CERN-TH.4464/86

ANOMALIES AND MODULAR INVARIANCE IN STRING THEORY

A.N. Schellekens

CERN - Geneva

and

N.P. Warner^{*)}

Mathematics Department
M.I.T., Cambridge MA 02139, U.S.A.

ABSTRACT

All known anomaly cancellations of heterotic string theories are derived directly from one-loop modular invariance, and are shown to be related to a property of modular functions of weight 2. Using modular invariance, we construct infinite classes of anomaly-free field theories in $8m + 2$ dimensions for any m . We obtain a generating function for the anomalies of string-related field theories in $8m + 2$ dimensions.

*) Work supported in part by National Science Foundation Grant #84-07109 and by the U.S. Department of Energy (D.O.E.) under contract #DE-AC02-76ER03069.

It is a deeply-held belief that the absence of gauge and gravitational anomalies in the field-theory limit of string theories¹⁾⁻³⁾ is closely related to modular invariance of string amplitudes. For a string theory to be consistent, it is essential to have modular invariance to all loops⁴⁾. However, to establish full modular invariance it is sufficient to check it up to two loops^{4),5)}. For a field theory, it is sufficient to check for the presence or absence of anomalies at one loop (we assume, of course, a generalization of the Adler-Bardeen theorem). This might suggest that one-loop modular invariance of a string theory is sufficient for the absence of anomalies in the corresponding field-theory limit. In this paper we will demonstrate that this is indeed the case. Moreover, we find that the string theory does not even have to be conformally invariant to provide an anomaly-free field theory. This enables us to construct infinite classes of such field theories in $8m + 2$ dimensions, of which the known anomaly-free theories are but the simplest examples^{2),6),7)}.

We consider generalizations of the fermionic formulation of the heterotic string⁸⁾. For simplicity, we will only consider here the case where the left- and right-moving sectors are separately modular invariant^{*}). The right-moving sector will be the usual Neveu-Schwarz-Ramond model (with G.S.O. projections) but in $8m + 2$ dimensions. The left-moving sector consists of $8m$ transverse scalars (the space-time co-ordinates) plus a certain number (N) of fermions that generate the internal gauge symmetry as in Ref. 8). In this letter, we will generalize the Green-Schwarz anomaly cancellation²⁾. The analogues of the type IIB anomaly cancellation¹⁾ can be derived from the heterotic anomaly cancellation by the method described in Ref. 9).

Each spin structure in the one-loop string diagram yields a set of fermionic modes and an associated partition function¹⁰⁾. Only certain combinations of these separate sectors are modular-invariant. A complete basis for the one-loop modular-invariant theories can be characterized as follows. One splits the N fermions into groups of $8\ell_i$ fermions ($\sum 8\ell_i = N$). Each group of $8\ell_i$ is put into a modular-invariant combination of either the odd spin structures or the even spin structures. In the notation of Ref. 10), the corresponding partition functions are

^{*}) The $O(16) \times O(16)$ string⁷⁾ is not included in this class, but the appropriate generalizations are straightforward.

$$\begin{aligned}
 \text{EVEN: } P_E^{\ell_i}(q) &\equiv P_{--}^{\ell_i}(q) + (-1)^{\ell_i} P_{-+}^{\ell_i}(q) \\
 &\quad + (-1)^{\ell_i} P_{+-}^{\ell_i}(q) \\
 \text{ODD: } P_O^{\ell_i}(q) &\equiv P_{++}^{\ell_i}(q).
 \end{aligned}$$

where $q = e^{i\pi\tau}$, and

$$\begin{aligned}
 P_{--}^{\ell}(q) &= q^{-\ell/3} \prod_{n=1}^{\infty} (1+q^{2n-1})^{8\ell} = \left(\frac{\theta_3(0|\tau)}{\eta(\tau)} \right)^{4\ell} \\
 P_{-+}^{\ell}(q) &= q^{-\ell/3} \prod_{n=1}^{\infty} (1-q^{2n-1})^{8\ell} = \left(\frac{\theta_4(0|\tau)}{\eta(\tau)} \right)^{4\ell} \\
 P_{+-}^{\ell}(q) &= 2^{4\ell} q^{2\ell/3} \prod_{n=1}^{\infty} (1+q^{2n})^{8\ell} = \left(\frac{\theta_2(0|\tau)}{\eta(\tau)} \right)^{4\ell} \\
 P_{++}^{\ell}(q) &= \text{Tr}[(-)^f \exp i \tau H_R] = 0
 \end{aligned}$$

