal and vertical planes respectively for the upstream end. At the downstream end, the field decreases along the longitudinal axis of the magnet, the impulsive variation symbolized by the δ -function changes its sign with respect to the upstream edge and so do the coefficients c_1 and c_2 .

These sign considerations will be illustrated by comparing rectangular and sector dipoles of same deflection angle θ :

magnet type	:	rectangular	;	sector
edge position	:	upstream, downstream	;	upstream, downstream
c_1	:	0	;	$-\tan(\theta/2), \tan(\theta/2)$
x_0/w_{z0}	:	$\tan(\theta/2), -\tan(\theta/2)$	;	$\tan(\theta/2), -\tan(\theta/2)$
$(e_{21})_h$	:	0	;	$-\tan(\theta/2)/(\rho_0 w_{z0}^2)$
$(e_{21})_v$	:	$-(1/\rho_0) \tan(\theta/2)$	;	0

The properties of sector and rectangular dipoles are just complementary. The focusing strengths of both ends add up because η_{x0} changes its sign. The symmetry of the end effects is substantially altered for the off-momentum orbits by the influence of the orbit dispersion which increases along a dipole.

The first order variations of the terms e_{21} with the relative momentum errors $\Delta p/p_0$ are obtained by performing the following substitutions:

$$\frac{1}{\varrho_0} \rightarrow \frac{1}{\varrho_0} \left(1 - \frac{\Delta p}{p_0}\right)$$

$$K_0 \rightarrow K_0 \left(1 - \frac{\Delta p}{p_0}\right)$$

$$x_0 \rightarrow x_0 + D_x \frac{\Delta p}{p_0}$$

$$\eta_{x0} \rightarrow \eta_{x0} + D'_x \frac{\Delta p}{p_0}$$

where D_x is the horizontal orbit distortion and D'_x its longitudinal derivative.

The focusing deficiency:

$$- \left[\frac{c_1 + c_2 x_0}{\varrho_0} + 2K_0 c_1 x_0 + \left(\frac{\eta_{x0}}{\eta_{z0}} \frac{1}{\varrho_0} + K_0 x_0 \right) \right] \frac{\Delta p}{p_0}$$

is compensated not only by the sextupole-like terms:

$$\left(\frac{c_2}{\varrho_0} + 2K_0 c_1 + \frac{\eta_{x0}}{\eta_{z0}} K_0 \right) D_x \frac{\Delta p}{p_0}$$

but also by a term which depends on D'_x

$$\frac{1}{\eta_{z0}} \left(\frac{1}{\varrho_0} + K_0 x_0 \right) D'_x \frac{\Delta p}{p_0}$$

In machines whose orbit dispersion is strongly modulated, these terms can be at the origin of large discrepancies between horizontal and vertical chromaticities.

For the large amplitudes of the betatron oscillation, the comparison of the cases of the pure transverse field (105, 107) and of the end effects (122, 124) shows that even for a rectangular magnet, there are terms specific to the edge :

$$\frac{1}{\varrho} \eta \eta'$$

in the horizontal plane, and

$$\frac{1}{e\pi^2 z_0} \eta \xi'$$

in the vertical plane. However, if it is assumed that the most important terms are those which depend upon position uniquely, the curved end of a dipole or the tilted face of a quadrupole are equivalent to a sextupole field.

4. BEAM ENVELOPE IN A NON-LINEAR FIELD

More important than the motion of an individual particle is the knowledge of the beam envelope which determines the physical aperture of a machine. The very concept of envelope implies that all the particles have a bounded motion. The linear theory of strong focusing synchrotrons proves the existence of a beam envelope in focusing structures made of two types of quadrupole fields of opposite signs K_1 and K_2 belonging to the famous *stability necktie*. As we have seen in the previous section, linear lattices do not exist: even dipole and quadrupole field induce non-linear terms in the equation of motion, a real magnet contains always non-linear components, deliberate sextupole or higher order fields are introduced to correct the chromaticity. An approach to the calculation of the non-linear beam envelope is sketched in this section. Action-angle instead of position-momentum variables are systematically used. The differential equations are solved only at the first iteration with the linear solution as a reference state. Last, the validity of the theory is tested on the CERN Antiproton Collector, an ideal machine to study large amplitude oscillations since its transverse acceptance reaches $200 \pi \text{ mm.mrad}$.

4.1 REFERENCE STATE

The reference state of the machine is the linear approximation to the equations of motion (93, 103, 124):

$$\frac{d^2\xi}{ds^2} + (K\pi z_0 + \frac{1}{\rho^2}) \xi = 0$$

$$\frac{d^2\eta}{ds^2} - (K\pi z_0 - \frac{\pi x_0}{\rho}) \eta = 0$$

The coefficients of ξ and η are functions of s ; nevertheless, by analogy with the equations of an harmonic oscillator, the solution can be written:

$$\xi = a_x(s) \cos[\mu_x(s) - \mu_{x_0}] \quad (127)$$

$$\eta = a_y(s) \cos[\mu_y(s) - \mu_{y_0}] \quad (128)$$

Courant and Snyder have introduced the function β which links amplitude and phase through the relations:

$$a(s) = \sqrt{2J \beta(s)} \quad (129)$$

$$\mu(s) = \int_0^s \frac{d\sigma}{\beta(\sigma)} \quad (130)$$

The β function is essentially determined by the focusing fields.

The two constants of the motion are the action J which is a beam property equal to half the emittance and the initial phase μ_0 . As the Courant and Snyder theory has been explicitly developed for synchrotrons in the original paper, it is sometimes believed that the theory is limited to circular machines. In fact, the concept of the β function can be applied either to an open lattice (transfer line) or to a closed lattice (ring). The difference between the two cases lies in the boundary conditions imposed to the motion equation. In the open case, the boundary conditions are the initial coordinates, whereas they are cyclic for the closed case (genuine Hill's equation).

