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NILPOTENT BRST INVARIANCE OF THE INTERACTING POLYAKOV STRING

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ABSTRACT

The BRST invariance currently popular in the construction of second quantized string theories is only nilpotent on-shell. We show that the Faddeev-Popov representation of Polyakov's interacting bosonic string admits a BRST invariance that is also nilpotent off-shell. The invariant vertices are just the Weyl- and reparametrization-invariant vertices of the original model. We also show that failure of the physical state condition is intimately related to the divergences of the theory and suggest that its rehabilitation be a requirement of any scheme adopted to regularize and renormalize the model.

1. INTRODUCTION

BRST invariance is crucial to the second quantization of string theories [1]. This symmetry was originally exploited by Kato and Ogawa [2] to prove the no-ghost theorem [3] for the first quantized free bosonic string. In their (canonical) approach physical states are required to be annihilated by the conserved generator Q of BRST transformations:

$$Q |phys.\rangle = 0. \quad (1.1)$$

In the critical dimension $D = 26$ Q is nilpotent, so a physical state remains physical if another state of the form $Q|\text{anything}\rangle$ is added to it, which is the origin of the enormous symmetries of the second quantized theory. Physical states are obtained by acting on a vacuum, which is itself physical, with operators that commute with λQ , where λ is a Grassmann number that parametrizes the transformation

$$|phys.\rangle = V_{phys} |0\rangle, \quad [\lambda Q, V_{phys}] = 0. \quad (1.2)$$

The constraint on physical states is equivalent to

$$\langle phys. | [\lambda Q, V] | phys.' \rangle = 0 \quad (1.3)$$

for arbitrary physical states and operators V . Since Q generates BRST transformations, this may be written

$$\langle 0 | V_{phys} \delta_\lambda V V_{phys}' | 0 \rangle = 0 \quad (1.4)$$

This constraint plays a central role in the theory.

One of the objects of second quantization is to construct a covariant approach to interactions between strings; however, Polyakov's formulation of the first quantized bosonic string [4] already includes string interactions in a natural way. Thus amplitudes for the scattering of particles from closed strings are expectation values of vertex operators,

$$A(k_1, \dots, k_n) = \sum_{h=0}^{\infty} \int \mathcal{D}x \mathcal{D}g e^{-S} \Gamma(x, g, k_1) \dots \Gamma(x, g, k_n), \quad (1.5)$$

where the summation is over all closed world-sheet topologies. A closed world-sheet of genus h corresponds to h string loops. In the critical dimension this expression is Weyl-invariant for appropriately chosen vertex operators. The BRST invariance is the symmetry of the Faddeev-Popov ghost representation of Jacobians resulting from the integration over metrics g . If we assume some underlying Hilbert space representation necessary for second quantization, then the amplitude $A(k_1, \dots, k_n)$ would be expressible as a vacuum expectation value,

$$A(k_1 \dots k_n) = \langle 0 | V_{\text{phys}}(k_1) \dots V_{\text{phys}}(k_n) | 0 \rangle, \quad (1.6)$$

where the V_{phys} are BRST-invariant extensions of the Weyl-invariant vertices. If the second quantization is BRST-invariant, then for consistency the transformation must be nilpotent and spurious operators (i.e. of the form $\delta_\lambda V$) must decouple from physical processes as in (1.4).

In the next section we describe the Faddeev-Popov ghost representation of Polyakov's interacting bosonic string. When the world-sheet has handles the antighosts have zero modes. We discuss their geometry in Section 3. They complicate the invariance of the theory; nonetheless we construct a nilpotent BRST transformation in Section 4 for the generic case $h \geq 2$. In Section 5 we discuss the decoupling of spurious vertices, and suggest that the failure for this to occur according to a mechanism proposed by Friedan et al. [5] implies the inconsistency of the unregulated purely bosonic theory, which may possibly be cured by a renormalization procedure. We prove that reparametrization and Weyl-invariant operators in the Polyakov theory are automatically BRST-invariant (Section 6), and in Section 7 we summarize these results.

2. FADDEEV-POPOV REPRESENTATION

We will now give a brief review of Polyakov's interacting bosonic string [4]; for a more detailed account, see Ref. [6].

The h -loop contribution to Polyakov's partition function for the bosonic string based on the action of Brink, Di Vecchia and Howe [7] is

$$\begin{aligned} Z_h &= \int_{m_h \times e_h} \mathcal{D}X \mathcal{D}g \frac{1}{\mathcal{V}} e^{-S(X)} \\ S(X) &= \frac{1}{2} \int_{\Sigma_h} d^2x \sqrt{g} (g^{ab} \partial_a X^\mu \partial_b X^\mu + \lambda_0) \end{aligned} \quad (2.1)$$

For closed strings we can take the world-sheet parameter space Σ_h to have h handles and no boundary, and we will consider only the case $h \geq 2$; m_h is the space of metrics g on Σ_h , and e_h the embeddings X of Σ_h in the target space-time which we take to have the critical dimension $D = 26$. The cosmological constant λ_0 is present to absorb divergences, and apart from this term the action is invariant under local Weyl rescalings of the metric. It is also invariant under reparametrizations, or diffeomorphisms. The measure $\mathcal{D}g$ is induced by the metric on variations of g :

$$(\delta_1 g, \delta_2 g) = \int_{\Sigma} d^2x \sqrt{g} (g^{ab} g^{rs} + c g^{ar} g^{bs}) \delta_1 g_{ar} \delta_2 g_{bs}, \quad (2.2)$$

with c a constant; $\mathcal{V}$ is the volume of diffeomorphisms and Weyl transformations acting on g .

An arbitrary metric is obtained by the action of a Weyl scaling and a diffeomorphism on a reference or fiducial metric which is a function of a finite number of additional variables [6]. Diffeomorphisms may be parametrized by sufficiently smooth vector fields on z . Such a vector field $\xi(z)$ specifies a family of curves, one through each point of Σ parametrized by a variable t and with tangent ξ :

$$\xi^a = \frac{dz^a}{dt}$$

If z_ξ is the point which is a unit interval in t along the curve through z , then the reparametrization consists of mapping scalar quantities defined at z to their values at z_ξ . The interval ds^2 at z is transformed as

$$ds^2(z) = g_{ab}(z) dz^a dz^b \rightarrow ds^2(z_\xi) = g_{mn}(z_\xi) dz_\xi^m dz_\xi^n = g_{mn}(z_\xi(z)) \frac{\partial z_\xi^m}{\partial z^a} \frac{\partial z_\xi^n}{\partial z^b} dz^a dz^b$$

If we denote the fiducial metric depending on the parameters y^A by $\hat{g}(z; y^A)$, then the arbitrary metric may be written

$$g_{ab}(z; \rho, \xi, y^A) = e^\rho \left(\frac{\hat{g}(z)}{\hat{g}(z_\xi)} \right)^{1/2} \frac{\hat{g}_{mn}(z_\xi; y^A)}{\det^2(\partial z_\xi^r / \partial z^s)} \frac{\partial z_\xi^m}{\partial z^a} \frac{\partial z_\xi^n}{\partial z^b}$$

The Jacobian and determinantal factors are present to cancel the scaling of $\hat{g}$ that results from the diffeomorphism and thus to disentangle the ρ and ξ degrees of freedom.

A variation of g may be decomposed into three pieces corresponding to this parametrization:

$$\delta g_{ab} = \delta \rho g_{ab} + P_{ab}(\delta \xi) + \delta y^A \psi_{Aab} \quad (2.3)$$

We will only deal with quantities that are covariant under reparametrizations, so we will only need this expression for a particular value of ξ . The most convenient value of ξ is zero, for which $P_{ab}(\delta \xi)$ is readily evaluated as

$$P_{ab}(\delta \xi) = g_{s(a} \nabla_{b)} \delta \xi^s - g_{ab} \nabla_s \delta \xi^s, \\ f_{(a} g_{b)} \equiv f_a g_b + f_b g_a \quad (2.4)$$

using

$$z_{\xi}^a = z^a + \delta \xi^a + O(\delta \xi^2) .$$

Here P is a map from vectors to symmetric, traceless tensors with two indices. If the integration over g is transformed to one over the parameters $q(z)$, $\xi^p(z)$, and y^A , then it would be convenient to choose the decomposition (2.3) to be orthogonal in order to facilitate the calculation of the Jacobian. However, such a choice does not allow q , ξ , and y^A to be an integrable coordinate system and so is unacceptable, as we now show.