We use the definitions of theta functions as given in Ref. 11). The operator f is the fermion number, and

$$\eta(\tau) = \eta(q^2) = q^{1/12} \prod_{n=1}^{\infty} (1-q^{2n})$$

is the Dedekind eta function. The partition function for the $8m$ scalar co-ordinates of the left-moving sector is, of course, given by

$$P_B(q) = q^{-2m/3} \prod_{n=1}^{\infty} (1-q^{2n})^{-8m} = \eta(q^2)^{-8m}$$

We will show that any linear combination of these theories will yield an anomaly-free field theory.

The massless fermions in the theory are obtained as the tensor product of the lowest states of the Ramond sector of the right-moving fermions, with the massless states of the left-moving string. Note that the lowest states of the right Ramond sector are massless in any dimension; however, because of the powers of $q^{1/3}$ in the partition functions, the left-moving sector will only have massless states if $N = 8m + 24m'$. The anomaly of the massless sector is determined from the Atiyah-Singer index^{12),13)} in $8m + 4$ dimensions:

$$\text{Anomaly} \equiv \hat{A}(R) \sum_i \text{Ch}(F, g_i) \text{Ch}(R, s_i)$$

where (g_i, s_i) denote the gauge and space-time rotation representations of the massless level of the left-moving string. The $\text{Ch}(R, s_i)$ term includes the information about all higher spin fermions. Since we are in a transverse gauge, the ghost contributions are automatically taken into account. The heterotic string has no self-dual tensor fields. Increasing the number of fermions, N , shifts the intercept of the left-moving string. Thus the massless level becomes progressively more complicated, involving more and more gauge and Lorentz representations. To solve this problem, we calculate the generating function of the Chern character at every level of the left-moving string, and restrict to the massless level only at the end.

The appropriate Chern characters of symmetric (respectively antisymmetric) tensors of rank p , denoted by (p) (resp. $[p]$), are given by¹⁴⁾

$$\begin{aligned} \sum_{p=1}^{\infty} x^p \text{Ch}(F, (p)) &= [\det(1 - x e^{\frac{iF}{2\pi}})]^{-1} \\ \sum_{p=1}^{\infty} x^p \text{Ch}(F, [p]) &= \det(1 + x e^{\frac{iF}{2\pi}}) \end{aligned}$$

where F on the right-hand side of these equations is evaluated in the vector representation. In addition, the Chern character for an $SO(2M)$ Weyl spinor S , is given by¹⁴⁾

$$\begin{aligned} \text{Ch}(F, S) &= \exp \left\{ \sum_{p=1}^{\infty} \frac{(2^{2p}-1)}{4p} \frac{B_{2p}}{(2p)!} \left[\text{Tr} \left(\frac{iF}{2\pi} \right)^{2p} \right] \right\} \dim S \\ &\pm \{\hat{M}\} \exp \left\{ \sum_{p=1}^{\infty} \frac{B_{2p}}{4p} \frac{1}{(2p)!} \left[\text{Tr} \left(\frac{iF}{2\pi} \right)^{2p} \right] \right\} \end{aligned}$$

where F on the right-hand side is again in a vector representation and

$$\{\hat{M}\} = \frac{1}{2} \frac{1}{M!} \text{Tr} \left[\left(\frac{iF}{2\pi} \right)^M \gamma_{2M+1} \right]$$

is evaluated in the spinor representation. The sign of the last term depends upon the chirality of the spinor. The coefficients B_{2p} are Bernoulli numbers. Finally, $\hat{A}(R)$ may be written as

$$\hat{A}(R) = \exp \left\{ - \sum_{p=1}^{\infty} \frac{B_{2p}}{4p} \frac{1}{(2p)!} \left[\text{Tr} \left(\frac{iR}{2\pi} \right)^{2p} \right] \right\}$$

