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STRING LORENTZ ALGEBRAS OUTSIDE THE CRITICAL DIMENSION?

A.N. Schellekens

CERN - Geneva

ABSTRACT

Three recently proposed new string theories have the paradoxical property of being very similar to the Neveu-Schwarz-Ramond (NRS) model in the light cone gauge, but having the wrong space-time dimensions. Inspired by this suggestion we examine whether Lorentz algebras can be constructed outside their usual critical dimension, by considering the lowest states of the Veneziano and NSR models, and using the complete set of operators that can be constructed out of products of oscillators.

In the known string theories, closure of the light-cone-gauge Lorentz algebra in the critical dimension is a consequence of the absence of conformal anomalies on the world sheet. Recently, new string theories in 18 and 26 dimensions have been proposed¹⁾⁻³⁾ for which no two-dimensional conformal field theory is known. The spectrum of these theories in the light cone gauge is completely known. It is also known that all massive states of the $D = 18$ theory fit into representations of the little group $SO(17)$ (for the $D = 26$ theory this is only known for the first three levels). A logical next step would be to construct a Lorentz algebra.

Although the physical states of these theories are generated by Neveu-Schwarz-Ramond oscillators, we cannot follow the conventional NSR construction, since that would inevitably lead to $D = 10$. Because there are no clues from a conformal field theory, the only viable approach seems to be to construct the Lorentz algebra explicitly for the lowest mass levels. If the result is sufficiently simple, one may be able to guess the complete answer, and perhaps even deduce a new conformal field theory from it. This approach would almost certainly be successful for the known string theories in their critical dimensions, and the question we will try to answer is whether one can see the beginnings of a simple and elegant algebra also for some other dimensions^{*)}.

We start with the observation that the $SO(d+1,1)$ (with $d = D-2$) Lorentz algebra of the bosonic strings and the superstring is always of the following form [see, e.g., Ref. 5)]

$$\begin{aligned} J^{ij} &= l^{ij} + K^{ij} \\ J^{+-} &= l^{+-} \quad ; \quad J^{i+} = l^{i+} \\ J^{i-} &= l^{i-} + \frac{1}{M} (K^{ij} p^j + \Delta^i) \end{aligned} \quad (1)$$

where $l^{\mu\nu}$ is the zero mode angular momentum operator, and K^{ij} and Δ^i can be expressed in terms of the bosonic and fermionic oscillators. In the rest frame of the massive excitations, the $SO(d+1)$ (little group) algebra is generated by K^{ij} and

$$\frac{1}{\sqrt{2}} (J^{i-} - J^{i+}) = \frac{1}{M} \Delta^i \equiv K^{i*} \quad (2)$$

where $M = p^0 = \sqrt{2} p^{\pm}$. Here K^{ij} satisfies an $SO(d)$ algebra, K^{i*} is a vector

*) It should be clear that this approach is completely unrelated to the one advocated by Polyakov⁴⁾. Indeed, our main interest is in theories above their canonical critical dimensions.

operator with respect to K^{ij} , and furthermore

$$[K^{i*}, K^{j*}] = i K^{\dot{j}} \quad \text{or equivalently} \quad (3)$$

$$[\Delta^i, \Delta^j] = i M^2 K^{\dot{j}} = i (2p^+ p^- - \vec{p}^2) K^{\dot{j}} \quad (4)$$

In general, the existence of an operator K^{i*} (or Δ^i) with these properties is necessary as well as sufficient for closure of the full $SO(d+1,1)$ Lorentz algebra. It is necessary because certainly there must exist an $SO(d+1)$ algebra in the rest frame of the massive particles; it is sufficient because one may check that the full Lorentz algebra (1) closes if Δ^i satisfies (4).

In all cases of interest, K^{ij} is a known bilinear function of the oscillators, but all that can be said about Δ^i is that it must be a vector operator, constructed out of the oscillators (because we are in the rest frame, we can safely assume that Δ^i does not depend on the bosonic zero mode operators). Our only assumption about Δ^i will be that it is "analytic" in the oscillators, i.e., can be written in terms of (normal ordered) products of arbitrary numbers of creation and annihilation operators. (In all known cases Δ^i is in fact trilinear in the oscillators). Fortunately, it is never necessary to calculate $[\Delta^i, \Delta^j]$. It is sufficient to require that the $SO(d+1)$ operators K^{ij} , K^{i*} have the correct matrix elements between all states at a given level.