A variation of the metric orthogonal with respect to the inner product (2.2) to Weyl scalings and diffeomorphisms is an element of $\ker P^\dagger$:

$$P^\dagger(\phi) = -2 \nabla^a \phi_{ab} . \quad (2.5)$$

Let $\{\phi_A\}$ be a basis for $\ker P^\dagger$. The defining equations

$$\nabla^a \phi_{ab} = 0 , \quad \phi_{ab} = \phi_{ba} , \quad g^{ab} \phi_{ab} = 0 \quad (2.6)$$

are unchanged by Weyl scalings, so we can take the ϕ_A to be invariant under $g_{ab} \rightarrow e^q g_{ab}$. The Riemann–Roch theorem is an index theorem for P :

$$\dim \ker P - \dim \ker P^\dagger = 3\chi , \quad (2.7)$$

where χ is the Euler number $\chi = 2 - 2h$. For $h \geq 2$, the kernel of P , which is the set of conformal Killing vectors, is empty. There are thus $6h - 6$ independent ϕ_A and consequently the same number of y^A . Deformations of the metric which are neither Weyl scalings nor diffeomorphisms are known as Teichmüller deformations, about which there exists extensive mathematical literature [8]. The parameter y^A are known as Teichmüller parameters, or moduli.

The metric g_{ab} must be a differentiable function of $q(z)$, $\xi^p(z)$, and the y^A if these are to be used as coordinates. This implies

$$\begin{aligned} [\delta_p, \delta_\xi] g_{rs} &= [\delta_p, \partial_A] g_{rs} = [\delta_\xi, \partial_A] g_{rs} = \\ [\partial_A, \partial_B] g_{rs} &= [\delta_p, \delta_{\xi^p}] g_{rs} = [\delta_{\xi^p}, \delta_{\xi^q}] g_{rs} = 0 \end{aligned}$$

where

$$\begin{aligned} \delta_p &\equiv \int d^2z \, \delta p(z) \frac{\delta}{\delta p(z)} , \quad \delta_\xi \equiv \int d^2z \, \delta \xi^p(z) \frac{\delta}{\delta \xi^p(z)} , \\ \partial_A &\equiv \frac{\partial}{\partial y^A} . \end{aligned} \quad (2.8)$$

Applying the first four conditions to Eq. (2.3) gives

$$\begin{aligned}
\delta_{\mathcal{P}} P_{rs}(\delta\xi) &= P_{rs}(\delta\xi) \delta_{\mathcal{P}}, \\
\delta_{\mathcal{P}} \psi_{Ars} &= \psi_{Ars} \delta_{\mathcal{P}}, \\
\delta_{\xi} \psi_{Ars} &= \psi_{A\rho(r} \nabla_{s)} \delta\xi^{\rho} - \psi_{Ars} \nabla_{\rho} \delta\xi^{\rho} + \delta\xi^{\rho} \nabla_{\rho} \psi_{Ars}, \\
\partial_A \psi_{Brs} &= \partial_B \psi_{Ars},
\end{aligned} \tag{2.9}$$

where the next to last equation follows from taking ψ_A to be traceless and a little algebra. The first condition is satisfied provided ξ is taken as a contravariant vector ξ^p (and not ξ_p). The second condition is not satisfied by choosing ψ_A to be in the kernel of $P^\dagger$ because the ϕ_A are invariant under Weyl scalings,

$$\delta_{\mathcal{P}} \phi_{Ars} = 0. \tag{2.10}$$

As we show in the next section, the third condition of (2.9) is also incompatible with identifying the ψ_A and ϕ_A .

Calculating the measure $\mathcal{D}g$ in terms of the new coordinates q , ξ , and y^A is done in analogy with finite-dimensional integration, so that if the coordinates are denoted collectively by λ_i ,

$$\mathcal{D}g = \text{Det}''\left(\frac{\partial g}{\partial \lambda_i}, \frac{\partial g}{\partial \lambda_j}\right) \prod_i d\lambda_i \tag{2.11}$$

The determinant may be factorized by projecting the ψ_A onto $\ker P^\dagger$ so that

$$\mathcal{D}g = \mathcal{D}_{\mathcal{P}} \mathcal{D}_{\xi} \prod_A dy^A (\text{Det}(P^\dagger P) \det \eta / \det H(P^\dagger))^{1/2}, \tag{2.12}$$

where

$$\begin{aligned}
\eta_{AB} &= \int d^2x \sqrt{g} \psi_{Ars} \phi_{Bmn} g^{rm} g^{sn}, \\
H_{AB} &= \int d^2x \sqrt{g} \phi_{Ars} \phi_{Bmn} g^{rm} g^{sn},
\end{aligned} \tag{2.13}$$

and $\mathcal{D}_q$ and $\mathcal{D}_{\xi}$ are induced by the inner products

$$\begin{aligned}
(\delta_1 \xi, \delta_2 \xi) &= \int d^2x \sqrt{g} \delta_1 \xi^r g_{rs} \delta_2 \xi^s, \\
(\delta_1 \mathcal{P}, \delta_2 \mathcal{P}) &= \int d^2x \sqrt{g} \delta_1 \mathcal{P} \delta_2 \mathcal{P}.
\end{aligned} \tag{2.14}$$

The measure $\mathcal{D}X$ is induced by the metric on variations of X ,

$$(\delta_1 X, \delta_2 X) = \int d^2z \sqrt{g} \delta_1 X^\mu \delta_2 X_\mu. \quad (2.15)$$

Since the action in Eq. (2.1) is only quadratic in X , this integration may be done immediately to give

$$\left(\frac{2\pi}{\int d^2z \sqrt{g}} \text{Det}' \Delta \right)^{-D/2}, \quad \Delta X = -\frac{1}{\sqrt{g}} \partial_a (g^{ab} \sqrt{g} \partial_b X). \quad (2.16)$$

The prime denotes omission of the zero mode of Δ which is responsible for the $2\pi/\int d^2z \sqrt{g}$.

Polyakov's original computation [4] for a simple topology showed that for an appropriate choice of λ_0 the q dependence of formula (2.16) cancelled with that of $\text{Det}(P^\dagger P)$ when $D = 26$. Friedan [9] showed that when $P^\dagger$ has zero-modes $\text{Det}(P^\dagger P)$ has an extra contribution which cancels $\sqrt{\det H(P^\dagger)}$ in Eq. (2.12). Because the integrability conditions (2.9) require that ψ_A scales with q like $\sqrt{g}$ whilst ϕ_B is invariant, it follows that η_{AB} is independent of q . The only dependence on q and ξ is contained in $\mathcal{D}_q \mathcal{D}_\xi$, so these integrations may be performed to produce a constant which cancels $\mathcal{V}$ in Eq. (2.1). The result is

$$Z_h = \int \prod_A dy^A \det'^{1/2} \eta \, e^{-S_0},$$

$$e^{-S_0} = \left(\frac{2\pi}{\int d^2z \sqrt{g}} \text{Det}' \Delta \right)^{-13} (\text{Det}(P^\dagger P) / \det H(P^\dagger))^{1/2} \quad (2.17)$$

The determinants are defined using the fiducial metric $\hat{g}$ depending on the Teichmüller parameters y^A . If we had attempted to take $\psi_A = \phi_A$ in Eq. (2.3) we would have arrived at the result of Alvarez [4] in which $\sqrt{\det \eta}$ is replaced by $\det H(P^\dagger)$, but this would have involved the use of variables which are not coordinates. Moore and Nelson [10] pointed out that the q -dependence of Alvarez' result coming from $\det H(P^\dagger)$ is spurious. As we have seen, the independence of Eq. (2.17) of the Weyl mode q is just a consequence of the integrability of the coordinate system.