The Chern characters for the levels of the string can be generated by introducing the "gauge" partition functions

$$\begin{aligned} P_{--}^{\ell}(q, F) &= q^{-\ell/3} \prod_{n=1}^{\infty} \det(1 + q^{2n-1} e^{iF/2\pi}) \\ P_{-+}^{\ell}(q, F) &= q^{-\ell/3} \prod_{n=1}^{\infty} \det(1 - q^{2n-1} e^{iF/2\pi}) \\ P_{+-}^{\ell}(q, F) &= 2^{4\ell} q^{2\ell/3} \exp \left\{ \sum_{p=1}^{\infty} \frac{(2^{2p}-1)}{4p} \frac{B_{2p}}{(2p)!} \left[\text{Tr} \left(\frac{iF}{2\pi} \right)^{2p} \right] \right\} \\ &\quad \cdot \prod_{n=1}^{\infty} \det(1 + q^{2n} e^{iF/2\pi}) \\ P_{++}^{\ell}(q, F) &= 2^{4\ell} q^{2\ell/3} \exp \left\{ \sum_{p=1}^{\infty} \frac{B_{2p}}{4p} \frac{1}{(2p)!} \left[\text{Tr} \left(\frac{iF}{2\pi} \right)^{2p} \right] \right\} \\ &\quad \cdot \prod_{n=1}^{\infty} \det(1 - q^{2n} e^{iF/2\pi}) \end{aligned}$$

where F is now in the vector representation of $SO(8\ell)$. Observe that if $F \equiv 0$, then these reduce to the usual partition functions. For $F = 0$, the coefficient of q^n is the complete Chern character of all the gauge group representations at the n^{th} level of the string, expressed in terms of F in the vector representation. Similarly, the appropriate Chern characters of the Riemann tensor are generated by

$$P_B^m(q, R) = q^{-2m/3} \prod_{n=1}^{\infty} [\det(1 - q^{2n} e^{iR/2\pi})]^{-1}$$

It is instructive to express these functions in terms of the skew eigenvalues x_{α} of $R/2\pi$ and y_{β} of $F/2\pi$ [see Ref. 13)]. One finds

$$\begin{aligned} P_{--}^{\ell}(q, F) &= \prod_{\beta=1}^{4\ell} \frac{\theta_3\left(\frac{y_{\beta}}{2\pi i} | \tau\right)}{\eta(\tau)} \\ P_{-+}^{\ell}(q, F) &= \prod_{\beta=1}^{4\ell} \frac{\theta_4\left(\frac{y_{\beta}}{2\pi i} | \tau\right)}{\eta(\tau)} \end{aligned}$$

$$\begin{aligned}
 P_{+-}^{\ell}(q, F) &= \prod_{\beta=1}^{4\ell} \frac{\theta_2\left(\frac{y_{\beta}}{2\pi i} \middle| \tau\right)}{\eta(\tau)} \\
 P_{++}^{\ell}(q, F) &= \prod_{\beta=1}^{4\ell} \frac{\theta_1\left(\frac{y_{\beta}}{2\pi i} \middle| \tau\right)}{\eta(\tau)} \\
 P_B^m(q, R) &= \prod_{\alpha=1}^{4m} \frac{2\sinh\left(\frac{x_{\alpha}}{2}\right)}{\theta_1\left(\frac{x_{\alpha}}{2\pi i} \middle| \tau\right)} \eta(\tau)
 \end{aligned}$$

Comparing these expressions with Ref. 10), we see that "gauging" the partition functions simply amounts to changing the first argument of the θ -functions from 0 to $y_{\beta}/2\pi i$. Expressions of this form are likely to appear naturally as a result of string loop calculations.

