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THE UNRUH EFFECT AND QUANTUM FLUCTUATIONS OF ELECTRONS IN STORAGE RINGS

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A B S T R A C T

The quantum fluctuation of electron orbits in ideal storage rings is a sort of Fulling-Unruh effect (heating by acceleration in vacuum). To spell this out, the effect is analyzed in an appropriate comoving, and so accelerating and rotating, co-ordinate system. The depolarization of the electrons is a related effect, but is greatly complicated by spin-orbit coupling. This analysis confirms the standard result for the polarization, except in the neighbourhood of a narrow resonance.

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1 INTRODUCTION

A physical system that is accelerated through vacuum is heated by interactions with the vacuum fluctuations. This is often referred to as the Unruh effect [1]. For linear uniform acceleration this effect has a universal character, since the excitation spectrum is thermal, with the temperature determined by the acceleration alone.

Also for circular uniform acceleration there is a similar effect, which we will refer to as the circular Unruh effect. Thus, a physical system in circular motion is heated by the vacuum fluctuations, but now the effect does not have the same universal character. The excitation spectrum is not purely thermal, but depends on details of the physical system [2,3].

In an earlier paper we related this effect to polarization effects of electrons in storage rings [3]. Using the g -factor of the electrons as a free parameter, we compared the equilibrium polarization $P(g)$, obtained under the assumption of a thermal spectrum of the spin excitations, with the corresponding function previously found by standard QED calculations [4-6]. These two functions showed clear similarities, but there were also some distinct differences. A part of this difference could be explained as a manifestation of the difference between the circular and the linear Unruh effect. But, as was pointed out, also complications due to Thomas precession of the circulating electrons are important.

In this paper we return to the question of spin polarization and the circular Unruh effect, and examine them in more detail. Our approach is similar to some of the previous QED calculations of the spin effects, but we employ a formulation based on the use of co-moving, accelerated coordinates, rather than using lab frame coordinates. We consider in some detail the effects of fluctuations in the particle position around the classical orbit. These fluctuations are shown to give important effects, but curiously enough the final result for the polarization - apart from a resonance - agrees with previous

calculations where the electrons are confined to the classical orbit.

While the orbital fluctuations are important for the spin effect, the effect of the spin back on the orbital motion is small. In fact the fluctuations in the path may be considered as a manifestation of the circular Unruh effect, independent of the spin. We examine the excitation spectrum for the vertical fluctuations and determine the associated 'temperature'. This is found to be slightly higher than for linear motion with the same (rest frame) acceleration.

2 HAMILTONIAN IN THE ACCELERATED FRAME

In the same way as for linear acceleration, discussed in ref.[7], an "accelerated" coordinate system (x', y', z) can be introduced, with the classical trajectory of the circulating electron defined by $x'=y'=z=0$. The time parameter τ is simply the proper time of this trajectory, and the coordinates (x', y', z) defines a cartesian rest frame of the classical particle at time τ . As stated above, this frame is chosen with that particle located at the origin. The orientation of the frame, furthermore, is defined with z as the coordinate orthogonal to the plane of the motion, and with the center of rotation, for all τ , lying on the x' -axis. As a consequence of this choice of orientation of the frame, the metric in the accelerated coordinate system is stationary.

The connection between the lab frame coordinates, with the center of rotation at the origin, and the accelerated coordinates is

$$\begin{aligned}x &= (R+x') \cos(\omega_0 \gamma \tau) - \gamma y' \sin(\omega_0 \gamma \tau) \\y &= (R+x') \sin(\omega_0 \gamma \tau) + \gamma y' \cos(\omega_0 \gamma \tau) \\t &= \gamma \tau + \beta \gamma y'\end{aligned} \tag{1}$$

with z as a common coordinate for the two systems. Here R is the

radius of the electron orbit, ω_0 is the angular velocity in the lab frame, and β and γ are the usual relativistic functions of the particle velocity

$$\beta = \frac{\omega_0 R}{c}, \quad \gamma = (1 - \beta^2)^{-1/2} \quad (2)$$

As opposed to the case of linear acceleration, the coordinate system (x', y', z, τ) defined above depends on two independent parameters. The first of these may be taken as the acceleration a' in the (inertial) rest frame,

$$a' = \gamma^2 \omega_0^2 R \quad (3)$$

and the other as γ .

Associated with this choice of coordinates is a quantum description of the system, where the state vector at time τ is identified with the state vector $\Psi(\tau)$ of the system in an inertial frame coinciding with the rest frame of the classical trajectory at time τ . Different frames, corresponding to different times τ , are connected by unitary transformations, representing the space-time transformations between the frames, and consequently these operators define the time evolution of the state $\Psi(\tau)$. An infinitesimal transformation corresponding to $\tau \rightarrow \tau + d\tau$ is given by

$$\Psi(\tau + d\tau) = -\frac{i}{\hbar} \left(H - \frac{a'}{\beta c} J_z + \frac{a'}{c} K_x \right) \Psi(\tau) d\tau \quad (4)$$

where H , J_z and K_x are generators for time translations, rotations and boosts in the inertial rest frame at time τ . Thus, the Hamiltonian associated with the accelerated coordinate system has the form

$$H' = H - \frac{a'}{\beta c} J_z + \frac{a'}{c} K_x \quad (5)$$

We assume that all external fields, as well as the metric, are stationary in the accelerated coordinate system. Consequently the Hamiltonian H' is a τ -independent operator.