The determinant of $P^\dagger P$ may be thought of as a Faddeev-Popov determinant arising from gauge-fixing the diffeomorphism invariance by setting the metric to equal the fiducial metric. The ghost representation may be obtained by considering the integral

$$\int \mathcal{D}(b, c) \exp - \int d^2z \sqrt{g} (b_{ab} \nabla^a c^b + b_{ab} \beta^{ab} - \gamma_a c^a). \quad (2.18)$$

where b and c are real Fermi fields, β and γ are Grassmannian sources, and b is traceless with respect to the fiducial metric; b and β may be decomposed into pieces in $\ker P^\dagger$ and orthogonal parts,

$$b = b^A \phi_A + b_\perp, \quad \beta = \beta_A \phi^A + \beta_\perp, \quad (2.19)$$

where the ϕ^A are dual to the ϕ_A :

$$\int d^2x \sqrt{g} \phi^A{}_{rs} \phi^s_{B} = \delta^A_B. \quad (2.20)$$

Since P is invertible for $h \geq 2$, the b integration may be transformed to one over the variables b^A and p defined by

$$P(p) = b_\perp. \quad (2.21)$$

The Jacobian of the transformation is $[\det H(P^\dagger) \cdot \text{Det}(P^\dagger P)]^{-1/2}$ so that Eq. (2.5) becomes

$$\begin{aligned} & \int \prod_A db^A \mathcal{D}p \mathcal{D}c (\text{Det}(P^\dagger P) \det H(P^\dagger))^{-1/2} \\ & \times \exp - \int d^2x \sqrt{g} \left(\frac{1}{2} P_{ab}(p) p^{ab}(c) + P_{ab}(p) \beta_\perp^{ab} \right. \\ & \quad \left. - \gamma_{ab} c^a + \beta_A b^A \right). \end{aligned} \quad (2.22)$$

The integrations over p and c yield another factor of $\text{Det } P^\dagger P$, whilst those over the b^A result in $\Pi \beta_A$. Let N be the Green function for P and $P^\dagger$ such that

$$\begin{aligned} -\nabla^c N_{bc}^a(z, z') &= \frac{1}{\sqrt{g}} \delta(z-z') \delta_b^a, \\ \nabla^c N_{rs}^b(z, z') - g^{ab}(z') \nabla_c N_{rs}^c(z, z') &= \frac{1}{\sqrt{g}} \delta(z-z') (\delta_r^b \delta_s^a - g^{ab} g_{rs}) \\ &\quad - 2 \phi_{rs}^A(z) \phi_A^{ab}(z'), \quad (2.23) \\ g^{rs}(z) N_{rs}^a(z, z') &= 0, \quad N_{rs}^a(z, z) = N_{sr}^a(z, z), \end{aligned}$$

$$\int d^2x \sqrt{g(z)} \phi_A^{rs}(z) N_{rs}^a(z, z') = 0.$$

In the first equation the covariant derivative at the point z acts only on the lower indices, whilst in the second it is taken at the point z' and acts only on the upper index. The term $\phi^A \phi_A$ is a projection operator which is present because $P^\dagger$ can only be inverted on the space orthogonal to its zero modes.

Using this Green function, the integral (2.5) may be evaluated as

$$\left(\text{Det}(P^\dagger P) / \det H(P^\dagger) \right)^{1/2} \prod_A \beta_A \exp \int d^2 z d^2 z' \sqrt{g(z)g(z')} \times \beta^{ab}(z) N_{ab}^P(z, z') \gamma_P(z'). \quad (2.24)$$

This contains the Faddeev-Popov determinant appearing in Z_h , so we conclude that any quantity of physical interest may be obtained by integrating over the y^A the appropriate functional derivatives of

$$\begin{aligned} Z^0[\xi, \beta, \gamma] &= \int \mathcal{D}(X, b, c) \exp - (S + S_T) \det^{1/2} \eta, \\ S &= \int d^2 z \sqrt{g} \left(\frac{1}{2} g^{ab} \partial_a X^\mu \partial_b X^\mu + \lambda_0 + b_{ab} \nabla^a c^b \right), \\ S_T &= \int d^2 z \sqrt{g} \left(- \xi^\mu X^\mu + b_{ab} \beta^{ab} - \gamma_a c^a \right). \end{aligned} \quad (2.25)$$

In particular

$$Z_h = \left\{ \int \prod_{A=1}^{6h-6} \frac{\pi}{\delta \beta^{ab}} (dy^A \int d^2 z \phi^{ab} \frac{\delta}{\delta \beta^{ab}}) Z^0 \det \eta \right\}_{\text{sources} = 0} \quad (2.26)$$

The powers of $\det \eta$ have been chosen for later convenience.

3. DEPENDENCE OF ZERO MODES ON TEICHMÜLLER PARAMETERS

The zero modes of $P^\dagger$, ϕ_A , and the parameters y^A are the new components that appear in the Polyakov approach when strings interact. We will need to know how the former depend on the latter, so we devote this section to calculating $\partial\phi/\partial y$ and $\delta\phi/\delta g$.

An arbitrary symmetric tensor may be decomposed into a piece in $\ker P^\dagger$, a piece proportional to the metric tensor, and a traceless tensor orthogonal to $\ker P^\dagger$. Thus

$$\partial_A \phi_{Brs} = \Gamma_{AB}^c \phi_{crs} + E_{AB} g_{rs} + F_{ABrs} \quad (3.1)$$

with Γ independent of z . Differentiating the condition of tracelessness of ϕ fixes E :

$$0 = \partial_A (g^{rs} \phi_{Brs}) = - \psi_A^{rs} \phi_{Brs} + 2 E_{AB} \quad (3.2)$$

F may be obtained by differentiating the condition that ϕ be annihilated by $P^\dagger$. Thus

$$0 = \partial_A (\nabla^r \phi_{Brs}) = \nabla^r (\partial_A \phi_{Brs}) - \nabla_p (\psi_A^{pr} \phi_{Brs}) - \frac{1}{2} (\nabla_s \psi_A^{mn}) \phi_{Bmn}$$

is an equation for $\nabla^r F_{rs}$ which is solved using the Green function N in Eq. (2.23):

$$F_{ABrs}(z) = \int d^2z' \sqrt{g(z')} N_{rs}^k(z, z') \left\{ \frac{1}{2} \psi_A^{mn}(z') \nabla_k \phi_{Bmn}(z') - \nabla_p (\psi_A^{pr}(z') \phi_{Brk}(z')) \right\}. \quad (3.3)$$

These expressions for E and F will be adequate. To go further would require specifying a parametrization of Teichmüller space.

The functional derivative $\delta/\delta g$ may be expressed in the coordinates ϱ , ξ , and y^A corresponding to the decomposition (2.3):

$$\delta g_{ab} = \delta \varrho g_{ab} + P_{ab}(\delta \xi) + \delta y^A \psi_{Aab}.$$

At the point $\xi = 0$ it takes the form

$$\begin{aligned} \frac{1}{\sqrt{g(z)}} \frac{\delta}{\delta g^{rs}(z)} &= -\frac{1}{2} g_{rs}(z) \frac{1}{\sqrt{g(z)}} \frac{\delta}{\delta \varrho(z)} - \eta^{AB} \phi_{ABrs}(z) \partial_B \\ &\quad - \frac{1}{2} \int d^2z' N_{rs}^p(z, z') \frac{\delta}{\delta \xi^p(z')} \\ &\quad + \frac{1}{2} \eta^{AB} \phi_{ABrs}(z) \int d^2z' d^2z'' \sqrt{g(z')} \psi_B^{rs}(z') N_{rs}^p(z', z'') \frac{\delta}{\delta \xi^p(z'')}, \end{aligned} \quad (3.4)$$

as may be verified by observing that with its use the chain rules

$$\delta_\xi = \int d^2z \delta_\xi g^{rs} \frac{\delta}{\delta g^{rs}}, \quad \delta_\varrho = \int d^2z \delta_\varrho g^{rs} \frac{\delta}{\delta g^{rs}}$$

and

$$\partial_A = \int d^2z \partial_A g^{rs} \frac{\delta}{\delta g^{rs}}$$

reduce to the identities in Eq. (2.8); η^{AB} is the inverse of η_{AB} .

We have already seen that $\delta\phi/\delta\phi$ vanishes [Eq. (2.10)], which is why the ψ_A and ϕ_A cannot be identified. To calculate $\delta\phi/\delta g$ we also need $\delta\phi/\delta\xi$; $\delta_{\xi}g$ consists of two parts, the first resulting from a diffeomorphism

$$g_{\mu\nu} \nabla_\mu \delta\xi^\nu, \quad (3.5)$$

the second being a Weyl scaling which cancels the scaling caused by the diffeomorphism

$$- g_{\mu\nu} \nabla_\mu \delta\xi^\nu. \quad (3.6)$$

Since ϕ_{AB} is a symmetric tensor invariant under Weyl transformations, the change induced by $\delta_{\xi}g$ should be simply that of the diffeomorphism (3.5)

$$\delta_{\xi}\phi_{AB} = \delta\xi^\mu \nabla_\mu \phi_{AB} + \phi_{A\mu} \nabla_B \delta\xi^\mu. \quad (3.7)$$

This may be checked by observing that it satisfies the equations obtained by differentiating with respect to ξ the defining conditions on ϕ [Eq. (2.6)]:

$$\begin{aligned} 0 &= \delta_{\xi}(\nabla^r \phi_{rs}) = \\ &= \nabla^r(\delta_{\xi}\phi_{rs}) - \nabla_r(P^{rr}(\delta\xi)\phi_{rs}) - \frac{1}{2}(\nabla_s P^{sm}(\delta\xi))\phi_{rm}, \end{aligned} \quad (3.8)$$

$$0 = \delta_{\xi}(g^{rs}\phi_{rs}) = -P^{rs}(\delta\xi)\phi_{rs} + g^{rs}\delta_{\xi}\phi_{rs}.$$

Comparing $\delta_{\xi}\phi$ in Eq. (3.7) with $\delta_{\xi}\psi$ in Eq. (2.9) shows again that ϕ and ψ cannot be identified.