To discuss anomaly cancellation, it is more convenient to exponentiate and re-express these functions in terms of traces of F and R . This can be achieved as follows

$$\prod_{\beta=1}^{4\ell} \theta_n\left(\frac{y_{\beta}}{2\pi i} \middle| \tau\right) = \prod_{\beta=1}^{4\ell} \theta_n(0 \middle| \tau) \exp \sum_{\beta=1}^{4\ell} \log \frac{\theta_n\left(\frac{y_{\beta}}{2\pi i} \middle| \tau\right)}{\theta_n(0 \middle| \tau)}$$

Expansions for the logarithm of the ratio of two θ -functions can be found in Ref. 15). For example,

$$\begin{aligned}
 \log \frac{\theta_4\left(\frac{y_{\beta}}{2\pi i} \middle| \tau\right)}{\theta_4(0 \middle| \tau)} &= -4 \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1-q^{2n}} \sinh^2\left(\frac{ny_{\beta}}{2}\right) \\
 &= - \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} n^{2k-1} \frac{q^n}{1-q^{2n}} \frac{1}{(2k)!} \text{Tr} \left(\frac{iF}{2\pi}\right)^{2k} \\
 (\text{because } \sum y_{\beta}^{2k} &= \frac{1}{2} \text{Tr} \left(\frac{iF}{2\pi}\right)^{2k})
 \end{aligned}$$

The coefficients can be expressed in terms of the Eisenstein series, defined as follows:

$$G_{2k}(q^2) = -\frac{B_{2k}}{4k} + \sum_{j=1}^{\infty} \sigma_{2k-1}(j) q^{2j}$$

where $\sigma_p(j)$ is the sum of the p^{th} powers of the divisors of j (including 1 and j). It is not difficult to show

$$\sum_{n=1}^{\infty} \frac{q^n}{1-q^{2n}} n^{2k-1} = -G_{2k}(q^2) + G_{2k}(q)$$

The result is therefore

$$P_{-+}^{\ell}(q, F) = P_{-+}^{\ell}(q) \exp \left\{ \sum_{k=1}^{\infty} [G_{2k}(q^2) - G_{2k}(q)] \frac{1}{(2k)!} \text{Tr} \left(\frac{iF}{2\pi} \right)^{2k} \right\}$$

By completely analogous calculations, we find

$$P_{--}^{\ell}(q, F) = P_{--}^{\ell}(q) \exp \left\{ \sum_{k=1}^{\infty} [G_{2k}(q^2) - G_{2k}(-q)] \frac{1}{(2k)!} \text{Tr} \left(\frac{iF}{2\pi} \right)^{2k} \right\}$$

$$P_{+-}^{\ell}(q, F) = P_{+-}^{\ell}(q) \exp \left\{ \sum_{k=1}^{\infty} [G_{2k}(q^2) - 2^{2k} G_{2k}(q^4)] \frac{1}{(2k)!} \text{Tr} \left(\frac{iF}{2\pi} \right)^{2k} \right\}$$

The other two partition functions require more care, because $\theta_1(0|\tau) = 0$. To calculate P_{++} , we factor out the zero mode and exponentiate an extra factor y_{β} , to obtain the following result

$$P_{++}^{\ell}(q, F) = 2\{4\ell\} (\eta(q^2))^{8\ell} \exp \left\{ - \sum_{k=1}^{\infty} G_{2k}(q^2) \frac{1}{(2k)!} \text{Tr} \left(\frac{iF}{2\pi} \right)^{2k} \right\}.$$

(Here we used $\{M\} = \frac{1}{2} \prod_{\beta=1}^M y_{\beta}$)

To calculate P_B , we combine it with the $\hat{A}$ -genus which multiplies the complete index. This provides precisely the Bernoulli numbers which are needed to express the result in terms of G_{2k} :

$$\hat{A}(R) P_B^m(q, R) = \eta(q^2)^{-8m} \exp \left\{ \sum_{k=1}^{\infty} G_{2k}(q^2) \frac{1}{(2k)!} \text{Tr} \left(\frac{iR}{2\pi} \right)^{2k} \right\}$$

This last expression appears as an overall factor in the anomaly, and provides the complete gravitational contribution. It is multiplied by factors $P_E^{\ell, i}(q, F_i)$ and/or $P_O^{\ell, i}(q, F_i)$, which are the obvious generalizations of our first equation to

$F_i = 0$. The resulting quantity we will call the anomaly-generating function $A(q, F, R)$.