Let us insist on some advantages of the action-angle variables. The amplitude a is the maximum position of a particle, it is thus a measure of the beam envelope and the β function can be used to define not only particle transfer but beam envelope transfer from one point to another point of the lattice. In closed machines, it is through the phase variable that the resonant properties of the motion are expressed and a parameter of fundamental importance is the number of oscillations per turn Q :

$$Q = \frac{\mu(C) - \mu(0)}{2\pi}$$

where C is the length of the circumference. There is one value of Q per plane and this value is the same for all particles in the linear approximation.

The representation of the motion is particularly simple in a phase space where coordinates are the normalized positions.

$$q_x = \frac{\xi}{\sqrt{\beta_x}} = \sqrt{2J_x} \cos(\mu_x - \mu_{x0}) \quad (131)$$

$$q_y = \frac{\eta}{\sqrt{\beta_y}} = \sqrt{2J_y} \cos(\mu_y - \mu_{y0}) \quad (132)$$

and the new momenta :

$$\dot{q}_x = - \frac{dq_x}{d\mu_x} = \sqrt{2J_x} \sin(\mu_x - \mu_{x0}) \quad (133)$$

$$\dot{q}_y = - \frac{dq_y}{d\mu_y} = \sqrt{2J_y} \sin(\mu_y - \mu_{y0}) \quad (134)$$

The phase-space trajectories are two circles of radii $\sqrt{2J_x}$ and $\sqrt{2J_y}$ for the particles which oscillate in the horizontal or in the vertical plane. More generally, for the particle oscillations which are not polarized in any of the reference planes, the phase-space trajectories lie on a 2-torus (Figure 6) defined as the direct product of the horizontal and vertical circles.

Fig. 6. Projections of a 2-torus. The normalization eliminates the s -dependence; in real space, the beam cross-section is modulated along the ring by the β -function.

The direct product of two sets A and B is the set of ordered pairs (a in A, b in B), therefore a torus can be considered as the set of couples of phases (μ_x, μ_y) with μ_x in $(0, 2\pi Q_x)$ and μ_y in $(0, 2\pi Q_y)$, the extremities of the segments being identified (Figure 7).

Fig. 7. Topological representation of a 2 torus.

The formulae (131) and (133) define a *Floquet transformation* which reduces an equation with variable coefficients of the type:

$$\frac{d^2x}{ds^2} + K(s)x = F(x)$$

to an equation of forced linear oscillator :

$$\frac{d^2q}{d\mu^2} + q = \beta^{3/2} F(q) \quad (135)$$

This simplification results from the relation which links the β function to its derivatives with respect to s :

$$\frac{d\alpha}{ds} = K\beta - \gamma \quad (136)$$

where the Courant and Snyder's notation is used:

$$\alpha = -\frac{1}{2} \frac{d\beta}{ds} \quad (137)$$

$$\gamma = \frac{1 + \alpha^2}{\beta} \quad (138)$$

4.2 PERTURBED STATE

The non-linear dynamical systems are usually presented in the framework of Hamiltonian mechanics. As several lectures are dedicated to these techniques, it is meaningless to repeat here what the reader can find excellently exposed elsewhere in the proceedings of the school. However, it is of common experience that physical insight into non-linear behaviours is hard to acquire, it may therefore be relevant to develop an alternative approach which shines the same subject from a different point of view. Here, we shall deduce the shape of the beam envelope and the betatron tune from the asymptotic properties of the single particle motion.

The equations to deal with are of type (135) and they can be replaced by the set of first order differential equations:

$$\frac{dq}{d\mu} = -\dot{q} \quad (139)$$

$$\frac{d\dot{q}}{d\mu} = q - \beta^{3/2} F(q, s) \quad (140)$$

The motion can be split into its reference state (index "o") which has been described in the previous section and a perturbed state (index "1") which is defined by the system:

$$\frac{dq_1}{d\mu} = 0 \quad (141)$$

$$\frac{d\dot{q}_1}{d\mu} = -\beta^{3/2} F(q, s) \quad (142)$$

The absence of change of position (141) is true when the perturbation F acts like a thin lens, which will always be assumed. The circumference C of the ring is unfolded over an infinite number of turns and the law of motion is searched in the limit case where the phase μ increases indefinitely. This is the reason why the treatment is called asymptotic. The function F is concentrated at discrete locations:

$$s = s_j + nC$$

and its expression is:

$$F(q,s) = f(q) \cdot \delta(s - s_j - nC) \quad (143)$$

The subscript "j" is attached to each of the N perturbations in the ring, the integer "n" is characteristic of the number of turns and varies from 1 to infinity. Using (130), the equation (142) is integrated with respect to s:

$$\Delta \dot{q} = - \sum_{n=1}^{\infty} \sum_{j=1}^N \sqrt{\beta_j} f_j(q) \quad (144)$$

The normalized betatron position q is unknown but can be considered as the limit, if it exists, of the functions:

$$q_k = \sqrt{2J_k} \cos(\mu_k + \mu_0) \quad (145)$$

defined by a recursive process. A change of momentum:

$$\Delta \dot{q}_k = - \sum_{n=1}^{\infty} \sum_{j=1}^N \sqrt{\beta_j} f_j(q_{k-1}) \quad (146)$$

induces new values of action and phase:

$$J_k = \frac{1}{2} [q_{k-1}^2 + (\dot{q}_{k-1} + \Delta \dot{q}_k)^2] \quad (147)$$

$$\mu_k = \tan^{-1} \frac{\dot{q}_{k-1} + \Delta \dot{q}_k}{q_{k-1}} \quad (148)$$

The iteration index "k" varies from 1 to infinity and the initial value of q is its reference state q_0 . The construction of the functions q_k is inspired by Picart's method. Physically, it means that a particle has a perfectly faithful memory of its history. A loss of memory can only be due to an external dissipative phenomenon like photon emission by synchrotron radiation, scattering on the residual gas of the vacuum chamber which contains the beam of particles, Brownian motion inside the beam, etc.