4. NILPOTENT BRST TRANSFORMATION

The Faddeev-Popov action

$$S = \int d^4x \sqrt{g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \chi^\mu \partial_\nu \chi^\mu + \frac{1}{2} \lambda_0 + b_{ab} \nabla^a c^b \right) \quad (4.1)$$

has a BRST symmetry corresponding to diffeomorphism invariance and parametrized by a Grassmann variable λ :

$$\begin{aligned}
\delta_\lambda X^\mu &= -\lambda c^a \partial_a X^\mu, & \delta_\lambda c^a &= -\lambda c^b \partial_b c^a, \\
\delta_\lambda b_{rs} &= 2\lambda \frac{1}{\sqrt{g}} \frac{\delta S}{\delta g^{rs}} = 2\lambda (T_{rs}(X) + T_{rs}^{gh}(b, c)), \\
2T_{ab}(X) &= \partial_a X^\mu \partial_b X^\mu - \frac{1}{2} g_{ab} g^{\mu\nu} \partial_\mu X^\mu \partial_\nu X^\mu, \\
2T_{rs}^{gh}(b, c) &= b_{p(r} \nabla_{s)} c^p + (\nabla_p b_{rs}) c^p - g_{rs} b_{pq} \nabla^p c^q.
\end{aligned} \tag{4.2}$$

If the integration measure in $Z^\xi[\xi, \beta, \gamma]$ respects this symmetry then, because the source term S_J does not, we are led to expect a Ward identity resulting from a shift in integration variables corresponding to Eq. (4.2):

$$\begin{aligned}
\Omega_{W.I.} Z^\xi[\xi, \beta, \gamma] &= 0, \\
\Omega_{W.I.} &= \int d^2x \sqrt{g} \left(\xi^\mu \hat{c}^a \partial_a \hat{X}^\mu - \gamma_a \hat{c}^b \partial_b \hat{c}^a \right. \\
&\quad \left. + 2\beta^{rs} (T_{rs}(\hat{X}) + T_{rs}^{gh}(\hat{b}, \hat{c})) \right) \\
\hat{X}^\mu &\equiv \frac{1}{\sqrt{g}} \frac{\delta}{\delta \xi^\mu}, \quad \hat{c}^a \equiv \frac{1}{\sqrt{g}} \frac{\delta}{\delta \gamma^a}, \quad \hat{b}_{rs} = \frac{1}{\sqrt{g}} \frac{\delta}{\delta \beta^{rs}}.
\end{aligned} \tag{4.4}$$

The zero modes spoil this identity. In Ref. [11] it was shown that instead of Eq. (4.3),

$$\left(\Omega_{W.I.} - \int d^2x \sqrt{g} \beta^{rs} \eta^{AB} F_{AB rs} \right) Z^\xi[\xi, \beta, \gamma] = 0. \tag{4.5}$$

This may be interpreted as the addition of a c-number to the energy momentum tensor in $\delta_\lambda b$ of Eq. (4.2). This would be an acceptable modification because it is only the transformation of the ‘geometrical ghost’ c^d that is fixed by cohomology and not that of the antighost b_{mn} [12].

With or without this modification the transformation (4.2) is only nilpotent on shell^{*)}, and so is inappropriate to our attempt to generalize the physical state condition (1.4). Applying (4.2) twice to the antighost b gives

$$\delta_{\lambda_2} \delta_{\lambda_1} b = 2\lambda_1 \lambda_2 P(c) \mathcal{L} \tag{4.6}$$

^{*)} I am indebted to Professor R. Stora for pointing this out.

where $\mathcal{L}$ is the Lagrangian density of S. The equation of motion for c derived from S is just the vanishing of $P(c)$, so (4.6) is zero on shell, which is not good enough in a functional formalism where all configurations of c must be integrated over. This failure of nilpotency is due to our having integrated out the auxiliary multiplier field B (the Nakanishi-Lautrup field) that imposes the gauge condition. Loosely speaking (i.e. neglecting the Weyl degree of freedom), a full BRST quantization procedure would start with an action that included the gauge-fixing term,

$$\int d^2x \sqrt{g} B_{rs} (\tilde{g}^{rs} - \hat{g}^{rs}) , \quad (4.7)$$

where $\tilde{g}$ is the dynamical variable to be integrated over, and $\hat{g}$ is our fiducial metric. The BRST transformation of b and B would then be

$$\delta_\lambda b = 2\lambda B , \quad \delta_\lambda B = 0 , \quad (4.8)$$

which ensures nilpotency. Integrating out the field B would give a delta-functional for $\tilde{g}$, facilitating the integration which results in (2.13).

Although the field B does not appear in our formalism it is clear that the fiducial metric itself acts as a source term for its traceless part, which is the only piece that interests us since b is traceless. This implies that instead of $\Omega_{w.l.}$ we should consider

$$\Omega = \int d^2x \sqrt{g} \left(\xi^\mu \hat{c}^a \partial_a \hat{X}^\mu - \gamma_a \hat{c}^b \partial_b \hat{c}^a - 2\beta^{rs} \frac{1}{\sqrt{g}} \frac{\delta}{\delta g^{rs}} \right) . \quad (4.9)$$

This modification is also suggested by the first form of δb in (4.2). Any observable quantity in the theory is represented by the action of a functional of $\hat{g}$, $\hat{X}$, $\hat{b}$, and $\hat{c}$ on $Z^{\hat{g}}$. We now *define* the BRST transformation of an arbitrary operator V as

$$\delta_\lambda V(g, \hat{X}, \hat{b}, \hat{c}) = [\lambda \Omega, V(g, \hat{X}, \hat{b}, \hat{c})] . \quad (4.10)$$

Nilpotency holds as may be seen by iterating the transformations

$$\begin{aligned} \delta_\lambda \hat{X}^\mu &= -\lambda \hat{c}^a \partial_a \hat{X}^\mu , & \delta_\lambda \hat{c}^a &= -\lambda \hat{c}^b \partial_b \hat{c}^a \\ \delta_\lambda \hat{b}_{rs} &= -2\lambda \frac{1}{\sqrt{g}} \frac{\delta}{\delta g^{rs}} \equiv 2\lambda \hat{B}_{rs} , & \delta_\lambda \hat{B}_{rs} &= 0 . \end{aligned} \quad (4.11)$$

Furthermore, the commutators of $\lambda\Omega$ with the sources

$$\begin{aligned} [\lambda\Omega, \xi^\mu] &= -\lambda \partial_a (\xi^\mu \hat{e}^a) , \quad [\lambda\Omega, \beta] = 0 , \\ [\lambda\Omega, \gamma_a] &= \lambda \partial_b (\gamma_a \hat{e}^b) + \lambda \gamma_b \partial_a \hat{e}^b , \\ [\lambda\Omega, g^{rs}] &= -2\lambda \beta^{rs} , \end{aligned} \quad (4.12)$$

all commute with $\lambda_2\Omega$, so that the BRST transformation is nilpotent even if V has explicit dependence on the sources. The price we have had to pay is that now the fiducial metric is not invariant. Note that the nilpotency of the transformation does not depend on the dimension D of space-time.

Having defined a nilpotent transformation we need to show that it is a symmetry of the quantum theory. In the remainder of this section we will show that Ω does indeed annihilate $Z^g[\xi, \beta, \gamma]$.