The anomaly-generating function can be expressed as a power series in only even powers of q . Its leading term is either $q^{-(m+m')}$ or $q^{-(m+m')+1}$, depending on which power is even (recall $N = 8m + 24m'$). To obtain the anomaly for the field-theory limit of the string, we have to extract the $8m + 4$ form appearing in the coefficient of q^0 . This apparently daunting task is actually surprisingly easy if one uses some of the properties of modular functions.

A function $F(\tau)$ is called a modular function of weight w if it is a meromorphic function for $\text{Im}(\tau) > 0$ (including the point $\tau = i\infty$) and it satisfies

$$F\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^w F(\tau)$$

for any integers a, b, c, d , such that $ad - bc = 1$ ⁽¹⁶⁾⁻¹⁸⁾. If F is in fact holomorphic in this region (including $\tau = i\infty$), then it is called a modular form. The functions $G_{2k}(\tau) = G_{2k}(q^2)$ are modular forms of weight $2k$ for $k > 2$. Our purpose is to prove first that the coefficients of any $(8m + 4)$ -form in the expansion of $A(q, F, R)$ is a modular function of weight 2, with the exception of those $(8m + 4)$ -forms that have factors of $\text{Tr}(F^2)$ or $\text{Tr}(R^2)$. To show this we need only consider the modular transformations, $\tau \rightarrow \tau + 1$ and $\tau \rightarrow -(1/\tau)$, since they generate the entire modular group.

Since $A(q, F, R)$ is an even function of q , it is obviously invariant under $\tau \rightarrow \tau + 1$. Under the transformation $\tau \rightarrow -(1/\tau)$, it was shown in Ref. 10) that $P_{+-}(q) \leftrightarrow P_{-+}(q)$ without a weight factor, while $P_{--}(q)$ is invariant. Furthermore, even though $\eta(q^2)$ is not an analytic function, it satisfies

$$\eta^8(-1/\tau) = \tau^4 \eta^8(\tau)$$

so that under this transformation η^8 has an effective weight factor of four. While $G_{2k}(q^2(\tau))$ is a modular function of τ , $G_{2k}(\pm q(\tau))$ and $G_{2k}(q^4(\tau))$ are not. However, for $k > 1$,

$$\begin{aligned} G_{2k}\left(q\left(-\frac{1}{\tau}\right)\right) &= 2^{2k} \tau^{2k} G_{2k}(q^4(\tau)) \\ G_{2k}(-q(-1/\tau)) &= \tau^{2k} G_{2k}(-q(\tau)) \end{aligned}$$

This shows that all the functions $P^\ell(q, F)$ transform under $\tau \rightarrow (-1/\tau)$ in an identical manner to $P^\ell(q)$, apart from the $\text{Tr } F^2$ factors. It is now straightforward to see that the coefficient of any p-form in $A(q, F, R)$ without $\text{Tr}(F^2)$ or $\text{Tr}(R^2)$ factors is a modular function of weight $(\frac{1}{2}p) - 4m$. Moreover, because of the analyticity properties of the theta functions and the functions G_{2k} , this coefficient will only have a pole at $\tau = i\infty$ or $q = 0$.

For $p = 8m + 4$ we get modular functions of weight 2, which have the remarkable property that they can all be written as τ -derivatives of polynomials of the zero weight modular function j ¹⁶⁾⁻¹⁸⁾

$$\begin{aligned} j(q^2) &= (240)^3 \frac{G_4^3(q^2)}{\eta^{24}(q^2)} \\ &= \frac{1}{q^2} + 744 + 196884 q^2 + 21493760 q^4 + \dots \end{aligned}$$

This can be understood by observing that a polynomial of order γ in $j(q^2)$ is a modular function of weight zero and leading pole $q^{-2\gamma}$. It is easy to see that $d/d\tau$ acting on such a polynomial produces a modular function of weight 2 with leading pole $q^{-2\gamma}$. We can remove all the q^2 -poles of an anomaly coefficient function by subtracting the τ -derivative of a suitable polynomial in j . What is left is a modular form of weight 2. However, there are no modular forms of weight 2¹⁶⁾ and therefore the remainder has to vanish. Because the weight 2 modular functions are derivatives of weight zero modular functions, the constant term in their q^2 expansion must vanish.