To illustrate this, we consider first the bosonic string, for which all calculations are much simpler than for the NSR model. The first massive level ($M^2 = 2$) consists of the states $\alpha_{-1}^i \alpha_{-1}^j |0\rangle$ and $\alpha_{-2}^i |0\rangle$, forming a traceless symmetric tensor, a vector and a scalar of $SO(d)$. These states can be combined to form one massive traceless symmetric tensor of $SO(d+1)$. We denote this state as $|I, J\rangle$ ($I, J = 1, \dots, d+1$), and decompose the indices as $I = (i, *)$ where "*" denotes the $(d+1)^{th}$ component. The $SO(d+1)$ invariant normalization of the massive states is

$$\langle I, J | K, L \rangle = \frac{1}{2} (\delta_{JK} \delta_{IL} + \delta_{JL} \delta_{IK}) - \frac{1}{d+1} \delta_{JJ} \delta_{KL} \quad (5)$$

We can now obtain the precise relation between the massive states and the $SO(d)$ states by imposing this normalization condition:

$$\begin{aligned}
 |i,j\rangle &= \frac{1}{\sqrt{2}} \left[\alpha_{-1}^i \alpha_{-1}^j - \frac{1}{d} \left(1 + \frac{\varepsilon}{\sqrt{d+1}} \right) \alpha_{-1}^k \alpha_{-1}^k \delta^{ij} \right] |0\rangle \\
 |i,*\rangle &= \frac{1}{2} \alpha_{-2}^i |0\rangle \\
 |*,*\rangle &= \frac{1}{\sqrt{2}} \frac{\varepsilon}{\sqrt{d+1}} \alpha_{-1}^k \alpha_{-1}^k |0\rangle
 \end{aligned} \tag{6}$$

where ε is an undetermined phase (all other phases are irrelevant). An $SO(d+1)$ generator must act on these states in the following way

$$K^{IJ} |K,L\rangle = i \delta^{JK} |J,L\rangle + i \delta^{IL} |K,J\rangle - (I \leftrightarrow J) \tag{7}$$

This dictates how K^{i*} must transform $|i,j\rangle$, $|i,*\rangle$ and $|*,*\rangle$ into each other. At this level, the most general expression for K^{i*} is

$$K^{i*} = \frac{1}{\sqrt{2}} \left[A \alpha_{-1}^i \alpha_{-1}^l \alpha_{-2}^l + B \alpha_{-1}^l \alpha_{-1}^l \alpha_{-2}^i \right] + h.c. \tag{8}$$

It is not difficult to determine A and B from (6) and (7)

$$\begin{aligned}
 A &= -i \\
 B &= \frac{i}{d} (1 + \varepsilon \sqrt{d+1}) \quad ; \quad \varepsilon = \pm 1
 \end{aligned} \tag{9}$$

The Lorentz algebra of the Veneziano model can be expanded to this order, and for its coefficients we find $A_V = -i$, $B_V = i/2$. These values agree with (9) only for $\varepsilon = +1$, $d = 24$, so that we do indeed reproduce the correct critical dimension at this level. Without restrictions on A and B we find, however, two Lorentz algebras for arbitrary d.

At the second level ($M^2 = 4$) the massive states are a symmetric rank 3 tensor $|IJK\rangle$ and an antisymmetric second rank tensor $|IJ\rangle$. The relation between the massive states and the light cone states is as follows

$$\begin{aligned}
 |i,j\rangle &= - \alpha_{-1}^{(i} \alpha_{-2}^{j)} |0\rangle \\
 |i,*\rangle &= \left[-\frac{1}{\sqrt{2(d+2)}} \sin\phi \alpha_{-1}^i (\alpha_{-1}^l \alpha_{-2}^l) + \frac{1}{\sqrt{3}} \cos\phi \alpha_{-3}^i \right] |0\rangle \\
 |i,j,k\rangle &= \left[\alpha_{-1}^i \alpha_{-1}^j \alpha_{-1}^k - \frac{3}{d+2} \left(1 + \frac{\cos\phi}{\sqrt{d+3}} \right) \alpha_{-1}^{(i} \delta^{jk)} \alpha_{-1}^l \alpha_{-1}^l \right. \\
 &\quad \left. - \sin\phi \sqrt{\frac{6}{(d+2)(d+3)}} \alpha_{-3}^{(i} \delta^{jk)} \right] |0\rangle \\
 |i,j,*\rangle &= \left[\alpha_{-1}^{(i} \alpha_{-2}^{j)} - \frac{1}{d} \left(1 + \sqrt{\frac{3}{d+3}} \right) \alpha_{-1}^l \alpha_{-2}^l \delta^{ij} \right] |0\rangle
 \end{aligned}$$