Introducing a Green function for the Laplacian

$$\Delta H(z, \bar{z}) = \frac{1}{\sqrt{g}} \delta(z - \bar{z}) \quad (4.13)$$

enables $Z^g[\xi, \beta, \gamma]$ to be evaluated as

$$\begin{aligned} Z^g[\xi, \beta, \gamma] &= \prod_A \beta_A \cdot (\det \eta)^{-1/2} \cdot e^{-S_0 + W} \\ W[\xi, \beta, \gamma] &= \\ &\int d^2z d^2\bar{z} \sqrt{g(z)g(\bar{z})} \left(\frac{1}{2} \xi^\mu(z) H(z, \bar{z}) \xi_\mu(\bar{z}) + \beta^{ab}(z) N_{ab}^p(z, \bar{z}) \gamma_p(\bar{z}) \right). \end{aligned} \quad (4.14)$$

We can now calculate the action of Ω on $Z^g[\xi, \beta, \gamma]$ directly. Let $\tilde{\Omega}$ denote that part of Ω that does not contain the trace of β :

$$\tilde{\Omega} = \Omega - \int d^2z \beta^{rs} g_{rs} g^{mn} \frac{\delta}{\delta g^{mn}} \quad (4.15)$$

then

$$\tilde{\Omega} Z^g[\xi, \beta, \gamma] = [\tilde{\Omega}, \prod_A \beta_A] e^{-S_0 + W} + \prod_A \beta_A \tilde{\Omega} e^{-S_0 + W} \quad (4.16)$$

Consider the second term first, 'Classical fields' may be defined as the action of the operators $\hat{X}$, $\hat{b}$, and $\hat{c}$ on W :

$$\begin{aligned}\tilde{X} &= \hat{X} W = \int d^2 z' \sqrt{g(z')} H(z, z') \xi(z') , \\ \tilde{b}_{rs} &= \hat{b}_{rs} W = \int d^2 z' \sqrt{g(z')} N_{rs}^p(z, z') \gamma_p(z') , \\ \tilde{c}^p &= \hat{c}^p W = - \int d^2 z' \sqrt{g(z')} \beta^{rs}(z') N_{rs}^p(z', z) .\end{aligned}\quad (4.17)$$

These relations may be inverted

$$\xi = \Delta \tilde{X} , \quad 2\gamma = P^+(\tilde{b}) , \quad 2\beta_{\perp} = -P(\tilde{c}) , \quad (4.18)$$

from which $W[\xi, \beta, \gamma]$ may be seen to equal the action evaluated for the classical fields $S[\tilde{X}, \tilde{b}, \tilde{c}]$, as is usual. In terms of these variables

$$\begin{aligned}\Pi_A \beta_A \cdot \tilde{\Omega} \left(e^{-S_0 + W} / \sqrt{|\eta|} \right) = \\ \Pi_A \beta_A \int d^2 z \sqrt{g} \left\{ \tilde{c}^a \partial_a \tilde{X}^r \Delta \tilde{X}^r - \frac{1}{2} \tilde{c}^b \partial_b \tilde{c}^a P_a^+(\tilde{b}) \right. \\ \left. - 2 \tilde{\beta}^{rs} \frac{1}{\sqrt{g}} \frac{\delta W}{\delta g^{rs}} + 2 \tilde{\beta}^{rs} \frac{1}{\sqrt{g}} \frac{\delta (S_0 + \frac{1}{2} \ln |\eta|)}{\delta g^{rs}} \right\} e^{-S_0 + W} / \sqrt{|\eta|}, \\ |\eta| \equiv \det \eta_{AB}\end{aligned}\quad (4.19)$$

where $\tilde{\beta}$ is the traceless part of β . The only divergences are contained in the last term. If these are regularized by some suitable scheme such as the zeta function method, then $S_0 + \frac{1}{2} \ln \det \eta$ is a finite function of the y^A alone and

$$\int d^2 z \tilde{\beta}^{rs} \frac{\delta (S_0 - \frac{1}{2} \ln |\eta|)}{\delta g^{rs}} = - \eta^{AB} \beta_A \partial_B (S_0 - \frac{1}{2} \ln |\eta|) \quad (4.20)$$

which cannot contribute to Eq. (4.19) because of the $\Pi \beta_A$ standing in front. Similarly $\tilde{\beta}$ may be replaced by $\beta_{\perp}$ in the previous term of (4.19). Now

$$-\frac{1}{\sqrt{g}} \left(\frac{\delta W}{\delta g^{rs}} - \frac{1}{2} g_{rs} g^{mn} \frac{\delta W}{\delta g^{mn}} \right) = T_{rs}(\tilde{X}) + T_{rs}^{gh}(\tilde{b}, \tilde{c}) , \quad (4.21)$$

so the remaining terms in the integral in (4.19) are

$$\int d^2x \sqrt{g} \left(\tilde{c}^a \partial_b \tilde{X}^a \Delta \tilde{X}^b - \frac{1}{2} \tilde{c}^b \partial_b \tilde{c}^a P_a^+ (\tilde{b}) \right. \\ \left. - P^{rs}(\tilde{c}) \left(T_{rs}(\tilde{X}) + T_{rs}^{gh}(\tilde{b}, \tilde{c}) \right) \right), \quad (4.22)$$

which vanishes because it is the change in the action $S(\tilde{X}, \tilde{b}, \tilde{c})$ under the transformation (4.2).

We now show that the commutator in (4.16),

$$[\tilde{\Omega}, \pi_A \beta_A], \quad (4.23)$$

vanishes when the trace of β is set to zero. The only part of $\tilde{\Omega}$ that contributes is the term containing β . To simplify it, it is convenient to write β as

$$\beta^{rs} = \beta_A \eta^{AB} \psi_B^{rs} + V^{rs}. \quad (4.24)$$

Integrating both sides against ϕ_{Ars} shows that V is orthogonal to $\ker P^\dagger$ and so may be written as

$$V^{rs} = -\frac{1}{2} P^{rs}(k)$$

for some k^P . The decomposition (2.19) together with (4.18) implies

$$P^{rs}(k-\tilde{c}) = 2\beta_A (\eta^{AB} \psi_B^{rs} - \phi^{Ars}) \quad (4.25)$$

which determines k as

$$k^P = \tilde{c}^P + \beta_A \eta^{AB} \int d^2x' \sqrt{g(x')} \psi_B^{rs}(x') N_{rs}^P(x', x). \quad (4.26)$$

Using this and expression (3.4) for $\delta/\delta g$ gives

$$\int d^2x \left(2\beta^{rs} \frac{\delta}{\delta g^{rs}} - \beta^{rs} g_{rs} g^{mn} \frac{\delta}{\delta g^{mn}} \right) = \\ -2\beta_A \eta^{AB} \partial_B + \int d^2x k^P \frac{\delta}{\delta \xi^P} \quad (4.27)$$

which may be used to write formula (4.23) as

$$\sum_A \left(\pi_{B \neq A} \beta_B \right) \left\{ 2\beta_C \eta^{CH} \partial_H - \int d^2x k^P \frac{\delta}{\delta \xi^P}, \int d^2x \sqrt{g} \beta^{rs} \phi_{Ars} \right\} \Big|_{\beta^{rs} g_{rs} = 0} \quad (4.28)$$

The product of β_B ensures that the only part of $\partial_H \phi_A$ contributing to the first term in the commutator comes from F_{HA} in Eq. (3.1) once the trace of β is set to zero. The second term in the commutator shifts ϕ_A through $\delta \xi = k$, so that formula (4.28) becomes

$$\sum_A \left\{ \prod_B \beta_B (-)^A \int d^4x \sqrt{g} \beta^{rs} \eta^{AH} F_{HAr s} - \prod_{B \neq A} \beta_B \int d^4x \sqrt{g} \beta^{rs} \delta_k \phi_{Ar s} \right\} \quad (4.29)$$

β integrated against F in the first term may be written in terms of $\tilde{\epsilon}$ using (4.17), but since k and $\tilde{\epsilon}$ differ only by something linear in β_B which is annihilated by $\Pi \beta_A$, k may be used instead to write the first term as

$$\sum_A \prod_B \beta_B (-)^A \int d^4x \sqrt{g} k^p \eta^{AH} (-\psi_H^{mn} \nabla_p \phi_{Amn} + \nabla_m (\psi_H^{nr} \phi_{Arp})). \quad (4.30)$$

Inserting the decomposition (4.24) and the expression (3.7) for $\delta_\xi \phi$ into the second term of (4.29) gives

$$\begin{aligned} & \sum_A \prod_B \beta_B (-)^A \int d^4x \sqrt{g} \eta^{AH} \psi_H^{rs} (k^p \nabla_p \phi_{Ar s} + \phi_{Ar p} (\nabla_s) k^p) \\ & - \frac{1}{2} \sum_A \prod_{B \neq A} \beta_B \int d^4x \sqrt{g} P^{rs}(k) (k^p \nabla_p \phi_{Ar s} + \phi_{Ar p} (\nabla_s) k^p). \end{aligned} \quad (4.31)$$

The first term cancels with (4.30) and the second term is separately zero, as a little algebra will show.