The elaboration of the functions q_k is complicated by the interdependence of the non-linear fields as soon as k exceeds 1. Analytical formulae may be established by resorting to automatic algebraic computation. Such methods have been developed successfully in the framework of Hamiltonian formalism by L. Michelotti using Deprit's algorithm. We shall restrict ourselves to the elementary case of the

first iteration where the correlations between multipoles are not taken into account; for brevity, the index will be omitted.

An iteration can be associated with an order of perturbation when the intensity of the non-linear field is small as compared to the reference quadrupole field, which must be the case in real accelerators and will be assumed in the following. To neglect the quadratic and higher order terms in the expansions of the action (147) and the phase (148) is then consistent with the treatment of the first iteration only and the variations of action and phase are simply:

$$\Delta J = \dot{q} \Delta \dot{q} \quad (149)$$

$$\Delta \mu = \frac{1}{2J} q \Delta \dot{q} \quad (150)$$

It has to be noted that it is not the nature of the non-linear field which determines the order of the perturbation. To emphasize this point, we shall discuss the influence of sextupole and octupole fields on the beam envelope and on the betatron tune.

4.3 BEAM ENVELOPE

The size of the beam envelope around the central orbit is twice the radius of the maximum betatron position:

$$a = \sqrt{2J\beta} \quad .$$

J is no longer a constant of motion, β stays the β -function defined for the reference state and there is therefore a strict dependence of the beam envelope on the action J .

4.3.1 PURE SEXTUPOLE FIELD

The horizontal and vertical deflections due to a sextupole field are:

$$\Delta \dot{q}_x = \frac{(K' \ell)}{2} \sqrt{\beta_x} (\xi^2 - \eta^2) \quad (151)$$

$$\Delta \dot{q}_y = -(K' \ell) \sqrt{\beta_y} \xi \eta \quad (152)$$

Using the expressions (127, 128), transforming the real space coordinates into normalized coordinates and linearizing the product of trigonometric functions provide the formulae for the changes of action at the location of the sextupole:

$$\begin{aligned} \Delta J_{x1} = \frac{(K' \ell)}{4} \sqrt{2J_x \beta_x} \{ & (J_x \beta_x - 2J_y \beta_y) \sin(\mu_x - \mu_{x0}) + J_x \beta_x \sin 3(\mu_x - \mu_{x0}) \\ & - J_y \beta_y [\sin(\mu_x + 2\mu_y - \mu_{x0} - 2\mu_{y0}) + \sin(\mu_x - 2\mu_y - \mu_{x0} + 2\mu_{y0})] \} \quad (153) \end{aligned}$$

$$\Delta J_{Y1} = -\frac{(K'\ell)}{2} \sqrt{2J_X\beta_X} J_Y\beta_Y [\sin(\mu_X+2\mu_Y-\mu_{X0}-2\mu_{Y0}) - \sin(\mu_X-2\mu_Y-\mu_{X0}+2\mu_{Y0})] \quad (154)$$

In first approximation, the perturbed motion results from the superposition of four modes which can be characterized by the couples of integers:

$$\bar{m}_1 = (1,0), \quad \bar{m}_2 = (3,0), \quad \bar{m}_3 = (1,2), \quad \bar{m}_4 = (1,-2) \quad (155)$$

and by the components on each mode:

$$C_{X1} = J_X\beta_X - 2J_Y\beta_Y, \quad C_{X2} = J_X\beta_X, \quad C_{X3} = -J_Y\beta_Y, \quad C_{X4} = -J_Y\beta_Y \quad (156)$$

$$C_{Y1} = 0, \quad C_{Y2} = 0, \quad C_{Y3} = J_Y\beta_Y, \quad C_{Y4} = -J_Y\beta_Y \quad (157)$$

By introducing the phase vector:

$$\bar{\mu} = (\mu_X - \mu_{Y0}, \mu_Y - \mu_{Y0}) \quad (158)$$

the changes of action take the compact form:

$$\Delta J_{X1} = \frac{(K'\ell)}{4} \sqrt{2J_X\beta_X} \sum_{r=1}^4 C_{Xr} \sin(\bar{m}_r \cdot \bar{\mu}) \quad (159)$$

$$\Delta J_{Y1} = -\frac{(K'\ell)}{2} \sqrt{2J_X\beta_X} \sum_{r=1}^4 C_{Yr} \sin(\bar{m}_r \cdot \bar{\mu}) \quad (160)$$

At each turn, the phase increases by $2\pi\bar{Q}$ and the cumulative change of action after n turns is:

$$\Delta J_{Xn} = \frac{(K'\ell)}{4} \sqrt{2J_X\beta_X} \sum_{r=1}^4 C_{Xr} \sum_n \sin \bar{m}_r \cdot (\bar{\mu} + 2\pi n\bar{Q}) \quad (161)$$

$$\Delta J_{Yn} = -\frac{(K'\ell)}{2} \sqrt{2J_X\beta_X} \sum_{r=1}^4 C_{Yr} \sum_n \sin \bar{m}_r \cdot (\bar{\mu} + 2\pi n\bar{Q}) \quad (162)$$

with

$$\bar{Q} = (Q_X, Q_Y) \quad (163)$$

The sums at the right hand sides are the imaginary parts of the geometrical series:

$$\sum_n e^{i\bar{m} \cdot (\bar{\mu} + 2\pi n \bar{Q})}$$

which tends towards:

$$\frac{e^{i\bar{m} \cdot \bar{\mu}}}{1 - e^{2\pi i \bar{m} \cdot \bar{Q}}}$$

when n increases indefinitely. The limits of the ΔJ 's are thus:

$$\Delta J_{x\infty} = \frac{(K' \ell)}{8} \sqrt{2J_x \beta_x} \sum_{r=1}^4 c_{xr} \frac{\cos \bar{m}_r \cdot (-\pi \bar{Q} + \bar{\mu})}{\sin(\pi \bar{m}_r \cdot \bar{Q})} \quad (164)$$

$$\Delta J_{y\infty} = - \frac{(K' \ell)}{4} \sqrt{2J_x \beta_x} \sum_{r=1}^4 c_{yr} \frac{\cos \bar{m}_r \cdot (-\pi \bar{Q} + \bar{\mu})}{\sin(\pi \bar{m}_r \cdot \bar{Q})} \quad (165)$$