To specify H' for the accelerated electrons, we take the expressions for H , J_z and K_x from standard Dirac theory, but we include in H also an anomalous magnetic term proportional to $\kappa=(g-2)/2$. This gives

$$H' = (1-a'x'/c^2)[\hat{\beta}mc^2 + \vec{\alpha} \cdot (\vec{p}' - \frac{e}{c} \vec{A}')] c + e\varphi' + \kappa \frac{e\hbar}{2mc} (i\hat{\beta}\vec{\alpha} \cdot \vec{E}' - \hat{\beta}\vec{\sigma} \cdot \vec{B}') + \frac{i\hbar a'}{2c} \alpha_x - \frac{a'}{\beta c} (L'_z + \frac{\hbar}{2} \sigma_z) \quad (6)$$

In this expression $\vec{\alpha}$ and β (not to be confused with $\beta=v/c$) denote the standard Dirac matrices, $\vec{p}'$ and L'_z are the operators

$$\vec{p}' = \frac{\hbar}{i} \left(\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'} \right) \quad (7)$$

$$L'_z = \frac{\hbar}{i} \left(x' \frac{\partial}{\partial y'} - y' \frac{\partial}{\partial x'} \right)$$

$(\vec{A}', \varphi')$ are the potentials and $(\vec{E}', \vec{B}')$ the (external and quantum) electromagnetic fields, in the (inertial) rest frame of the classical trajectory at time τ . The potential relation for the electric field is

$$\vec{E}' = -\nabla\varphi' - \frac{1}{c} (1-a'x'/c^2)^{-1} \left[\frac{\partial}{\partial \tau} \vec{A}' + \frac{a'}{\beta c} (\vec{k}_x \vec{A}' - (x' \frac{\partial}{\partial y'} - y' \frac{\partial}{\partial x'}) \vec{A}') - \frac{a'}{c^2} \varphi' \vec{i}' \right] \quad (8)$$

- as modified by the continuous change of inertial system (with $\vec{i}'$, $\vec{j}'$ and $\vec{k}'$ as unit vectors in the rest frame). However we shall need it only to zero order in $\hbar$. To this order the complications cancel. The classical potentials we may choose to vanish at the unperturbed classical particle orbit.

We consider here only "weak focussing" storage rings, with the z-axis an axis of rotational symmetry. The constants defining the problem are then

$$e, c, \hbar, m, R, n, \gamma$$

where R is the radius of the unperturbed circular classical orbit.
The magnetic field on this orbit is given by

$$eB = m\gamma\beta c^2/R \quad (9)$$

and the fall-off index n defined by

$$n = (R/B)(\partial B/\partial r) \quad (10)$$

The seven parameters can be replaced by the dimensionless combinations

$$\frac{e^2}{\hbar c}, \frac{\hbar}{mc} \frac{1}{R}, n, \gamma$$

and the dimensional

$$c, m, R$$

We assume

$$\frac{\hbar}{mc} \frac{1}{R} \lll 1 \quad (11)$$

- as can be brought about by making R large enough for fixed values of the other constants. We find it helpful to think in terms of a unit of length which is fixed in fractions of R . We might even take

$$m = c = R = 1 \quad (12)$$

In these units the parameters

$$\hbar, e = \sqrt{\alpha\hbar c} \quad (13)$$

become very small. We are concerned then with small quantum corrections to the basic classical circular motion. We expect a

non-relativistic approximation, in the moving system, to be adequate, and make a Foldy-Wouthuysen transformation. In making the transformation we tentatively treat

$$\vec{p}' - \frac{e}{c} \vec{A}' \equiv \vec{\pi}' \quad (14)$$

as of order $\sqrt{\hbar}$. See for example eqs.(60) and (61) below, for vertical fluctuations, and note the commutator

$$[\pi'_r, \pi'_s] = -i\hbar \epsilon_{rst} \frac{e}{c} B'_t \quad (15)$$

Remember that despite the factor e ($\propto \sqrt{\hbar}$) here,

$$eB' = m\gamma^2 \beta c^2 / R + \dots \quad (16)$$

is of zero order. The F-W-transformed Hamiltonian has for the spin-dependent part

$$\begin{aligned} H_{\text{spin}} = & - \frac{e\hbar}{4mc} \vec{\sigma} \cdot [(1-a'x'/c) g\vec{B}' + \frac{1}{mc} (g-1) \vec{E}' \times \vec{\pi}'] \\ & - \frac{1}{2} \frac{a'\hbar}{\beta c} \sigma_z + \frac{1}{4} \frac{a'\hbar}{mc^2} (\sigma_y \pi'_z - \sigma_z \pi'_y) \end{aligned} \quad (17)$$

3 SPIN EXCITATIONS

The spin Hamiltonian (17) has the form

$$H_{\text{spin}} = \frac{1}{2} \hbar \vec{\omega} \cdot \vec{\sigma} \quad (18)$$

where $\vec{\omega}$ can be split into a classical part $\vec{\omega}_1$ and a quantum part $\delta\vec{\omega}$. $\vec{\omega}_1$ is given by

$$\vec{\omega}_1 = - \frac{ge}{2mc} \vec{B}'_c - \frac{a'}{\beta c} \vec{k} \quad (19)$$

where $\vec{B}'_c$ is the classical field on the (classical) particle orbit. By relating this to the accelerating magnetic field in the lab frame, we find

$$\vec{B}'_c = \frac{m}{e} \frac{a'}{\beta} \vec{k} \quad (20)$$

This gives

$$\vec{\omega}_1 = \frac{a'}{2\beta c} (g-2) \vec{k} \quad (21)$$

For $g=2$ $\vec{\omega}_1$ vanishes. This demonstrates the well-known fact that for this g -value the spin vector has a precession frequency equal to the frequency of the orbital motion.