Finally we see that $\tilde{\Omega}$ annihilates Z when the trace of β vanishes. Since $\tilde{\Omega}$ and Ω differ only by a piece proportional to the trace of β ,

$$\Omega Z^g \Big|_{\beta^{rs} g_{rs} = 0} = 0, \quad (4.32)$$

which expresses the BRST invariance of the generating functional. That Ω annihilates Z only when the trace of β vanishes will not be a problem because it is not the source of any dynamical field. The source of b is only the traceless part of β .

5. EXPECTATION VALUES AND THE DECOUPLING OF UNPHYSICAL OPERATORS

In the previous section we said that any observable quantity in the theory is represented by the action of a functional of $\hat{g}$, $\hat{X}$, $\hat{b}$, and $\hat{c}$ on $Z^{\hat{g}}$. We need to make this more precise. The expectation

value of a function of X and g in the original Polyakov theory, unadorned by ghosts, may be defined as

$$\langle F(X, g) \rangle \equiv \sum_h \alpha^h \langle F \rangle_h / \sum_h \alpha^h Z_h$$

$$\langle F \rangle_h = \int_{m_h \times E_h} \mathcal{D}X \mathcal{D}g \frac{1}{V} [\exp - S(X)] F(X, g), \quad (5.1)$$

where α is a coupling constant. For consistency F should be required not to introduce any dependence on the Weyl degree of freedom of g . From $F(X, g)$ we can make an operator $F(\hat{X}, \hat{g})$, whose action on the generating functional would be to produce $\langle F \rangle_h$ once the parameters y^A were integrated over and the sources set to zero, were it not for the $\Pi\beta_A$ in (4.14). To eliminate the $\Pi\beta_A$ we should insert an additional operator. The choice of the operator is not unique; we could, for example, use as in Eq. (2.26).

$$\prod_A \int d^2x \sqrt{g} \phi^{rs} \hat{b}_{rs}. \quad (5.2)$$

We simply require a prescription that reduces to Polyakov's for expectation values of functions of X and g . However, since Weyl invariance is respected by the dynamics of the theory it is inelegant to break it by an artefact like (5.2) if it can be avoided. Instead we will use

$$\prod_A \int d^2x \sqrt{g} \psi_A^{rs} \hat{b}_{rs} \quad (5.3)$$

which is Weyl-invariant if we take $\hat{b}_{rs}$ to be. Furthermore it is this operator and not (5.2) that will be useful in understanding the physical-state condition. Now

$$\left\{ \prod_A \int d^2x \sqrt{g} \psi_A^{rs} \hat{b}_{rs} \right\} \Pi\beta_B = \det \eta \quad (5.4)$$

which accounts for our prejudice in choosing the powers of $\det \eta$ in Z^B . We now define the h -loop contribution to the expectation value of a general operator which may contain ghost fields as well as X and g , as

$$\langle F \rangle_h = \left(\int \left(\prod_A dy^A \int d^2x \sqrt{g} \psi_A^{rs} \hat{b}_{rs} \right) F[\hat{X}, \hat{g}, \hat{c}, \hat{g}] Z^{\hat{g}}[\xi, \beta, r] \right)_{\xi, \beta, r=0} \quad (5.5)$$

Since our BRST transformation involves the operator $\hat{B}$ for which $\hat{g}$ is a source, we need to define expectation values of quantities containing $\hat{B}$. As $\hat{B}$ and $\hat{g}$ do not commute there is a potential ordering-ambiguity, which we resolve by adopting a normal-ordering prescription by standing $\hat{B}$ to the left of $\hat{g}$. The expectation value of an operator is then defined as in Eq. (5.5) but with the normal-ordered expression on the right-hand side.

If we speculate on the existence of an underlying Hilbert space formulation of the first or second quantized theory based on BRST invariance, then $\langle F \rangle_h$ would be the h-loop contribution to the vacuum expectation value of an operator $\mathcal{F}$ corresponding to F :

$$\langle 0 | \mathcal{F} | 0 \rangle . \quad (5.6)$$

BRST transformations would be implemented on the Hilbert space by a nilpotent operator Q which annihilates the vacuum

$$Q^2 = 0 \quad , \quad Q | 0 \rangle = 0 . \quad (5.7)$$

Physical states would be defined as the cohomology of Q , i.e. they would be annihilated by Q and identified if they differ by Q acting on another state. Thus physical states are created from the vacuum by operators commuting with λQ :

$$| \text{phys} \rangle = \Phi_{\text{phys}} | 0 \rangle \quad , \quad [\lambda Q, \Phi_{\text{phys}}] = 0 . \quad (5.8)$$

The BRST invariance of $|0\rangle$ is equivalent to the condition that

$$\langle 0 | [\lambda Q, \Phi'] | 0 \rangle = 0 \quad (5.9)$$

for arbitrary Φ' . The commutator is just the BRST transformation of Φ' , so we can translate this Hilbert space statement into the language of Polyakov's model by requiring that $\langle \delta_\lambda V \rangle_h$ vanish for an arbitrary operator V .

Equation (5.9) is the key condition in the BRST quantization procedure. It also ensures the decoupling from physical processes of spurious states, i.e. states created by operators of the form $\delta_\lambda \mathcal{U}$ since

$$\begin{aligned} \langle \text{phys}' | \delta_\lambda \mathcal{U} | \text{phys} \rangle &= \\ \langle 0 | \Phi'_{\text{phys}} [\lambda Q, \mathcal{U}] \Phi_{\text{phys}} | 0 \rangle &= \\ \langle 0 | [\lambda Q, \Phi'_{\text{phys}} \mathcal{U} \Phi_{\text{phys}}] | 0 \rangle . \end{aligned} \quad (5.10)$$

We previously defined the BRST transform of an arbitrary operator in the Polyakov model to be its commutator with $\lambda\Omega$, so

$$\langle \delta_\lambda V \rangle_h = \left(\int \prod_A \left(dy^A \int d^2z \sqrt{g} \psi_A^{rs} \hat{b}_{rs} \right) [\lambda\Omega, V(\hat{x}, \hat{b}, \hat{c}, \hat{g})] z^{\hat{g}}[\xi, \beta, \gamma] \right)_{\xi=\beta=\gamma=0} \quad (5.11)$$

is required to vanish for arbitrary V to ensure the success of the BRST quantization procedure for the putative Hilbert space approach. Note the equivalent roles of Ω and Q . Ω annihilates $Z^{\hat{g}}$ when the trace of β vanishes [Eq. (4.32)] and neither V nor $\Pi\hat{b}$ contain derivatives with respect to the trace of β , so the second term in the commutator vanishes when the sources are set to zero. This allows us to rearrange Eq. (5.8) as

$$\langle \delta_\lambda V \rangle_h = \left(\int \prod_A dy^A \cdot \left[\prod_B \hat{b}_{\psi_B}, \lambda\Omega \right] V z^{\hat{g}} \right)_{\xi=\beta=\gamma=0}, \quad (5.12)$$

$$\hat{b}_{\psi_B} \equiv \int d^2z \sqrt{g} \psi_B^{rs} \hat{b}_{rs},$$

since the second term in this commutator has sources standing to the left of all derivatives and so vanishes when the sources are set to zero.