It has just been proven that, in general, the asymptotic change of action exists at the first iteration level. Singular cases occur when the denominators of the right hand sides vanish; in that case, the particle is trapped in a *resonance* and its motion may become unstable or, at least, reach amplitudes which exceed the aperture of the machine. A resonance is characterized by a linear relationship between the horizontal and vertical tunes:

$$\bar{m}_r \cdot \bar{Q} = p, \quad r = 1, 2, 3, 4 \quad (166)$$

where the integer p is the *harmonic number*. Its order is the sum of the absolute values of the two components of the vector $\bar{m}_r$. It is drawn in a *working diagram* (Fig. 8) which is used to define the operational tunes (Q_x, Q_y) of the machine.

Fig. 8. Third order resonances between two integer values of the betatron tunes.

The last step consists of adding the contributions of all the non-linear lenses. The final change of action is the same for all beam sections around the ring, this is the virtue of normalized coordinates. To derive the actual beam envelope distortion in the real space, an observer is placed at the origin of the particle oscillation. Observer (index "i") and lenses (index "j") locations are distinguished by re-writing the vector $\bar{\mu}$:

$$\bar{\mu}_{ij} = \bar{\mu}_j - \bar{\mu}_i, \quad (167)$$

the lenses being always downstream to the observer. The change of amplitude Δa_i is such that:

$$a_i + \Delta a_i = \sqrt{2(J + \Delta J)\beta_i}$$

and

$$\Delta a_i = \sqrt{\frac{\beta_i}{2J}} \Delta J. \quad (168)$$

Grouping together formulae (168), (164), (165), (155), (156) and (157), the detailed expression of the envelope distortions is:

$$\begin{aligned} \Delta a_{xi} = & \frac{\sqrt{\beta_{xi}}}{8} \sum_j \sqrt{\beta_{xj}} (K' \ell)_j \left\{ (J_x \beta_{xj} - 2J_y \beta_{yj}) \frac{\cos(-Q_x \pi + \mu_{xij})}{\sin(Q_x \pi)} \right. \\ & + J_x \beta_{xj} \frac{\cos 3(-Q_x \pi + \mu_{xij})}{\sin 3(Q_x \pi)} \\ & - J_y \beta_{yj} \frac{\cos[-(Q_x + 2Q_y) \pi + \mu_{xij} + 2\mu_{yij}]}{\sin(Q_x + 2Q_y) \pi} \\ & \left. - J_y \beta_{yj} \frac{\cos[-(Q_x - 2Q_y) \pi + \mu_{xij} - 2\mu_{yij}]}{\sin(Q_x - 2Q_y) \pi} \right\} \quad (169) \end{aligned}$$

$$\begin{aligned} \Delta a_{yi} = & - \frac{\sqrt{J_x J_y \beta_{yi}}}{4} \sum_j \sqrt{\beta_{xj}} \beta_{yj} (K' \ell)_j \left\{ \frac{\cos[-(Q_x + 2Q_y) \pi + \mu_{xij} + 2\mu_{yij}]}{\sin(Q_x + 2Q_y) \pi} \right. \\ & \left. - \frac{\cos[-(Q_x - 2Q_y) \pi + \mu_{xij} - 2\mu_{yij}]}{\sin(Q_x - 2Q_y) \pi} \right\} \quad (170) \end{aligned}$$

One shall note in the case of the sextupole field a few facts which are general as soon as the multipole field order exceeds the quadrupole term:

- i) Horizontal and vertical motions are coupled even when the horizontal plane is a plane of symmetry. This characteristic has already been mentioned in the summary of the pure transverse field effects (section 3.3.3). Its importance will be put in evidence in the example treated in section 4.3.
- ii) The change of action depends on the amplitude of the particle since the actions J_x and J_y remain explicitly in formulae (169), (170).
- iii) There are no longer one but four modes of oscillation. The number of modes increases with the degree of the non-linearity. The action changes related to the coupled resonances are linked through the expressions:

$$2\Delta J_x - \Delta J_y = 0 \quad (171)$$

for the *sum* resonance:

$$Q_x + 2Q_y = 0 ,$$

and

$$2\Delta J_x + \Delta J_y = 0 \quad (172)$$

for the *difference* resonance

$$Q_x - 2Q_y = 0 .$$

The motion is therefore bounded near a difference resonance and can be unlimited near a sum resonance, at least in the framework of the first iteration of the equation of motion. These features will be confirmed in the study of the end field and of the octupole field.

4.3.2 SOLENOIDAL COUPLING

The nonlinear motion of a particle at the end of a magnet is affected by the longitudinal or *solenoidal* component of the magnetic field. The nature of this contribution has no equivalence in a pure transverse field because the coupling between horizontal and vertical motions depends not only on position but also on velocity:

$$\Delta \xi' = \frac{1}{\rho} \eta \eta' \quad (173)$$

$$\Delta \eta' = - \frac{1}{\rho} \eta \xi' . \quad (174)$$

In the action-angle variables, the expressions of η , η' and ξ' are:

$$\eta = \sqrt{2J_y \beta_y} \cos (\mu_y - \mu_{y0}) \quad (175)$$

$$\eta' = - \sqrt{2J_y / \beta_y} [\alpha_y \cos (\mu_y - \mu_{y0}) + \sin (\mu_y - \mu_{y0})] \quad (176)$$

$$\xi' = -\sqrt{2J_X/\beta_X} [\alpha_X \cos(\mu_X - \mu_{X0}) + \sin(\mu_X - \mu_{X0})] \quad (177)$$