The transition probabilities between the spin up and the spin down in the z -direction are determined by the x' - and y' -components of the quantum contribution to $\vec{\omega}$. Thus, starting with spin down (or up), and allowing the interaction to work for a long time T , we calculate the norm of the spin-flipped state, and identify the flip up and flip down transition probabilities per unit time as

$$\Gamma_{\pm} = \frac{1}{4T} \left| \int_{-T/2}^{T/2} d\tau \exp(\pm i\omega_1 \tau) \delta\omega_{\mp}(\tau) |0\rangle \right|^2 \quad (22)$$

where

$$\delta\omega_{\pm} = \delta\omega_x \pm i\delta\omega_y \quad (23)$$

In the above expression for $\Gamma_{\pm}$ we have assumed that the spin interaction can be treated as a perturbation relative to the spin independent part of the Hamiltonian. In particular this means that $|0\rangle$ is the state of the radiation field unperturbed by the spin

interaction. However, it is not the vacuum of the free electromagnetic field, because of the interaction between these fields and the orbital motion.

From the form of eq.(22) we also note that a contribution to $\delta\omega$, which is a total time derivative $\dot{F}$ of a bounded function F of τ , can be substituted according to the rule

$$\dot{F} \rightarrow \mp i\omega_1 F \quad (24)$$

This follows from the fact that the difference between these two expressions gives a contribution to the squared norm in eq.(22) which does not increase with T , and therefore is suppressed for large T . We shall make use of the substitution rule (24) below.

To linear order in the quantum fluctuations $\delta\vec{\omega}$ is given by

$$\delta\vec{\omega} = - \frac{e}{2mc} \{ g\delta\vec{B}' - (a'x'/c^2)g\vec{B}'_C - (g-2) \frac{a'}{ec} \vec{1}'x'\vec{\pi}' \} \quad (25)$$

Here $\delta\vec{B}'$ (and $\delta\vec{E}'$) denote the quantum contributions to the fields. They can be written as

$$\begin{aligned} \delta\vec{E}' &= \vec{E}'_q + \delta\vec{E}'_C \\ \delta\vec{B}' &= \vec{B}'_q + \delta\vec{B}'_C \end{aligned} \quad (26)$$

with $\vec{E}'_q$ and $\vec{B}'_q$ as the quantum fields along the classical orbit, and $\delta\vec{E}'_C$ and $\delta\vec{B}'_C$ as the contributions from the (inhomogeneous) classical fields due to fluctuations in the particle position.

The quantum field fluctuations $\vec{E}'_q$ and $\vec{B}'_q$ are of order $\sqrt{\hbar}$. The classical fields $\vec{E}'_C$ and $\vec{B}'_C$ are of order $e^{-1} \propto (\sqrt{\hbar})^{-1}$. The fluctuations due to particle position fluctuations of order $\sqrt{\hbar}$, are then of order

$$\sqrt{\hbar} (\sqrt{\hbar})^{-1} = 1$$

So it could seem that the $\hat{B}'_q$ contributions in (25) would be unimportant in comparison with the $\delta\hat{B}'_c$ and $\vec{\pi}'/e$ terms. But we will see below that these latter contribute to spin flip transition amplitudes only in one order higher in $\sqrt{\hbar}$ than their own order. For similar reasons terms of order $\hbar^2$ neglected in (17) and (25) contribute only in order $(\sqrt{\hbar})^5$.

The expression for $\delta\omega_{\mp}$ then is

$$\begin{aligned}\delta\omega_{\mp} &= -\frac{e}{2mc} (g\delta B'_{\mp} \mp i(g-2) \frac{a'}{ec} \pi'_z) \\ &= -\frac{e}{2mc} (g\delta B'_{\mp} \mp 2i\beta\omega_1 \frac{m}{e} \ddot{z})\end{aligned}\quad (27)$$

In the last expression we have made use of eq.(21), and also the fact that, to lowest order, π'_z is equal to $m\dot{z}$. By use of the substitution rule (24) $\delta\omega_{\mp}$ may also be written as

$$\delta\omega_{\mp} = -\frac{e}{2mc} (g\delta B'_{\mp} + 2\beta \frac{m}{e} \ddot{z})\quad (28)$$

We now make use of the Heisenberg equation of motion in the z-direction

$$m\ddot{z} = e(E'_{qz} + \delta E'_{cz})\quad (29)$$

and also the connection between the fluctuations in the classical magnetic and electric fields

$$\delta B'_{c\mp} = -\frac{1}{\beta} \delta E'_{cz}\quad (30)$$

The last equation follows from the fact that these fields are produced by the same magnetic field in the lab frame. Eqs. (29) and (30) gives

$$\delta\omega_{\mp} = - \frac{e}{2mc} \{gB'_{q\mp} + 2\beta E'_{qz} - \frac{1}{\beta} (g-2\beta^2) \delta E'_{cz}\} \quad (31)$$

Since we are mainly interested in large values of γ , we shall in the following approximate β by 1. The expression for $\delta\omega_{\mp}$ then simplifies to

$$\delta\omega_{\mp} = - \frac{e}{2mc} \{gB'_{q\mp} + 2\beta E'_{qz} - (g-2)\delta E'_{cz}\} \quad (32)$$