The commutator may be written as a sum of terms involving the product of all but one of the $\hat{b}_{\psi_B}$ multiplied by the anticommutator of Ω with the remaining $\hat{b}_{\psi_A}$, which may be evaluated as

$$\begin{aligned} \{ \hat{b}_{\psi_A}, \Omega \} &= \\ \left\{ \int d^2z \psi_A^{rs} \frac{\delta}{\delta \beta^{rs}}, - \int d^2z' 2\beta^{mn} \frac{\delta}{\delta g^{mn}} \right\} &= \\ - 2 \int d^2z \psi_A^{rs} \frac{\delta}{\delta g^{rs}} & \\ - 2 \int d^2z d^2z' \beta^{mn}(z) \frac{\delta \psi_A^{rs}(z')}{\delta g^{mn}(z)} \cdot \frac{\delta}{\delta \beta^{rs}(z')} &. \end{aligned} \quad (5.13)$$

The first term is just $2\partial_A$. The whole of this anticommutator commutes with the remaining $\hat{b}_{\psi_B}$ and so may be pushed to their left because

$$\begin{aligned}
& \left[\{ \hat{b}_{\psi_A}, \Omega \}, \int d^2z \psi_B^{rs} \frac{\delta}{\delta \beta^{rs}} \right] = \\
& 2 \int d^2z \partial_A \psi_B^{rs} \frac{\delta}{\delta \beta^{rs}} + 2 \int d^2z d^2z' \psi_B^{rs}(z) \frac{\delta \psi_A^{rs'}(z')}{\delta g^{rs'}(z)} \frac{\delta}{\delta \beta^{rs'}(z')} = \quad (5.14) \\
& 2 \int d^2z \left(\partial_A \psi_B^{rs} - \partial_B \psi_A^{rs} \right) \frac{\delta}{\delta \beta^{rs}},
\end{aligned}$$

which vanishes owing to the last integrability condition in Eq. (2.9). So the commutator of Eq. (5.12) is

$$\begin{aligned}
& \left[\pi_A \hat{b}_{\psi_A}, \lambda \Omega \right] = \\
& \sum_A (-)^A 2\lambda \left(\partial_A + \int d^2z d^2z' \beta^{mn}(z) \frac{\delta \psi_A^{mn'}(z')}{\delta g^{mn'}(z)} \frac{\delta}{\delta \beta^{mn'}(z')} \right) \pi_{B \neq A} \hat{b}_{\psi_B}. \quad (5.15)
\end{aligned}$$

The second term in the bracket vanishes when the sources are set to zero. The remaining ∂_A acts on everything to the right, including any function multiplying the commutator, so that the expectation value of $\delta_\lambda V$ is

$$\langle \delta_\lambda V \rangle_\lambda = \lambda \left(\int \pi_A d\gamma^A \cdot \sum_B \partial_B \left\{ (-)^B 2 \pi_{C \neq B} \hat{b}_{\psi_C} V z^{\hat{B}} \right\} \right) \quad (5.16)$$

$\xi, \beta, \gamma = 0$

Following Friedan et al. [5] we could argue that the integrand is a total divergence so that the integral may be transformed to one over the boundary [note that they use the transformation (4.2), whereas for this argument to work it is essential to use the nilpotent transformation generated by Ω Eq. (4.12)]. The boundary of Teichmüller space has been studied by mathematicians [8]. It contains points corresponding to manifolds with nodes; that is, degenerate configurations corresponding to emission of tachyons and tadpoles obtained by shrinking a geodesic on the surface to zero and thus changing the topology. These configurations are responsible for the divergences of the functional determinants in the integrand of Eq. (5.16) so that the integrand itself diverges on the boundary [13]. We are thus unable to argue that $\langle \delta_\lambda V \rangle$ vanishes for arbitrary V in the absence of a scheme for regularizing the divergences in the theory. The vanishing of $\langle \delta_\lambda V \rangle$ should be a central requirement of such a scheme.

This difficulty is not some peculiarity of our approach to the BRST transformation but has its origin in the implementation of the Virasoro conditions in the Polyakov model unadorned with

ghosts. In the usual approach to the dual string [14] the expectation value of the energy momentum tensor is required to vanish for physical states. The h-loop contribution to the expectation value of a moment of the energy-momentum tensor in Polyakov's model is

$$\begin{aligned} & \int \mathcal{D}X \mathcal{D}g \frac{1}{\sqrt{g}} \int d^2z \sqrt{g} u^{ab}(z) T_{ab}(X(z)) e^{-S(X)} \\ &= - \int \mathcal{D}X \mathcal{D}g \frac{1}{\sqrt{g}} \int d^2z u^{ab}(z) \frac{\delta}{\delta g^{ab}} e^{-S(X)} \end{aligned} \quad (5.17)$$

which is again the integral of a total derivative and would vanish if we could neglect the contribution of the boundary. Since we cannot, Eq. (5.17) is non-zero and the Virasoro gauge conditions do not hold for the unregulated theory.

6. PHYSICAL VERTICES

In Polyakov's model without ghosts, scattering amplitudes are expectation values of reparametrization invariant functionals of X and g . For consistency these should not introduce any dependence on the Weyl degree of freedom of g into the functional integral. The simplest example is the tachyon vertex

$$\int_{\Sigma} d^2z \sqrt{g} e^{ik^\mu X_\mu(z)}. \quad (6.1)$$

This is manifestly reparametrization-invariant but appears to depend on the magnitude of $\det g$. It is a singular operator in the sense that its expectation value involves the value of the Green function $H(z, z')$ in (4.13) taken at two coincident points. As Polyakov pointed out in his original paper [4], the appropriately regularized value of $H(z, z)$ introduces a dependence on the Weyl degree of freedom. This cancels the dependence on $\sqrt{g}$ in formula (6.1) provided $k^2 = 8\pi$. This is the functional expression of the key property of the vertex operators of the dual resonance model.

We now show that for any functional $F[X, g]$ that has an expectation value in Polyakov's original model without ghosts and that is both reparametrization-invariant and Weyl-invariant, the corresponding operator $F[\hat{X}, \hat{g}]$ is BRST-invariant. It is thus unnecessary to introduce ghost coordinates into the vertex operators of the original model to make them BRST-invariant. As with Weyl invariance, the BRST invariance of formula (6.1) is not manifest. To be more precise, since we are dealing with singular operators that stand in need of regularization (which is accomplished by allowing them to act on a regularized generating functional) we will show that

$$[\lambda \Omega, F[\hat{X}, \hat{g}]] \mathcal{Z}^g[\xi, \rho, \gamma] = 0. \quad (6.2)$$

This is adequate for our purposes because our normal ordering prescription allows us to move any operators independent of $\hat{B}$ to be adjacent to Z^g in the definition of the expectation value (5.5).

Using the definition of Ω , the right-hand side of (6.2) is

$$- \lambda \int d^2x \sqrt{g} \left(\hat{c}^a (\partial_a \hat{x}^\mu) F^\mu[\hat{x}, g] + 2 \beta^{rs} \frac{1}{\sqrt{g}} \frac{\delta F[\hat{x}, g]}{\delta g^{rs}} \right) Z^g, \quad (6.3)$$

$$F^\mu[x, g] \equiv \frac{1}{\sqrt{g}} \frac{\delta F[x, g]}{\delta x^\mu}.$$

This is not obviously zero; to see that it is, we must exploit the Weyl and reparametrization invariance of FZ^g . Under a reparametrization the sources transform as

$$\begin{aligned} \delta_v g_{ab} &= \nabla_{(a} \delta v_{b)} \quad , \quad \delta_v \xi^\mu = \delta v^a \partial_a \xi^\mu \quad , \quad \delta_v g^{ab} = - \nabla^{(a} \delta v^{b)} \\ \delta_v \gamma_a &= (\nabla_a \delta v^b) \gamma_b + \delta v^b \nabla_b \gamma_a \quad , \\ \delta_v \beta^{rs} &= - \beta^{p(r} \nabla_p \delta v^{s)} + \delta v^b \nabla_b \beta^{rs} \quad . \end{aligned} \quad (6.4)$$

We can express the invariance of Z^g as

$$\begin{aligned} \delta_v Z^g[\xi, \beta, \gamma] &= \\ \int d^2x \left\{ \delta_v g^{ab} \frac{\delta}{\delta g^{ab}} + \delta_v \xi^\mu \frac{\delta}{\delta \xi^\mu} + \delta_v \beta^{rs} \frac{\delta}{\delta \beta^{rs}} + \delta_v \gamma_a \frac{\delta}{\delta \gamma_a} \right\} Z^g &= 0. \end{aligned} \quad (6.5)$$

In the definition of Z^g [Eq. (2.25)], S and $\det \eta$ are Weyl invariant; only the source term S_J breaks this invariance. When the functional integration has been done to arrive at (4.14) the combination of determinants is Weyl-invariant for $D = 26$ and only W inherits the breaking of the symmetry from S_J . However, S_J is invariant if the source terms are scaled so as to compensate for the scaling of g , i.e.

$$\begin{aligned} \delta_\lambda g_{ab} &= g_{ab} \delta \lambda \quad , \quad \delta_\lambda \xi^\mu = - \xi^\mu \delta \lambda \quad , \quad \delta_\lambda g^{ab} = - g^{ab} \delta \lambda \\ \delta_\lambda \gamma_a &= - \gamma_a \delta \lambda \quad , \quad \delta_\lambda \beta^{rs} = - \beta^{rs} \delta \lambda \quad . \end{aligned} \quad (6.6)$$