The changes of action associated with one deflection of this nature are:

$$\Delta J_X = \dot{q}_X \Delta \dot{q}_X$$

$$\Delta J_Y = \dot{q}_Y \Delta \dot{q}_Y$$

with

$$\dot{q}_X = \sqrt{2J_X} \sin(\mu_X - \mu_{X0}) ,$$

$$\dot{q}_Y = \sqrt{2J_Y} \sin(\mu_Y - \mu_{Y0}) ,$$

$$\Delta \dot{q}_X = -\sqrt{\beta_X} \Delta \xi' ,$$

$$\Delta \dot{q}_Y = -\sqrt{\beta_Y} \Delta \eta' .$$

The algebra of the previous section is repeated with a slight variant: the real part of the geometrical series is also needed:

$$\operatorname{Re} \frac{e^{i \bar{m} \cdot \bar{\mu}}}{1 - 2\pi i \bar{m} \bar{Q}} = \frac{\sin \bar{m}(\pi \bar{Q} - \bar{\mu})}{2 \sin \pi \bar{m} \bar{Q}}$$

The final result is:

$$\begin{aligned} \Delta a_{xi} = \frac{J_Y \sqrt{\beta_{xi}}}{4} \sum_j \frac{\sqrt{\beta_{xj}}}{e_j} \left\{ 2\alpha_{yj} \frac{\cos(-Q_X \pi + \mu_{xij})}{\sin Q_X \pi} \right. \\ + \frac{\alpha_{yj} \cos[-(Q_X - 2Q_Y)\pi + \mu_{xij} - 2\mu_{yij}] - \sin[-(Q_X - 2Q_Y)\pi + \mu_{xij} - 2\mu_{yij}]}{\sin(Q_X - 2Q_Y)\pi} \\ \left. + \frac{\alpha_{yj} \cos[-(Q_X + 2Q_Y)\pi + \mu_{xij} + 2\mu_{yij}] - \sin[-(Q_X + 2Q_Y)\pi + \mu_{xij} + 2\mu_{yij}]}{\sin(Q_X + 2Q_Y)\pi} \right\} \quad (178) \end{aligned}$$

$$\begin{aligned} \Delta a_{yi} = -\frac{\sqrt{J_X J_Y \beta_{yi}}}{4} \sum_j \frac{\beta_{yj}}{e_j \sqrt{\beta_{xj}}} \\ \left\{ -\frac{\alpha_{xj} \cos[-(Q_X - 2Q_Y)\pi + \mu_{xij} - 2\mu_{yij}] + \sin[-(Q_X - 2Q_Y)\pi + \mu_{xij} - 2\mu_{yij}]}{\sin(Q_X - 2Q_Y)\pi} \right. \\ \left. + \frac{\alpha_{xj} \cos[-(Q_X + 2Q_Y)\pi + \mu_{xij} + 2\mu_{yij}] + \sin[-(Q_X + 2Q_Y)\pi + \mu_{xij} + 2\mu_{yij}]}{\sin(Q_X + 2Q_Y)\pi} \right\} \quad (179) \end{aligned}$$

As expected, the end field generates special effects. In addition to the β -function, the α -function brings a contribution. On the other hand, the invariants of the motion are much more difficult to derive than in the case of the transverse field. It will only be shown that the motion remains bounded on a difference resonance. The change of action ΔJ_- can be considered as the sum of two terms, the first one, ΔJ_{1-} , is related to the α -function and the second one, ΔJ_{2-} , is independent of α :

$$\Delta J_- = \Delta J_{1-} + \Delta J_{2-} \quad (180)$$

ΔJ_{1-} is composed of the cosine terms exclusively in (178, 179) and verifies the condition:

$$\frac{\alpha_x}{\beta_x} \Delta J_{1x-} - \frac{\alpha_y}{\beta_y} \Delta J_{1y-} = 0 \quad (181)$$

As a consequence of the alternating gradient focusing, horizontal and vertical β -functions vary in opposite directions and at a given curvilinear position s , α_x and α_y are of opposite signs; the coefficients of ΔJ_{1x-} and ΔJ_{1y-} are thus of the same sign and ΔJ_{1-} is bounded. ΔJ_{2-} is related to the sine terms and its horizontal and vertical components are constrained by the relation:

$$\frac{\Delta J_{2x-}}{\beta_x} + \frac{\Delta J_{2y-}}{\beta_y} = 0 \quad (182)$$

which ensures that ΔJ_{2-} and therefore ΔJ_- are bounded.

For the sum resonance, the relations corresponding to (181) and (182) are:

$$\frac{\alpha_x}{\beta_x} \Delta J_{1x+} + \frac{\alpha_y}{\beta_y} \Delta J_{1y+} = 0 \quad (183)$$

$$\frac{\Delta J_{2x+}}{\beta_x} - \frac{\Delta J_{2y+}}{\beta_y} = 0 \quad (184)$$

and nothing prevents ΔJ_+ from growing indefinitely.

4.3.3 OCTUPOLE FIELD AND RESONANCE PROLIFERATION

Once the deflections in an octupole

$$\Delta \xi' = - \frac{(K''\ell)}{6} (\xi^3 - 3\xi\eta^2) \quad (185)$$

$$\Delta \eta' = \frac{(K''\ell)}{6} (3\xi^2\eta - \eta^3) \quad (186)$$

are recalled, the calculations performed in the sextupole case can be repeated identically and the envelope distortions observed at s_i and created by octupoles located at s_j are given by:

$$\Delta a_{xi} = \frac{\sqrt{2J_x \beta_{xi}}}{48} \sum_j \beta_{xj}(K''\ell)_j \left\{ 2(J_x \beta_{xj} - 3J_y \beta_{yj}) \frac{\cos 2(-Q_x \pi + \mu_{xij})}{\sin 2Q_x \pi} \right. \\ + J_x \beta_{xj} \frac{\cos 4(-Q_x \pi + \mu_{xij})}{\sin 4Q_x \pi} \\ - 3J_y \beta_{yj} \frac{\cos 2[-(Q_x - Q_y) \pi + \mu_{xij} - \mu_{yij}]}{\sin 2(Q_x - Q_y) \pi} \\ \left. - 3J_y \beta_{yj} \frac{\cos 2[-(Q_x + Q_y) \pi + \mu_{xij} + \mu_{yij}]}{\sin 2(Q_x + Q_y) \pi} \right\} \quad (187)$$