We remarked already that our perturbation is not away from the free electromagnetic vacuum, but a state already influenced by the spin independent part of the particle - field interaction. To allow the use of vacuum matrix elements below, we write

$$E'_{qz} = E'_{fz} + E'_{sz} \quad (33)$$

where E'_{fz} is the free field, and E'_{sz} is the dissipative part of the self field (as in classical theory [8]):

$$E'_{sz} = \frac{2}{3} \frac{e^2}{c^3} \{\ddot{z} - (a'^2/c^2)\dot{z}\} \quad (34)$$

(As usual an infinity is absorbed into the physical electron mass.) The second term of (33) is of order $e^2/\hbar \propto (\sqrt{\hbar})^3$, as compared with $\sqrt{\hbar}$ for the first. So we omit it in (31) and (32). However, the damping force is important in (29), because of its essential role in limiting the growth of quantum fluctuations in z . We further assume that the classical field $\delta E'_{cz}$, which provides the focussing of the electrons in the z -direction, can be approximated by a harmonic oscillator field

$$\delta E'_{cz} = - \Omega^2 \frac{m}{e} z \quad (35)$$

Expressed in terms of the index n , we have

$$\Omega^2 = \frac{a'}{R} n \quad (36)$$

Inserting eqs. (34) and (35) into the equation of motion (29), we get

$$\ddot{z} - \frac{e^2}{mc^3} (\ddot{z} - \frac{a'^2}{c^2} \dot{z}) + \Omega^2 z = \frac{e}{m} E'_{fz} \quad (37)$$

By use of the substitution rule (24) this may be solved for z ,

$$z = \{\Omega^2 - \omega_1^2 \mp i \frac{2}{3} \frac{e^2}{mc^3} \omega_1 (\frac{a'^2}{c^2} + \omega_1^2)\}^{-1} \frac{e}{m} E'_{fz} + \Lambda \quad (38)$$

Here Λ denotes a term which is irrelevant (and will be suppressed) in the expression for $\delta\omega_{\mp}$. This expression we may now re-write by use of eq.(38). We find

$$\delta\omega_{\mp} = - \frac{e}{2mc} \{gB'_{f\mp} + (2+f_{\mp}(g)) E'_{fz}\} \quad (39)$$

with $f_{\mp}(g)$ given by

$$f_{\mp}(g) = \frac{(g-2)\Omega^2}{\Omega^2 - \omega_1^2 \mp i\Delta\omega_1} \quad (40)$$

where

$$\Delta = \frac{2}{3} \frac{e^2}{mc^3} (\frac{a'^2}{c^2} + \omega_1^2) \quad (41)$$

For $\gamma \gg 1$, using (3), (21), (36), and (41),

$$f_{\mp} = 2\kappa [1 - \gamma^2 \kappa^2 / n \mp i \frac{2}{3} \frac{r}{R} e \frac{\gamma^4}{n} (\kappa + \kappa^3)]^{-1} \quad (42)$$

with

$$\begin{aligned} \kappa &= g/2 - 1 = 0.00116 \\ r_e &= e^2/mc^2 = 2.8 \cdot 10^{-13} \text{ cm} \end{aligned} \quad (43)$$

There is a resonance at

$$\gamma = \sqrt{n/\kappa} = 861/\sqrt{n} \quad (44)$$

But as γ increases beyond this, $f_{\mp}$ becomes rapidly negligible, and (39) simplifies to

$$\delta\omega_{\mp} = - \frac{e}{2mc} (gB'_{f\mp} + 2E'_{fz}) \quad (45)$$

By use of eq.(45), the transition probabilities $\Gamma_{\pm}$, which can be written as

$$\Gamma_{\pm} = \frac{1}{4} \int_{-\infty}^{\infty} d\tau \exp(\mp i\omega_1 \tau) \langle \delta\omega_{\pm}(\tau/2) \delta\omega_{\mp}(-\tau/2) \rangle \quad (46)$$

now can be expressed in terms of integrals of free field correlation functions along the classical orbit. These can readily be calculated. However, instead of doing so, we shall rather compare with earlier calculations of polarization effects in storage rings. In particular we refer to the lucid article by Jackson [6] on this subject. His discussion is based on the use of a spin Hamiltonian directly related to the (classical) Thomas-BMT spin equation, with the electrons confined to the classical orbit. (For more details about the relation between our approach and the one found in Jackson's paper, see the Appendix A.)

To leading order in $1/\gamma$, the expression (45) for $\delta\omega_{\mp}$, indeed agrees with the expression for the spin Hamiltonian found in Jackson's paper, when this is written in terms of fields in the co-moving frame rather than in the lab frame. This may seem to demonstrate that all the

additional complications due to fluctuations in the path, which we have considered here, give no contribution to the transition probabilities - except for the resonance at $\omega_1 = 0$. However there is a subtlety involved which we would like to point out. This is most clearly seen if we consider the case $g=2$. We first note that the correct expression for $\delta\omega_{\mp}$ could be obtained already from eq.(28) if we there included only the free fields in the expressions for $\delta B_{\mp}'$ and $\dot{z}$. However the second term, which in the lab frame formulation is interpreted as a Thomas precession term [6], in reality gives no contribution to the transition probabilities $\Gamma_{\pm}$. This follows from the fact that total time derivatives give no contribution to $\Gamma_{\pm}$ when $\omega_1 = 0$. Instead both terms in eq.(45) are derived from the first term in eq.(28) when $g=2$. For other values of g both terms in eq.(28) give contributions to the transition probabilities, but there is not a direct connection between each of the two terms in eq.(28) and the corresponding two terms in eq.(45).