We have now shown that all anomalies involving only terms of higher order than $\text{Tr } R^2$ or $\text{Tr } F^2$ cancel completely. The coefficients of the remaining terms have factors that depend upon G_2 , which transforms as follows

$$G_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 G_2(\tau) + \frac{i}{4\pi} c(c\tau+d)$$

so that it is not a modular function. Therefore one cannot expect the remaining terms to vanish. However, it is not hard to derive the following transformation properties:

$$2G_2(q^2(-1/\tau)) - G_2(q(-1/\tau)) = \tau^2 [2G_2(q^2(\tau)) - 4G_2(q^4(\tau))]$$

$$2G_2(q^2(-1/\tau)) - G_2(-q(-1/\tau)) = \tau^2 [2G_2(q^2(\tau)) - G_2(-q(\tau))]$$

If one now writes $\text{Tr}(R^2)$ and $\text{Tr}(F^2)$ as a linear combination of $[\text{Tr}(F^2) - \text{Tr}(R^2)]$ and $[\text{Tr}(F^2) + \text{Tr}(R^2)]$, it is a consequence of these identities that the coefficient of the latter is a modular function of weight 2. The q^0 term is therefore absent and the anomaly is proportional to $\text{Tr}(F^2) - \text{Tr}(R^2)$.

The foregoing argument assumed that the gauge group was simple. It is straightforward to show that the same result also holds for general groups if one replaces $\text{Tr}(F^2)$ by $\sum_i \text{Tr}(F_i^2)$. Because all traces are taken in the vector representation, this is the usual Green-Schwarz factorization.

Note that nothing in the preceding arguments restricts the values of m and m' , so that even if the corresponding string theory has conformal anomalies, two-loop modular anomalies or tachyons, the massless sector is still anomaly-free. Therefore we obtain an infinite number of anomaly-free theories. As was pointed out in Ref. 10), some fermions may have the wrong statistics. While this may cause problems of consistency in the string theory, it is always possible to generate the correct signs in the field-theory limit by changing the chirality of those fermions. Such difficulties never occur if $m \equiv 1 \pmod{2}$ and $\lambda_1 \equiv 0 \pmod{2}^{*)}$.

Apart from generating many anomaly-free theories, our results also relate anomaly cancellations in groups of seemingly unrelated theories. Essentially these groups are characterized by the order of the q^2 pole of $A(q, F, R)$. If there is no pole at all, the anomaly coefficient will have to be an entire modular function of weight 2, and must therefore be zero. The only interesting example in this class is $m = 1$, $m' = 0$. The gauge group is $SO(8)$, and for the even and odd spin structure we obtain massless Weyl fermions in the representations

*) In type IIB theories there are contributions to the anomaly from self-dual tensors, which are bosons. The anomalies cancel only if one uses the kind of statistics for the fermions that is dictated by the partition function¹⁰⁾.

$$\text{EVEN:} \quad 2(8_v) - 8_s - 8_c$$

$$\text{ODD:} \quad 8_s - 8_c$$

The minus signs can be interpreted either as opposite statistics or opposite chirality. The first of these theories is related to the anomaly cancellation in the type IIB superstring⁹⁾.

If there is a simple pole in q^2 , there are several possible choices for m and m' ; for example:

- 1) $m = 1, m' = 1$ This includes $E_8 \times E_8$ and $SO(32)$ in ten dimensions, depending on how one splits the fermions into groups ℓ_i .
- 2) $m = 2, m' = 0$ This is the 18-dimensional E_8 theory^{6),19),20)}.
- 3) $m = 3, m' = -1$ Note that $N = 8m + 24m' = 0$. This is the 24-dimensional theory discussed in Ref. 6).
- 4) $m = 1, m' = 2$ This includes a new $SO(56)$ theory in ten dimensions. The field theory consists of a Weyl spinor in the rank 3 antisymmetric tensor representation of $SO(56)$ and a gravitino in the vector representation of $SO(56)$.