$$\Delta a_{yi} = \frac{\sqrt{2J_y \beta_{yi}}}{48} \sum_j \beta_{yj}(K''\ell)_j \left\{ 2(3J_x \beta_{xj} - J_y \beta_{yj}) \frac{\cos 2(-Q_y \pi + \mu_{yij})}{\sin 2Q_y \pi} \right. \\ - J_y \beta_{yj} \frac{\cos 4(-Q_y \pi + \mu_{yij})}{\sin 4Q_y \pi} \\ - 3J_x \beta_{xj} \frac{\cos 2[-(Q_x - Q_y) \pi + \mu_{xij} - \mu_{yij}]}{\sin 2(Q_x - Q_y) \pi} \\ \left. - 3J_x \beta_{xj} \frac{\cos 2[-(Q_x + Q_y) \pi + \mu_{xij} + \mu_{yij}]}{\sin 2(Q_x + Q_y) \pi} \right\} \quad (188)$$

and the motion invariants on the sum and difference coupled resonances are:

$$\Delta J_{x+} - \Delta J_{y+} = 0 \quad (189)$$

$$\Delta J_{x-} + \Delta J_{y-} = 0 \quad (190)$$

The diagram of resonances is represented in Fig. 9. One notes the multiplication of the resonances with the increasing multipole order. About the proliferation of resonances there is a property of rational numbers which is worth reminding:

The rational number of smallest denominator belonging to the interval $(p_1/q_1, p_2/q_2)$ is $(p_1 + p_2)/(q_1 + q_2)$.

Since the set of the rational numbers is dense in the set of the real numbers, it is hopeless to find a working point which does not

Fig. 9. Resonance diagram for octupole fields.

sit on a resonance. However, it is a fact of experience that envelope distortions are the weakest when the working point is remote from low order resonances. The property of the rational numbers which has just been quoted gives a recipe to establish the numerology of resonances in a given area of the working diagram. A typical path of convergence for the determination of the working point is the following:

Interval of resonance orders	Range of tune adjustment
3, 4	$\frac{1}{3} - \frac{1}{4} = 0.083$
7, 4	$\frac{2}{7} - \frac{1}{4} = 0.035$
7, 11	$\frac{2}{7} - \frac{3}{11} = 0.013$
18, 11	$\frac{5}{18} - \frac{3}{11} = 0.005$

Exercise: Find an alternative path starting between second and third order resonances.

The scheme seems to be applicable to one-dimension resonances only but it can be extended to the two-dimension case of a working point close to the main diagonal which is the locus of all the resonance vertices. The neighbourhood of the main diagonal is actually a privileged area to tune a machine because all the sum resonances are

well confined there; as a counterpart, linear or, worse, non-linear coupling are maxima when the horizontal and vertical tunes are the same and they may have serious adverse effects on the beam envelope.

4.4 BETATRON TUNE

In the spirit of the asymptotic treatment of the betatron motion, the betatron tune is defined as the mean value of the number of betatron oscillations per turn over a great number of turns:

$$\Delta Q = \frac{1}{2\pi N} \sum_{j=1}^N \Delta \mu_j \quad (191)$$

or, using (150),

$$\Delta Q = \frac{1}{4\pi N J} \sum_{j=1}^N q_j \Delta \dot{q}_j \quad (192)$$

with the same index notation as for the beam envelope distortion.

Let us first apply this definition to the well known case of a quadrupole perturbation:

$$\Delta \xi' = -(\Delta K \cdot \ell) \xi \quad (193)$$

Substituting the expressions:

$$\xi = \sqrt{\beta_x} q_x$$

$$q_x = \sqrt{2J_x} \cos \mu_x$$

$$\Delta \dot{q}_x = -\sqrt{\beta_x} \Delta \xi'$$

into (192) gives:

$$\Delta Q = \frac{1}{4\pi N} \sum_{n=0}^N \sum_j (\Delta K \cdot \ell)_j \beta_j [1 + \cos 2(\mu_{ij} + 2\pi n Q)]$$

As it has been seen in section 4.3.1, the sum,

$$\sum_{n=0}^N \cos 2(\mu_{ij} + 2\pi n Q)$$

is bounded unless the tune belongs to a half-integer resonance, its mean value tends towards zero and

$$\Delta Q_{\infty} = \lim_{N \rightarrow \infty} \Delta Q_N = \frac{1}{4\pi} \sum_j (\Delta K \cdot \ell)_j \beta_j \quad (194)$$

When the same calculation is repeated for sextupole deflections, it turns out that there is no change of betatron tune because the right hand side of ΔQ contains only trigonometric sums which are all bounded outside resonance condition.

The situation changes with octupole fields (4.3.3) where there is a net change of betatron tune which varies with the square of the oscillation amplitude whereas the quadrupole change of tune was independent of the amplitude:

$$\Delta Q_x = \frac{1}{16\pi} \sum_j (K'' \ell)_j \beta_{xj} (J_x \beta_{xj} - 2 J_y \beta_{yj}) \quad (195)$$

$$\Delta Q_y = - \frac{1}{16\pi} \sum_j (K'' \ell)_j \beta_{yj} (2 J_x \beta_{xj} - J_y \beta_{yj}) \quad (196)$$

It is not difficult to generalize these results to a general multipole field and one shall retain that:

The only field components which produce a change of betatron tune are the even multipoles like quadrupole, octupole, etc.

The theory of non-linear betatron oscillations will be stopped here. The correlations between multipoles which constitute the major new physical fact in the treatment of higher order iterations will be left in the shade. Their importance will nevertheless be revealed in the practical example treated in the next section and it is so overwhelming that it seems to swamp any other consideration. One can then wonder whether a first approximation theory is of some interest. The answer is, fortunately, positive because a correction of beam envelope distortions based on a first approximation theory not only diminishes the magnitude of the terms which are specific to this theory but also cuts the correlations between multipoles.