These complications in our account are not due to the use of the comoving rather than the laboratory frame. Of course, this involves us in a slightly different definition of spin ([3], eq.(A7))

$$\sigma_z' = \sigma_z + \frac{\gamma\beta}{\gamma+1} (\dot{z}/c)\sigma_y \dots \quad (47)$$

(using a prime momentarily to distinguish our spin from the other). But the quantum fluctuation in z is of order $\sqrt{\hbar}$, so that the extra term here does not show up when transition rates are calculated to lowest order in $\hbar$. It is not surprising then that after some manipulation we can get back to Jackson's formulae: our (28) is equivalent to his (43) (remembering that we use proper time, refer fields to a moving frame and that our frame rotates).

At this point Jackson, neglects quantum fluctuations away from the classical orbit. Not doing so gives us the extra third term in (32). On the face of it this term is actually of lower order in $\sqrt{\hbar}$ than the others (remembering that E_0' is of order $e^{-1} \propto (\sqrt{\hbar})^{-1}$). That it does not in fact dominate the others is because it is only the noise in it

that causes spin flip: If $\delta E_c'$ were purely harmonic it could make transitions only if it had the right frequency, $\Omega = \omega_1$. That for large γ the extra term is actually negligible compared with the others, we see only after calculation. We insist that this calculation involves the classical guiding fields off as well as on the classical orbit.

To illustrate this last point, suppose the magnetic guiding field to be uniform, $n=0$. This removes the g -contribution to the third term in (32), and to the numerator of f , eq.(40). To restore the vertical stability add an auxiliary electric field E' which vanishes on the classical orbit. (We do not tell you how to do this in practice.) Make this auxiliary focussing rather strong, so that quantum fluctuations in z are greatly suppressed, and $\Omega \gg \omega_1$. Then $f \approx -2$, and we are left with only the first term in (39). We find that in this case the equilibrium polarization

$$P = \frac{\Gamma_+ - \Gamma_-}{\Gamma_+ + \Gamma_-} \quad (48)$$

has the value

$$P = \frac{64}{113} \sqrt{3} \approx 0.98 \quad (49)$$

as compared with

$$P = \frac{8}{15} \sqrt{3} \approx 0.92 \quad (50)$$

for the purely magnetic "weak focussing" storage ring

Returning to the purely magnetic ring, our analysis confirms the accepted result outside the narrow resonance. The existence of such a resonance, where vertical quantum fluctuations become important for polarization, was noted by Derbenev and Kondratenko [9]. But they treated it as a separate "depolarization" effect. When we treat the vertical fluctuation effect coherently with the basic polarization

mechanism, we find something a little more complicated (Appendix B). As γ passes through the resonance from below, the polarization first falls, from 92% to -17%, but then increases to 99% before finally decreasing again to 92%.

4 FLUCTUATIONS IN THE ORBITAL MOTION

The difference between the expressions (49) and (50) for the equilibrium polarization can be interpreted as showing the effect that the fluctuations in the orbital motion have on the spin. This effect is important, as it increases the depolarization of the electrons from 2% to 8%. Consequently, the spin excitations, considered separately, do not give a direct measurement of the circular Unruh effect. One should instead consider the coupled system of spin and orbital motion. However, for the orbital motion, the effect from the spin back on the motion is of less importance. Therefore, we shall in this section simply leave out the spin and consider the vertical fluctuations of the electrons from the point of view of the Unruh effect.

For large accelerations ($a'/c \gg \Omega$) the radiation reaction is completely dominated by the z -term. The equation of motion then can be written as

$$\ddot{z} + 2\Gamma\dot{z} + \Omega^2 z = \frac{e}{m} E'_{fz}(\tau) \quad (51)$$

where the damping factor is

$$\Gamma = \frac{1}{3} \frac{e^2}{m} \frac{a'^2}{c^5} \quad (52)$$

The damping term usually is very small, in the sense $\Gamma \ll \Omega$. Making use of this, we find the following (approximate) solution of eq.(51)

$$z(\tau) = z_0(\tau) + \frac{e}{m\Omega} \int_{-\infty}^{\infty} d\tau' e^{-\Gamma(\tau-\tau')} \sin[\Omega(\tau-\tau')] E'_{fz}(\tau') \quad (53)$$

The first term on the right is a free oscillator Heisenberg operator. It represents the quantum zero point motion, which would be present even in the absence of the radiation coupling. This part is relatively unimportant for high energy storage rings (where $a'/c \gg \Omega$) and we will neglect it in the following. The second term represents the radiation induced fluctuations. It is the radiation damping, which cuts off the integral at $\tau' \approx \Gamma^{-1}$, which prevents these fluctuations from building up indefinitely.

From eq.(53) we get the following expression for the fluctuation $\langle z^2 \rangle$ in the z-amplitude,

$$\langle z^2 \rangle = \left(\frac{e}{m\Omega}\right)^2 \int_{-\infty}^0 d\tau \int_{-\infty}^0 d\tau' e^{\Gamma(\tau-\tau')} \sin\Omega\tau \sin\Omega\tau' G(\tau-\tau'-i\epsilon) \quad (54)$$

where G is the correlation function

$$G(\tau-\tau') = \langle E'_{fz}(\tau) E'_{fz}(\tau') \rangle \quad (55)$$

Changing to new variables $T=\tau+\tau'$ and $t=\tau-\tau'$, we get

$$\langle z^2 \rangle = \frac{1}{2} \left(\frac{e}{m\Omega}\right)^2 \int_{-\infty}^{+\infty} dt G(t-i\epsilon) \int_{-\infty}^{-|t|} dT e^{\Gamma T} (\cos\Omega t - \cos\Omega T) \quad (56)$$