$$|i,*,* \rangle = \left[\frac{\cos \phi}{\sqrt{d+3}} \alpha_{-1}^i (\alpha_{-1}^{\ell} \alpha_{-1}^{\ell}) + \sin \phi \sqrt{\frac{2d+4}{3d+9}} \alpha_{-2}^i \right] |0 \rangle$$

$$|*,*,* \rangle = \sqrt{\frac{3}{d+3}} \alpha_{-1}^{\ell} \alpha_{-2}^{\ell} |0 \rangle$$

As before, these relations are dictated by the normalization conditions on $|IJK\rangle$ and $|IJ\rangle$ with one new complication namely the presence of two $SO(d)$ vectors at this level. This necessitates the introduction of a mixing angle ϕ , which determines the linear combinations of the two vectors that belong to $|IJK\rangle$ or $|IJ\rangle$. Several phases could have been introduced, but they are all either irrelevant or determined by the closure conditions. The expression for K^{i*} contains first of all the terms already determined at the previous level [given by (8), (9), but with the factor $1/\sqrt{2}$ replaced by $\frac{1}{2}$, due to the factor $1/M$ in (2)], plus some new terms:

$$\begin{aligned} \delta K^{i*} = & a_1 \alpha_{-1}^{\ell} \alpha_{-2}^{\ell} \alpha_{-1}^i + a_2 \alpha_{-1}^{\ell} \alpha_{-2}^i \alpha_{-3}^{\ell} + a_3 \alpha_{-1}^i \alpha_{-2}^{\ell} \alpha_{-3}^{\ell} \\ & + b_1 \alpha_{-1}^i \alpha_{-1}^{\ell} \alpha_{-1}^m \alpha_{-1}^m + b_2 \alpha_{-1}^i \alpha_{-1}^{\ell} \alpha_{-1}^m \alpha_{-2}^{\ell} \alpha_{-2}^m \\ & + b_3 \alpha_{-1}^m \alpha_{-1}^m \alpha_{-1}^{\ell} \alpha_{-1}^i \alpha_{-2}^{\ell} + b_4 \alpha_{-1}^m \alpha_{-1}^m \alpha_{-1}^{\ell} \alpha_{-1}^i \alpha_{-2}^i \\ & + h.c. \end{aligned}$$

As before we require that the Lorentz generators act properly on the massive states, as we find that this requirement determines all coefficients, as well as the mixing angle ϕ in terms of d and phase ϵ left over from the first level:

$$\begin{aligned} a_1 &= \frac{i}{2d} \sin \phi \left[\sqrt{2d+4} + 2 \sqrt{\frac{2(d+3)}{3(d+2)}} \right] \\ a_2 &= -\frac{i}{2} \left(\sin \phi \sqrt{\frac{2(d+3)}{3(d+2)}} - \frac{1}{\sqrt{3}} \cos \phi \right) \\ a_3 &= -\frac{i}{2} \left(\sin \phi \sqrt{\frac{2(d+3)}{3(d+2)}} + \frac{1}{\sqrt{3}} \cos \phi \right) \\ b_1 &= \frac{i}{2(d+2)} - \frac{i}{2d} \cos \phi \left[\sqrt{3} + \frac{2}{d+2} \sqrt{d+3} \right] \\ b_2 &= 0 \\ b_3 &= \frac{i}{2\sqrt{2(d+2)}} \sin \phi + \frac{i \cos \phi}{2\sqrt{d+3}} + \frac{i}{2(d+2)} \left(1 + \frac{\cos \phi}{\sqrt{d+3}} \right) - \frac{i}{2d} (1 + \epsilon \sqrt{d+1}) \\ b_4 &= \frac{-i}{2\sqrt{2(d+2)}} \sin \phi + \frac{i \cos \phi}{2\sqrt{d+3}} + \frac{i}{2(d+2)} \left(1 + \frac{\cos \phi}{\sqrt{d+3}} \right) \end{aligned}$$