4.5 CORRECTION OF BEAM ENVELOPE DISTORTIONS

The non-linear beam envelope distortions are mainly due, and by far, to the sextupoles used to correct the chromaticity. An ideal scheme of chromaticity correction would be self-compensated for amplitude dependent effects and, in fact, all the big machines are equipped with sextupole configurations which tend to minimize the beam envelope distortions. In small machines, once the chromaticity and the off-momentum orbits are corrected, there is no more freedom to take extra constraints into account; the problem is then resolved by placing sextupoles in zero orbit dispersion regions so that they act specifically on particle amplitude independently of the momentum.

4.5.1 SELF-COMPENSATED SCHEMES OF CHROMATICITY CORRECTION

The basic "brick" of a synchrotron lattice is a FODO cell made of two quadrupoles of opposite focusing strengths with sextupoles adjacent to them. The betatron phase advances and the β -functions repeat identically to themselves from one sextupole to the next. Moreover, the phase advances are the same in the horizontal and vertical planes. As a consequence, the phases of the four modes which contribute to the change of action (153, 154) vary like μ and 3μ . Typical phase advances per cell are $\pi/3$ or $\pi/2$ and the compensation of sextupole distortions occurs over six or four cells (Fig. 10).

Fig. 10. Vector diagrams of sextupole induced beam distortions in the case of ideal FODO cells.

Real lattice cells deviate from these ideal conditions, the sextupole strengths are then calculated by using computer codes which contain algorithms of envelope distortion minimization.

4.5.2 PRINCIPLE OF A NON-LINEAR BEAM ENVELOPE CORRECTION

The self-compensation of a sextupole scheme is nothing but the application of the "bump" method which is so widely used in closed orbit manipulations. The analogy between closed orbit and beam envelope distortion is in fact clearly apparent in the structure of the formulae (169, 170).

In the case of an orbit distortion due to random dipole errors, the orbit has to be observed all around the ring. Here, the causes of the envelope distortions, the chromaticity sextupoles, are known in position and strength, the beam envelope can thus be reconstructed everywhere from the information of amplitude and phase for each mode at a given point of observation. As the available space in the lattice may be very limited, the method of the *best correctors* which permits an efficient correction with few elements will be used.

The process of correction is the following:

- i) The actions J_x , J_y determine the particle oscillation amplitude.

- ii) The point of observation fixes the origin of the phases:

$$\mu_{x0} = \mu_{y0} = 0.$$

- iii) The m chromaticity sextupoles define an 8-component vector $\bar{b}$. Each component b_i ($i = 1, 8$) is either the real or the imaginary part of the complex number:

$$\sum_{j=1}^m (K' \ell)_j \sqrt{\beta_{xj}} C_{xrj} \exp i \bar{m}_r \cdot (-\bar{Q}\pi + \bar{\mu}) \quad r = 1, 4 \quad (197)$$

Vectors $\bar{m}$, $\bar{\mu}$, $\bar{Q}$ and coefficients C_{xrj} are given by (155, 158, 163) and (156, 157) respectively.

- iv) The n correction sextupoles are preferably located in zero orbit dispersion straight sections near each quadrupole and about the middle of each free section. Their number is not restricted. Each of them is characterized by a vector a_j whose eight components are listed in the same order as for the vector $\bar{b}$:

$$a_{ij} = \text{Re or Im } \sqrt{\beta_{xj}} C_{xrj} \exp i \bar{m}_r \cdot (-\bar{Q}\pi + \bar{\mu}) \quad \begin{matrix} r = 1, 4 \\ i = 1, 8 \\ j = 1, n \end{matrix} \quad (198)$$

The coefficients a_{ij} compose the correction matrix A .

- v) The strengths of the correction sextupoles are the components of a k -vector $\bar{x}_k$ ($k = 1, 8$). The index k denotes both the dimension of the correction vector and the iteration number of the approximate solution to the system:

$$A\bar{x} + \bar{b} = 0. \quad (199)$$

At each iteration, the norm of the residual vector

$$\bar{r}_k = A\bar{x}_k + \bar{b} \quad (200)$$

is minimized and the sextupole correction is incorporated into a particle tracking program which tests the validity of the solution. The interpretation of the tracking results are based on the variations of the horizontal and vertical actions or of quantities proportional to them like the Courant and Snyder's invariant:

$$W = \frac{x^2}{\beta} + \beta x'^2 \quad (201)$$

defined at a symmetry point of the machine where the observer sits. The initial coordinates (x_0, x_0') define a value W_0 of W . In a linear machine, the ratio W/W_0 remains equal to unity at each passage of the particle; when sextupole or higher multipole fields are present, W/W_0 varies from turn to turn and can be considered as the amplification factor of the action J .

The iterative solution of (199) is stopped when the excursions of W_x/W_{x0} and W_y/W_{y0} have reached a minimum. As in the case of closed

Fig. 11. A quadrant of the Antiproton Collector lattice.

Fig. 12. Beta and dispersion functions of the Antiproton Collector lattice.

orbit distortions, but for different reasons, the exact solution of (199) does not give, in general, the best results. The formulae which serve as a basis for the beam envelope correction correspond indeed to a first approximation of the equation of motion; they may reduce the envelope distortions if their strengths are weak enough and if their correlations between themselves and with the chromaticity sextupoles remain at a low level. These constraints are ignored by the system (199) especially when the iteration number increases and the individual corrections tend to counteract each other.