Keeping only the leading term from the T-integration, we find

$$\langle z^2 \rangle = \frac{1}{2\Gamma} \left(\frac{e}{m\Omega}\right)^2 \int_{-\infty}^{+\infty} dt e^{-\Gamma|t|} \cos\Omega t G(t-i\epsilon) \quad (57)$$

The correlation function G(t) is determined from the correlation functions of the free electromagnetic field along the classical trajectory. We find (in Gaussian units)

$$G(t) = \frac{4\hbar}{\pi c^3} t^{-4} \left(1 + \frac{1}{12} \frac{a'^2}{c^2} t^2\right)^{-3} \left(1 + \frac{1}{12} \frac{a'^2}{c^2} t^2 + \frac{1}{72} \frac{a'^4}{c^4} t^4\right) \quad (58)$$

The form of this function shows that the integral in eq.(57) converges rapidly for $|t| > c/a'$. Since we assume $a'/c \gg \Omega \gg \Gamma$, the first two factors in the integral therefore can be approximated by 1. This gives

$$\langle z^2 \rangle = \frac{2\hbar}{\pi \Gamma c^3} \left(\frac{e}{m\Omega}\right)^2 \int_{-\infty}^{+\infty} dt t^{-4} \left(1 + \frac{1}{12} \frac{a'^2}{c^2} t^2\right)^{-3} \left(1 + \frac{1}{12} \frac{a'^2}{c^2} t^2 + \frac{1}{72} \frac{a'^4}{c^4} t^4\right) \quad (59)$$

The integral is readily solved by closing the integration path in the upper complex half plane and using Cauchy's theorem

$$\begin{aligned} \langle z^2 \rangle &= \frac{13}{288} \sqrt{3} \frac{a'^3 e^2}{\Gamma m^2 c^6 \Omega^2} \\ &= \frac{13}{96} \sqrt{3} \frac{a' \hbar}{m c \Omega^2} \\ &= \frac{13}{96} \sqrt{3} \frac{R \hbar}{m c} \frac{1}{n} \end{aligned} \quad (60)$$

where n is the focussing parameter introduced in eq.(10). This is the standard result for the vertical fluctuations [10,11].

From eq.(60) the mean energy can be found. We first note that

$$\langle \dot{z}^2 \rangle = \Omega^2 \langle z^2 \rangle \quad (61)$$

Therefore the energy is

$$\begin{aligned} \langle E \rangle &= m \Omega^2 \langle z^2 \rangle \\ &= \frac{13}{96} \sqrt{3} \frac{a' \hbar}{c} \end{aligned} \quad (62)$$

As already pointed out, we may consider these excitations as a manifestation of the circular Unruh effect. Assuming the energy

spectrum to have a thermal form, we find for the corresponding temperature,

$$\begin{aligned} kT &= \frac{13}{96} \sqrt{3} \frac{a' \hbar}{c} \\ &\approx 0.23 \frac{a' \hbar}{c} \end{aligned} \quad (63)$$

where k is Boltzman's constant. This may be compared with the temperature formula for linear acceleration [1],

$$\begin{aligned} kT &= \frac{1}{2\pi} \frac{a' \hbar}{c} \\ &\approx 0.16 \frac{a' \hbar}{c} \end{aligned} \quad (64)$$

The effective temperature of the vertical fluctuations for the circulating electrons consequently is slightly higher than for linear acceleration. This agrees well with the comparison between circular and linear acceleration in our earlier paper [3].

Whereas the excitation spectrum would be thermal for linear acceleration, irrespectably of the nature of the accelerated system, this is not so for circular motion. However, for the vertical fluctuations the spectrum is in fact thermal (to the approximation we have considered) [10,11]. The reason for this is that the vertical motion defines a linear system, linearly coupled to the electromagnetic field. To see this, we first note that the probability distribution for z has a gaussian form, as a consequence of the fact that the fluctuation spectrum of the electromagnetic field is gaussian. The probability density therefore can be written as

$$\rho(z) = N \exp\left(-\frac{1}{2} \beta m \Omega^2 z^2\right) \quad (65)$$

where β now is a new parameter and N is a normalization constant. Furthermore, when $\Omega \gg \Gamma$, the probability density $\rho(z)$ in phase space

only depends on the energy of the free oscillator

$$E = \frac{1}{2} m \dot{z}^2 + \frac{1}{2} m \Omega^2 z^2 \quad (66)$$

This follows from the fact that the oscillator traverses the trajectory of constant energy many times before the radiation field has any effect on the motion. Consequently the full distribution $\rho(z, \dot{z})$ is determined by the reduced probability density $\rho(z)$. For the energy function $\rho(E)$ eq.(65) implies the following form,

$$\rho(E) = \beta e^{-\beta E} \quad (67)$$

Thus, the spectrum is indeed thermal, with β given by

$$\begin{aligned} \beta &= (kT)^{-1} \\ &= \left[\frac{13}{96} \sqrt{3} \frac{a' \hbar}{c} \right]^{-1} \end{aligned} \quad (68)$$

5 CONCLUDING REMARKS

We have in this paper examined further the connection between the Unruh effect and the radiative excitations of electrons in storage rings, which we introduced in an earlier paper [3]. The main point has been to examine how the orbital fluctuations influence the spin excitations. To study this connection we have formulated the problem in co-moving accelerated coordinates, which are natural coordinates when viewing the excitations as an Unruh effect. In the co-moving frames we find that the fluctuations in the vertical directions do have an important effect on the fluctuations. However, the resulting transition probabilities confirm (apart from a narrow resonance) previous calculations of the electron polarization, in a lab frame formulation, where the Thomas precession of the electrons instead is

important.