with $\sin \phi = \frac{\sqrt{2(d+2)}}{2d} (1 + \epsilon \sqrt{d+1})$

Notice that formally we could have defined an infinite number of Lorentz algebras by choosing a value for ϕ and defining the matrix elements of K^{i*} accordingly. If we want to write the algebra in terms of creation and annihilation operators, however, only two values of ϕ are allowed for given d . This shows that the fact that we can fit the states at any level into $SO(d+1)$ multiplets does not guarantee the existence of an operator K^{i*} in terms of oscillators.

The result can be checked by comparing with the Veneziano model which gives $a_1 = i/6$, $a_2 = -i/4$, $a_3 = -i/2$, $b_i = 0$. If we would not already know that the critical dimension of the model is $d = 24$, it would immediately be clear from the solution that this value is special because it is the only dimension for which all fifth order terms vanish, as well as the only reasonable value of d for which all coefficients are rational numbers. Other dimensions are, however, not rigorously ruled out. For $d < 13$ the $\epsilon = +1$ solution is inconsistent, as it would lead to $\sin\phi > 1$. It is conceivable that additional consistencies at higher levels would finally leave only $d = 24$ as a solution. It is, however, also possible that there is an enormous proliferation of new solutions.

Unfortunately, extending the calculation to the third level would be extremely tedious, and therefore we will now move on to the spinning string. The first massive state in the Ramond sector is an $SO(d+1)$ vector-spinor $|I, A\rangle$, where A is an $SO(d+1)$ spinor index. With respect to the $SO(d)$ subgroup an $SO(d+1)$ spinor decomposes into two conjugate Weyl spinors with indices a and $\dot{a}$ (assuming $d = \text{even}$). The massive state $|I, A\rangle$ is irreducible in $SO(d+1)$ and has the following normalization

$$\langle I, A | J, B \rangle = \delta^{IJ} \delta_{AB} - \frac{1}{d+1} \gamma_{AC}^J \gamma_{CB}^I \quad (10)$$

where γ^I is the set of $SO(d+1)$ Dirac matrices, consisting of the $SO(d)$ Dirac matrices γ^i and the matrix γ^* ($= \gamma_5$). The light cone gauge states are obtained by acting with the bosonic and fermionic oscillators α_{-1}^i and d_{-1}^i on the vacua $|a\rangle$ and $|\dot{a}\rangle$ and making a GSO projection.

Using (10), we derive the relation between the massive states and the light cone states

$$\begin{aligned}
 |i, a\rangle &= \alpha_{-i}^i |a\rangle - \frac{1}{d} (1 + \varepsilon_1 \sqrt{\frac{1}{d+1}}) \alpha_{-i}^i |b\rangle (\gamma^L \gamma^i)_{ba} \\
 |i, \bar{a}\rangle &= -\bar{\alpha}_{-i}^i |\bar{a}\rangle + \frac{1}{d} (1 + \varepsilon_2 \sqrt{\frac{1}{d+1}}) \bar{\alpha}_{-i}^i |\bar{b}\rangle (\gamma^L \gamma^i)_{\bar{b}\bar{a}} \\
 |*, a\rangle &= \frac{-\varepsilon_2}{\sqrt{d+1}} \bar{\alpha}_{-i}^i |\bar{a}\rangle \gamma_{aa}^L \\
 |*, \bar{a}\rangle &= -\frac{\varepsilon_1}{\sqrt{d+1}} \alpha_{-i}^i |a\rangle \gamma_{a\bar{a}}^L
 \end{aligned}$$

where ε_1 and ε_2 are phases to be determined later. [A third phase was chosen to agree with the conventions used in Ref. 5).] The required $SO(d+1)$ transformation is

$$K^{IJ} |K, A\rangle = i \delta^{JK} |J, A\rangle - i \delta^{JK} |J, A\rangle + |K, B\rangle \Sigma_{BA}^{JK}$$

where $\Sigma^{IJ} = -i/4 [\gamma^I, \gamma^J]$. The most general expression for K^{i*} at this level is

$$K^{i*} = \frac{1}{\sqrt{2}} [a_1 d_{-i}^i d_0^k \alpha_{-i}^k + a_2 d_{-i}^i d_0^k \alpha_{-i}^k + a_3 d_{-i}^i d_0^k \alpha_{-i}^i]$$

with $d_0^i = 1/\sqrt{2} \gamma_0^i$. After some computation we obtain the values of the coefficients for arbitrary d .