Z^ξ is invariant under this operation, so that

$$\int d^2x \delta\lambda \left\{ g^{ab} \frac{\delta}{\delta g^{ab}} + \xi^\mu \frac{\delta}{\delta \xi^\mu} + \beta^{\mu\nu} \frac{\delta}{\delta \beta^{\mu\nu}} + \gamma_a \frac{\delta}{\delta \gamma_a} \right\} Z^g[\xi, \beta, \gamma] = 0. \quad (6.7)$$

Adding these two Ward identities with the choice $\delta\lambda = \nabla_a \delta V^a$ gives

$$\Gamma[\delta V] Z^g[\xi, \beta, \gamma] = 0,$$

where

$$\begin{aligned} \Gamma[\delta V] = & \int d^2x \left\{ -P^{ab}(\delta V) \frac{\delta}{\delta g^{ab}} + (\delta_V \xi^\mu + (\nabla_a \delta V^a) \xi^\mu) \frac{\delta}{\delta \xi^\mu} \right. \\ & + (\delta_V \beta^{\mu\nu} + (\nabla_a \delta V^a) \beta^{\mu\nu}) \frac{\delta}{\delta \beta^{\mu\nu}} \\ & \left. + (\delta_V \gamma_a + (\nabla_\mu \delta V^\mu) \gamma_a) \frac{\delta}{\delta \gamma_a} \right\}. \end{aligned} \quad (6.8)$$

Since the expectation value of $F[X, g]$ is also Weyl- and reparametrization-invariant we must have, in addition,

$$\Gamma[\delta V] F[\hat{X}, g] Z^g[\xi, \beta, \gamma] = 0, \quad (6.9)$$

which together with Eq. (6.8) implies that the commutator of Γ and F annihilates Z^ξ ; F contains no ghost variables, so this is just

$$\begin{aligned} [\Gamma[\delta V], F[\hat{X}, g]] Z^g[\xi, \beta, \gamma] = & \int d^2x \sqrt{g} \left\{ -P^{ab}(\delta V) \frac{1}{\sqrt{g}} \frac{\delta F[\hat{X}, g]}{\delta g^{ab}} - \delta V^a (\partial_a F^\mu[\hat{X}, g]) \hat{X}^\mu \right. \\ & \left. - (\nabla_a \delta V^a) F^\mu[\hat{X}, g] \hat{X}^\mu \right\} Z^g[\xi, \beta, \gamma] = 0 \end{aligned} \quad (6.10)$$

In the expression (6.3) the operator $\hat{c}$ may be replaced by the classical field $\tilde{c}$ in terms of which β may also be expressed using (4.18). Only $\beta_\perp$ contributes since the β_A are annihilated by the $\Pi\beta_A$ in Z^ξ , so formula (6.3) becomes

$$\int d^2x \sqrt{g} \left(-\lambda \tilde{c}^a (\partial_a \hat{X}^\mu) F^\mu[\hat{X}, g] + P^{ab}(\lambda \tilde{c}) \frac{1}{\sqrt{g}} \frac{\delta F[\hat{X}, g]}{\delta g^{ab}} \right) Z^g. \quad (6.11)$$

Taking $\delta V = \lambda \tilde{c}$ in Eq. (6.10) and adding it to this cancels the terms in $\delta F / \delta g$. The remainder is a total divergence which gives zero when integrated over Σ_h , and we conclude that formula (6.3) vanishes.

Operators that are reparametrization- and Weyl-invariant in Polyakov's original formulation without ghosts are thus automatically BRST-invariant.

7. CONCLUSION

We can summarize our results as follows:

- 1) We defined the h-loop contribution to the expectation value of a quantity F as an integral over Teichmüller parameters

$$\langle F \rangle_h = \left(\int \left(\prod_{A=1}^{6h-6} dy^A \hat{b}_{\psi A} \right) F[\hat{X}, \hat{b}, \hat{c}, \hat{g}(y^A)] \mathcal{Z}^g[\zeta, \beta, \gamma] \right)_{\zeta, \beta, \gamma=0}$$

$\mathcal{Z}^g$ is a generating functional for the Faddeev-Popov representation of Polyakov's model when the diffeomorphism invariance is gauge-fixed by choosing the metric to be $\hat{g}(y^A)$. $\hat{X}$, $\hat{b}$, and $\hat{c}$ are functional derivative operators with respect to the sources ζ , β , and γ , respectively. When F is independent of ghost operators $\hat{b}$ and $\hat{c}$, the expectation value reduces to that of the original Polyakov theory

$$\langle F \rangle_h = \int_{\mathcal{M}_h \times \mathcal{E}_h} \mathcal{D}x \mathcal{D}g \frac{1}{\mathcal{V}} \left\{ \exp - \frac{1}{2} \int d^2x \sqrt{g} (g^{\mu\nu} \partial_\mu x^\alpha \partial_\nu x^\alpha + \lambda) \right\} F[x, g].$$

In the critical dimension of space-time this is well known to be independent of the Weyl degree of freedom of g for Weyl-invariant operators F.

- 2) The Faddeev-Popov action admits a BRST symmetry which is expressed in terms of the generating functional by the Ward identity

$$\Omega \mathcal{Z}^g[\zeta, \beta, \gamma] = 0$$

$$\Omega = \int d^2x \sqrt{g} \left(\zeta^\alpha \hat{c}^\alpha \partial_\alpha \hat{X}^\mu - \gamma_\alpha \hat{c}^\alpha \partial_\alpha \hat{c}^\alpha - 2\beta^{\mu\nu} \frac{1}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} \right)$$

when β is traceless.

- 3) If we use Ω to define the BRST transform of an operator as

$$\delta_\lambda F[\hat{X}, \hat{b}, \hat{c}, g] = [\lambda \Omega, F[\hat{X}, \hat{b}, \hat{c}, g]],$$

then the transformation is nilpotent,

$$\delta_{\lambda_1} \delta_{\lambda_2} F = 0$$

even if F also has explicit dependence on the sources.

4) Any Weyl and reparametrization-invariant operator $F[X, g]$ in the original Polyakov model is also BRST-invariant without the need to introduce dependence on the ghosts

$$[\lambda \Omega, F[\hat{x}, g]] Z^g[\xi, \beta, \gamma] = 0.$$

5) The expectation value of the BRST transform of an arbitrary operator may be transformed to a surface integral over the boundary of Teichmüller space because

$$\langle \delta_{\lambda} F \rangle_h = \left(2\lambda \sum_B \int (\pi_A dy^A) \partial_B \left\{ (-)^B \prod_{c \neq B} \hat{b}_{\psi_c} F[\hat{x}, \hat{b}, \hat{c}, \hat{g}] Z^g \right\} \right)_{\xi, \beta, \gamma = 0}$$

The boundary is not empty but contains those configurations of g responsible for the divergences of Z . So in general this expectation value does not vanish. This is not special to the BRST formulation but is a manifestation of the failure of the Virasoro conditions

$$\int \mathcal{D}x \mathcal{D}g \frac{1}{\mathcal{V}} \int T_{ab}(x) u^{ab} \sqrt{g} d^2x e^{-S(x)} = 0$$

for those components of $T = \delta S / \delta g$ corresponding to variations of g induced by varying the Teichmüller parameters y^A .

If we speculate on the existence of an underlying Hilbert space formulation of the interacting Polyakov string, then the expectation value $\langle F \rangle_h$ would be the h -loop contribution to the vacuum expectation value of some corresponding operator $\mathcal{F}$

$$\langle F \rangle_h = \langle 0 | \mathcal{F} | 0 \rangle_h$$

BRST transformations would be generated by an operator Q ,

$$\delta_{\lambda} \mathcal{F} = [\lambda Q, \mathcal{F}]$$

which must be nilpotent [at least in the critical dimension [7]] because of point (3):

$$Q^2 = 0$$

Since the expectation value of the BRST transform of an operator $\langle \delta_\lambda F \rangle_h$ would be

$$\langle 0 | [\lambda Q, \mathcal{F}] | 0 \rangle$$

we conclude that the physical-state condition for the vacuum

$$Q | 0 \rangle = 0$$

cannot hold in the unregulated theory. However, a prescription for regularizing and renormalizing the theory would tame the divergences of Z^k and may cancel the contribution of the boundary to $\langle \delta_\lambda F \rangle_h$; or, put more strongly, this cancellation might be made a requirement of such a scheme. Note that such divergences may be absent for certain susperstring models [15].

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