The influence of the vertical fluctuations on the spin excitations explains, in part, the difference between the correct polarization function $P(g)$ and the one obtained by a naive application of the acceleration - temperature formula [3]. As a consequence of this complication, a comparison between the temperature formula and the excitation spectrum of the coupled system of spin and vertical motion would be more relevant. However, since the reaction of the spin back on the orbital motion is not so important, we have considered the excitations in the z-coordinate independent of the spin. The result is that we for these excitations find a thermal spectrum, with a temperature which is higher by a factor ~ 1.4 than the temperature obtained by use of the temperature formula for linear acceleration. This difference between circular and linear motion fits well with results found in ref.[3], for circulating particles coupled to a scalar radiation field. The effective temperature found there was higher than the "Unruh temperature" (64) by a factor in the interval 0.9-1.8.

The spin depolarization effect, although small, is a measurable effect, and has indeed been measured [12]. The reported result is in good agreement with the theoretical value $P=0.92$. But, due to the complications discussed above this measurement cannot be considered as a direct demonstration of the (circular) Unruh effect. Therefore a measurement of the vertical fluctuations would be of interest, as a more direct experimental demonstration of this effect. However, these fluctuations are small, and it is not clear whether it would be possible to separate this effect from other perturbations in the orbit. Let us finally point out that the horizontal fluctuations, which we have not considered here, are larger than the vertical ones, essentially by a factor γ^2 [10,11]. This we attribute to the fact that the horizontal motion is not in a confining potential well, but is stabilized only by magnetic and coriolis forces.

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APPENDIX A

Here we derive our essential equations from those of Jackson [6]. His eqs.(38) and (40) give us the Thomas-BMT equation and the corresponding Hamiltonian:

$$\frac{d\vec{\sigma}}{dt} = (\vec{\omega} - \vec{\omega}_0) \times \vec{\sigma} \quad (A1)$$

$$H_{int} = \frac{\hbar}{2} \vec{\sigma} \cdot (\vec{\omega} - \vec{\omega}_0) \quad (A2)$$

$$\vec{\omega} = -\frac{e}{mc} \left\{ \left(\kappa + \frac{1}{\gamma} \right) \vec{B} - \frac{\kappa\gamma}{\gamma+1} \vec{\beta}(\vec{\beta} \cdot \vec{B}) - \left(\kappa + \frac{1}{\gamma+1} \right) \vec{\beta} \times \vec{E} \right\} \quad (A3)$$

$$\kappa = \frac{g}{2} - 1 \quad (A4)$$

We have made a slight change from Jackson, in rotating the coordinate system with the unperturbed classical particle:

$$\vec{\omega}_0 = -e\vec{B}/mc\gamma \quad (A5)$$

For the unperturbed circular orbit

$$\frac{d\vec{\sigma}}{dt} = \gamma\kappa \vec{\omega}_0 \times \vec{\sigma} \quad (A6)$$

Assuming that in zero order

$$\vec{E} = 0 = \vec{\beta} \cdot \vec{B}$$

then

$$\begin{aligned} \delta\vec{\omega} = & -\frac{e}{mc} \left\{ \left(\kappa + \frac{1}{\gamma} \right) \delta\vec{B} - \frac{\kappa\gamma}{\gamma+1} \vec{\beta}(\vec{\beta} \cdot \delta\vec{B}) - \left(\kappa + \frac{1}{\gamma+1} \right) \vec{\beta} \times \delta\vec{E} - \frac{\kappa\gamma}{\gamma+1} \vec{\beta}(\delta\vec{\beta} \cdot \vec{B}) \right. \\ & \left. + \dots \right\} \end{aligned} \quad (A7)$$

The neglected terms ... (from $\delta\gamma$) contribute only to the z-component, and not therefore to the σ_z -flip matrix elements; we will drop such terms when convenient. Jackson neglects completely quantum

fluctuations of the particle from its classical orbit. So he does not have the $\delta\vec{\beta}$ term in (A7), and he substitutes free quantum fields for $\delta\vec{B}$ and $\delta\vec{E}$. This gives his equation (45) (assuming that his last term $\gamma/(\gamma+1)$ should read $1/(\gamma+1)$). Here we continue with (A7) as it stands. Introducing the field variations in the comoving system,

$$\begin{aligned}\delta\vec{B}' &= \gamma(\delta\vec{B} - \vec{\beta} \times \delta\vec{E}) - \frac{\gamma^2}{\gamma+1} \vec{\beta}(\vec{\beta} \cdot \delta\vec{B}) \\ \delta\vec{E}' &= \gamma(\delta\vec{E} + \vec{\beta} \times \delta\vec{B}) - \frac{\gamma^2}{\gamma+1} \vec{\beta}(\vec{\beta} \cdot \delta\vec{E})\end{aligned}\tag{A8}$$

we have

$$\delta\vec{\omega} = -\frac{e}{mc} \left\{ \frac{\gamma}{2} \frac{\delta\vec{B}'}{\gamma} + \frac{\vec{\beta} \times \delta\vec{E}'}{\gamma+1} + \left(1 - \frac{g}{2}\right) \frac{\gamma}{\gamma+1} \vec{\beta}(\delta\vec{\beta} \cdot \vec{B}) \right\}\tag{A9}$$