There are obviously many more examples, but it should be clear from the last one that these new theories are rather bizarre. It is rather remarkable that all previously-known anomaly cancellations can be derived from the properties of just one modular function, namely $(d/d\tau)(j(\tau))$. This function j is related to the monster group F_1 , as is evident from the fact that all its higher coefficients can be written in terms of monster group representations.

Another surprising result is that all anomalies that we find can be cancelled by the simplest Green-Schwarz mechanism. Only a single two-form $B_{\mu\nu}$ is needed. Conversely, the fact that the anomaly does not cancel directly, but only by the Green-Schwarz mechanism, is due to the fact that the Eisenstein function G_2 is not a modular function -- entire modular functions of weight two do not exist.

ACKNOWLEDGEMENTS

We would like to thank L. Alvarez-Gaumé, W. Lerche and H. Stark for discussions. One of us (A.N.S.) would like to thank the Mathematics Department of M.I.T. for its hospitality.

REFERENCES

- 1) L. Alvarez-Gaumé and E. Witten, Nucl. Phys. B234 (1983) 269.
- 2) M.B. Green and J. Schwarz, Phys. Lett. 149B (1984) 117.
- 3) J. Thierry-Mieg, Phys. Lett. 156B (1985) 199.
- 4) E. Witten, "Global Anomalies in String Theory", in "Geometry Anomalies Topology", W.A. Bardeen and A.R. White eds., World Scientific (1985).
- 5) L. Alvarez-Gaumé, G. Moore and C. Vafa, "Theta Functions, Modular Invariance and Strings", Harvard preprint HUTP-86/A017 (1986).
- 6) J. Thierry-Mieg, Phys. Lett. 171B (1986) 163.
- 7) L. Alvarez-Gaumé, P. Ginsparg, G. Moore and C. Vafa, "An $O(16) \times O(16)$ Heterotic String", Harvard preprint HUTP-86/A013 (1986);
L. Dixon and J. Harvey, "String Theories in Ten Dimensions Without Space-Time Supersymmetry", Princeton preprint (1986).
- 8) D. Gross, J. Harvey, E. Martinec and R. Rohm, Phys. Rev. Lett. 54 (1985) 502; Nucl. Phys. B256 (1985) 253; Nucl. Phys. B267 (1986) 75.
- 9) A.N. Schellekens, "Anomaly Cancellation in Ten Dimensions and Beyond", CERN preprint TH.4412/86 (1986), to appear in Phys. Lett. B.
- 10) N. Seiberg and E. Witten, "Spin Structures in String Theory", Princeton preprint (1986).
- 11) J.H. Schwarz, Phys. Reports 89 (1982) 223.
- 12) M.F. Atiyah and I.M. Singer, Ann. of Math. 87 (1968) 485, 546; 93 (1971) 119, 139;
M.F. Atiyah and G.B. Segal, Ann. of Math. 87 (1968) 531.
- 13) L. Alvarez-Gaumé and P. Ginsparg, Ann. of Phys. 161 (1985) 423;
O. Alvarez, I. Singer and B. Zumino, Comm. Math. Phys. 96 (1984) 409.
- 14) A.N. Schellekens and N.P. Warner, to appear.
- 15) M. Abramowitz and I. Segun, "Handbook of Mathematical Functions", Dover, New York (1965).
- 16) B. Schoeneberg, "Elliptic Modular Functions", Springer (1970).
- 17) J.-P. Serre, "A Course in Arithmetic", Springer (1973).
- 18) R. Rankin, "Modular Forms and Functions", Cambridge Univ. Press (1977).
- 19) F. Englert, H. Nicolai and A.N. Schellekens, CERN preprint TH.4360/86 (1986), to appear in Nucl. Phys. B.
- 20) W. Nahm, Bonn preprint Bonn-HE-85-37 (1985).