4.5.3 EXAMPLE OF CORRECTION

The principles of the previous section will be illustrated in the real case of the CERN Antiproton Collector (ACOL). Antiprotons being very scarcely produced, the machines which have to store them have large transverse and momentum acceptances, the case of ACOL is extreme:

$$\epsilon_x = \epsilon_y = 200\pi \text{ mm.mrad} , \quad \frac{\Delta p}{p} = 6.10^{-2} .$$

The actions J_x and J_y are half the acceptances. Reasonable physical apertures of vacuum chamber and magnets are obtained by resorting to a strong focusing structure*. The lattice (Fig. 11) is made of 28 FODO cells. The seventy degree phase advance per cell ensures a minimum size of the beam envelope. The bending magnets are distributed in such a way that the orbit dispersion vanishes at the ends of the arcs. Orbit dispersion and β -functions are plotted in Figure 12 for one quadrant. The focusing magnets in the arcs have quadrupole and sextupole components combined in their pole profile. The chromaticities and the orbit dispersion in the long straight sections are maintained as close to zero as possible over the full momentum range by the sextupole fields located in the arcs. The high values of betatron tune:

$$Q_x = 5.465 , \quad Q_y = 5.45$$

lead to corresponding high chromaticities, the sextupole strengths

* The classification of focussing strengths is based on the dispersion of the revolution frequencies with the relative momentum error $\frac{dp}{p}$:

$$\eta = \left(\frac{df}{f} \right) / \left(\frac{dp}{p} \right) = \frac{1}{\gamma^2} - \bar{\alpha} ,$$

γ being the usual relativistic parameter and $\bar{\alpha}$ the average orbit dispersion. The focussing strength of an alternating gradient synchrotron is defined on the $\bar{\alpha}$ -axis:

are then rather large and the correction of non linear beam envelope distortions is of primary importance.

The correction sextupoles can only be accommodated next to the quadrupoles of the long straight sections. In practice, the sextupole effect would be created by tilting the end face of the quadrupole using an adequate shimming of the outside part of the pole pieces. The correction sextupoles must be considered as virtual as long as the system (199) has not been solved. In the present example, the convergence of the solution is shown in Figure 13 where the absolute value of the peak-to-peak residual distortion $|r_{\max} - r_{\min}|$ is plotted against the iteration number.

Fig. 13. Convergence of the *best correctors* algorithm

It is therefore likely that a 3 or 4 sextupole scheme would give the best corrections. This prediction is confirmed by particle tracking which shows insignificant differences between the 3 and 4 sextupole solution but dramatic degradation of the beam envelope beyond the fourth iteration even for the exact solution of (199). The 3-sextupole scheme is therefore retained.

In order to have a good sampling of the trajectories on the 2-torus, the particles were tracked with the following combinations of initial values :

x_0	x'_0	y_0	y'_0	
1	0	0	0	} one dimension motion
0	1	0	0	
0	0	1	0	
0	0	0	1	
1	0	1	0	} coupled motion
1	0	0	1	
0	1	1	0	
0	1	0	1	

Typical patterns of evolution of the actions over 200 turns are shown in Figures (14) to (17). The upper parts concerns the effect of the chromaticity sextupoles only and the lower parts the corrected machine.

Fig 14. Action tracking, horizontal motion

Fig. 15. Action tracking, vertical motion.

Fig. 16. Action tracking, coupled motion: x_0, y'_0

Fig. 17. Action tracking, coupled motion: x'_0, y'_0

Several conclusions can be drawn :

- i) The determination of the envelope distortion can only be deduced from the tracking of particles starting at various independent positions in the phase space. This phenomenon is easy to understand in the case of a purely horizontal motion, its persistence for more complicated motions is thus not surprising. The theory of the resonance :

$$3 Q_x = p$$

shows that there exists in the phase space three unstable fixed points joined by separatrices which form an equilateral triangle (Figure 18). All the points of the circle corresponding to a linear machine of given acceptance are not equidistant from the separatrices and the initial conditions influence the deformation of the trajectories by the non linear fields.

Fig. 18. Phase space structure of third order resonance.

- ii) The beam envelope distortions are dominated by coupled motions. The amplification factor reaches 1.8 in Figure 16 against 1.1 in Figure 14.
- iii) The correction is valid for all initial conditions. It reduces the amplification factor from 1.8 to 1.20 in the vertical plane and from 1.4 to 1.35 in the horizontal plane. Its efficiency is limited by higher order terms which are likely to be enhanced by the vicinity of the line :

$$2(Q_x - Q_y) = 0$$

as the periodicity of 20 turns in the evolution of the actions suggests it.

- iv) The higher order terms in the action perturbation manifest themselves by an oscillation of the actions about a non zero average value. Sometimes, the action distortions are completely folded either on the negative or on the positive side. Even for a pure vertical motion (Figure 15), one would expect no

distortion of the vertical action; in fact, one observes a tiny horizontal distortion which is normal and, in addition, a noticeable vertical distortion.

- v) The beam envelope distortions reach their stationary amplitude very quickly, after three turns typically. The particle tracking can therefore be limited to a small number of turns. It is true that exotic terms may handicap the long term stability of the beam but it is a matter of common understanding that improving the beam quality requests a reduction of the low order effects first. These remarks are evident, they are nevertheless worth mentioning. So much activity is developed in fine and precise numerical studies that one might think that salvation in non linear dynamics lies in the use of super computers; elaborate codes are only necessary to study topics which are totally out of the scope of this paper like bifurcations, stochastic layers or chaos.

5. CONCLUSION

In the spirit of the dipole and quadrupole perturbations well known in the linear theory of alternating gradient synchrotrons, the beam envelope distortion and tune shifts induced by multipole fields like sextupoles or octupoles have been studied. After a detailed discussion of the particle deflexions in a three dimensional magnetic field, the non linear equations of motion have been established. Their solution, when it exists, results from the application of repeated iterations; the closed formulae derived here belong to the first iteration level where the complications generated by the correlations between multipoles have not to be taken into account. At this level, any multipole field can be treated and a solution exists unless resonance conditions are fulfilled; closed orbit and β -function distortions, linear tune shift appear as special cases of a general treatment of the action and angle variables. The formulae have been used to evaluate and reduce the envelope distortion created by chromaticity sextupoles in a real machine. Although it is efficient, the correction is limited by higher order terms which should be studied in the framework of a higher level of iteration of the equation of motion.

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