Suppose that instead of the old spin vector $\vec{\sigma}$ we use the combination

$$\vec{\sigma} + \vec{A} \times \vec{\sigma} \rightarrow \vec{\sigma}\tag{A10}$$

(and rename it $\vec{\sigma}$, to avoid proliferation of notation), where $\vec{A}$ is a small quantity. To first order one finds the precession equation

$$\frac{d\vec{\sigma}}{dt} = (\gamma\kappa\vec{\omega}_0 + \delta\vec{\omega}) \times \vec{\sigma}\tag{A11}$$

where $\delta\vec{\omega}$ is augmented by

$$\vec{A} + \vec{A} \times \gamma\kappa\vec{\omega}_0\tag{A12}$$

The corresponding addition to the Hamiltonian (A2) is the time derivative

$$\frac{d}{dt} (\vec{A} \cdot \vec{\sigma})\tag{A13}$$

and does not contribute to transition probabilities per unit time when $\vec{A}$ remains bounded.

Take

$$\vec{A} = - \frac{\gamma^2}{\gamma+1} \delta\vec{\beta} \times \vec{\beta} \quad (\text{A14})$$

$$\begin{aligned} \dot{\vec{A}} &= - \frac{\gamma^2}{\gamma+1} \delta\dot{\vec{\beta}} \times \vec{\beta} \\ &= - \frac{1}{\gamma+1} \frac{e}{mc} \delta\vec{E}' \times \vec{\beta} \end{aligned} \quad (\text{A15})$$

$$\vec{A} \times \gamma \kappa \vec{\omega}_0 = \frac{\gamma^2}{\gamma+1} \left(\frac{g}{2} - 1 \right) \frac{e}{mc} \delta\vec{\beta} \cdot \vec{B} \vec{\beta} \quad (\text{A16})$$

Then instead of (A9),

$$\begin{aligned} \delta\vec{\omega} &= \frac{e}{mc} \left\{ - \frac{g}{2} \frac{\delta\vec{B}'}{\gamma} + \left(\frac{g}{2} - 1 \right) \gamma \vec{\beta} (\delta\vec{\beta} \cdot \vec{B}) \right\} \\ &= \frac{e}{mc} \left\{ - \frac{g}{2} \frac{\delta\vec{B}'}{\gamma} + \left(\frac{g}{2} - 1 \right) \delta\vec{\beta} \times \vec{E}' + \dots \right\} \end{aligned} \quad (\text{A17})$$

where ... is an uninteresting term in the z-direction. Eq.(A17) corresponds to eqs.(17) and (27) of the present paper, an overall factor γ being accounted for by the use of proper time in the main text rather than the lab time of this appendix.

Instead of (A14) take

$$\vec{A} = + \frac{\gamma}{\gamma+1} \delta\vec{\beta} \times \vec{\beta} \quad (\text{A18})$$

Then from (A9) and (A12) we come to

$$\delta\vec{\omega} = - \frac{e}{mc\gamma} \left(\frac{g}{2} \delta\vec{B}' + \vec{\beta} \times \delta\vec{E}' \right) \quad (\text{A19})$$

- which is (28) of the main text.

Consider finally the choice

$$\vec{A} = \delta\vec{\beta} \times \vec{\beta}/\beta^2 \quad (\text{A20})$$

$$\begin{aligned} \vec{A} &= \delta\vec{\beta} \times \vec{\beta}/\beta^2 \\ &= \frac{e}{mc\gamma} (\delta\vec{E} \times \vec{\beta} + \beta^2 \delta\vec{B} - \vec{\beta} \vec{\beta} \cdot \delta\vec{B})/\beta^2 \end{aligned} \quad (\text{A21})$$

This is especially convenient in the case $\kappa=0$. For that case, adding (A21) to the RHS of (A7), we come to

$$\delta\vec{\omega} = - \frac{e}{mc\gamma} \{ \vec{\beta} \vec{\beta} \cdot \delta\vec{B} + \frac{1}{\gamma} \vec{\beta} \times \delta\vec{E} \} / \beta^2 \quad (\text{A22})$$

In the case of a purely magnetic guiding classical field, purely transverse to the unperturbed $\vec{\beta}$, $\delta\vec{B}$ and $\delta\vec{E}$ in (A22) can be immediately identified with the quantum free fields. The fluctuations of the classical field have been eliminated. The possibility of doing this is related to the fact that in a purely magnetic field, with $g=2$, $\vec{\beta}$ and $\vec{\sigma}$ turn in the same way.

APPENDIX B

We have calculated the polarization resulting from (46) when f is retained in (39). We approximate $g-2$ by zero, except in the resonance factor f . We suppose R so large that the last, imaginary, term in (42) is negligible. The calculation is by standard techniques, reducing to integrals over expectation values of products of electromagnetic fields. The final integrals are among those tabulated by Sokolov and Ternov [11, p.151]. The result is

$$P = [8/(5/3)](1-f/6)[1-f/18+13f^2/360]^{-1} \quad (\text{B1})$$

Outside the immediate neighbourhood of

$$\gamma = \gamma_r = \sqrt{n/(g/2-1)} \quad (\text{B2})$$

P reduces to the standard

$$8/(5/3) = 0.924 \quad (B3)$$

Near $\gamma = \gamma_r$,

$$f = (g/2-1)/(1-\gamma/\gamma_r) \quad (B4)$$

As γ moves through the resonance from below, the polarization (B1) first drops from 0.924 to -0.169, but then increases to 0.992 before falling again to 0.924.

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