$$a_1 = \frac{-2i}{d} (1 + \varepsilon_1 \sqrt{d+1})$$

$$a_2 = i$$

$$a_3 = \frac{-2i}{d} (1 - \varepsilon_1 \sqrt{d+1}) \quad (\varepsilon_1 = -\varepsilon_2 = \pm 1)$$

The spinning string Lorentz algebra has $a_1 = -i$, $a_2 = i$, $a_3 = i/2$, which is indeed obtained for $d = 8$ (and $\varepsilon_1 = 1$). Just as in the Veneziano model, we get two Lorentz algebras for any d . The coefficients which would be relevant for the 18-dimensional string are equal to $1/8(1 \pm \sqrt{17})$ which does not look like a very promising start for a new Lorentz algebra.

The Neveu-Schwarz sector looks even more discouraging outside its critical dimension. The spectrum for $D = 10, 18$ and 26 is rather different in this case, due to shifts in the intercept, opposite GSO projections and, in the case of the 26-dimensional theory, the removal of 24 Veneziano models¹⁾. In ten dimensions, the Lorentz algebra closes on even as well as odd G -parity states, but that need not be the case for $D = 18$ or $D = 26$. We will therefore only construct the

Lorentz-algebra on those levels that are kept in the GSO projection. The first massive level for $D = 18$ consists of a rank 4 antisymmetric tensor, a mixed symmetry rank 3 tensor, a rank 2 symmetric tensor and a singlet of $SO(d+1)$. There are three $SO(d)$ states which occur twice, so that three mixing angles are needed. The $SO(d+1)$ generator K^{i*} can be expressed in terms of twenty operators, one of seventh order, ten of fifth order and nine of third order. It would be pointless to present the complete solution here, and we will just give a representative example. The three coefficients a_i of the three contractions of the operator $\alpha_{-1}^b \alpha_{-\frac{1}{2}}^b$ are given by

$$a_1 = i\epsilon \sqrt{3M^2} \frac{1 + s_1 \sqrt{d}}{d-1}$$

$$a_2 = -i\epsilon \sqrt{3M^2} \frac{1}{2} \frac{\sqrt{d}}{\sqrt{d+5}} c_1^2$$

$$a_3 = -i\epsilon \sqrt{\frac{M^2}{3}}$$

where s_1 and c_1 are the sine and cosine of one of the mixing angles and $\epsilon = \pm 1$. Because of the shifts in intercept, $M^2 = 3$ for $d = 8$ and $M^2 = 2$ for $d = 16$. The value of s_1 can be determined by solving a quartic equation

$$(2d^4 - 5d) - 2\sqrt{d}(d+2)s_1 - (d-1)(4+3d)s_1^2 - 6\sqrt{d}(d-2)s_1^3 + (6d-3d^2)s_1^4 = 0$$

This equation has a sensible solution for $d = 8$ ($s_1 = (1/3)\sqrt{2}$) which leads to the correct result ($a_1 = i$, $a_2 = a_3 = -i$) for $\epsilon = +1$. For $d = 16$, however, there are two solutions ($s_1 = 0.549689622$ and $s_1 = -0.768565099$) which are genuine roots of a quartic equation (there is a second solution of this type for $d = 8$).

Our hope was to find a new Lorentz algebra for $D = 18$, different from the one in ten dimensions, but equally simple. Clearly this has not been realized and, in fact, there is nothing to suggest that 18 is a preferred dimension for the spinning string. (Of course, one should keep in mind that one assumption has been made, namely that the operator Δ^i is "analytic" in the oscillators.)

These results are valid for the Ramond sector of the $D = 26$ "monsterstring" (and look acceptable) but not for the Neveu-Schwarz sector (due to the GSO projection, but more importantly, the removal of states at every level). In the NS sector a new calculation would have to be done at the next level (i.e., half a

unit above the one just considered), which seems forbiddingly complicated. Thus there remains a small ray of hope for the fate of this theory.

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