ANOMALY CANCELLATION AND SELF-DUAL LATTICES

A.N. Schellekens

CERN - Geneva

and

N.P. Warner*)

Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139

ABSTRACT

We give a simple proof that a heterotic string in 8m+2 dimensions whose bosonic sector has been compactified on any self dual lattice gives rise to a field theory in which the anomaly can be cancelled by the simplest Green-Schwarz mechanism. We obtain a generating function for the anomalies of these heterotic string theories. The fact that the anomalies do not cancel completely is related to an anomalous modular transformation property of the generating function.

*) Work supported in part by National Science Foundation Grant #84-07109 and by the U.S. Department of Energy (D.O.E.) under contract #DE-AC02-76ER03069.

CERN-TH. 4465/86
June 1986
In an earlier paper [1] we have shown how the fermionic formulation of the heterotic string [2] can be used to understand the relationship between Green-Schwarz anomaly cancellation [3] and one loop modular invariance. Rather surprisingly anomaly cancellation is possible in $(8m+2)$ dimensions, for any integer $m$, irrespective of conformal invariance of the corresponding string theory. The fermionic formulation appears to be limited to gauge groups that are products of $SO(2n)$. In ten dimensions this covers all known heterotic strings (the $E_8 \times E_8$ anomaly cancellation follows from that of its $SO(16) \times SO(16)$ subgroup). However, in other dimensions there are many other heterotic strings based on the larger even self dual lattices, whose corresponding gauge groups involve groups other than $SO(2n)$. This raises the question whether the massless sectors of these theories are anomaly free. Moreover, even in ten dimensions it is interesting to know what anomaly cancellation has to do with the self duality of the lattice.

In this paper we will consider a heterotic string in $8m+2$ dimensions, where the right moving sector is the Neveu-Schwarz-Ramond string, with G.S.O. projections, and the left moving sector is a purely bosonic string in $8m+2+\ell$ dimensions. The right moving sector is thus separately modular invariant. The extra $\ell$ dimensions of the left moving sector will be compactified as usual on an $\ell$-torus, $T^\ell$. This torus is obtained by dividing $R^\ell$ by some lattice, $\Lambda$. The lattice is generated by a set of basis vectors $\vec{e}_i \equiv (e_i^a)$, where $i = 1, \ldots, \ell$ indexes the basis and $a$ is a vector index in $R^\ell$:

$$\Lambda = \{ n_i \vec{e}_i : n_i \in Z \}.$$ 

The metric $A_{ij}$ on $\Lambda$ is defined by

$$A_{ij} = \vec{e}_i \cdot \vec{e}_j$$

(For a root lattice $\vec{e}_i$ are the simple roots and $A_{ij}$ is the Cartan matrix.)

By means of the Frenkel-Kac construction [4], the length two vectors on this lattice and the canonical momenta on $T^\ell$, can be used to construct the generators of some gauge group, $G$. All the states of the compactified theory lie in representations of this group. We will show that the $G$-anomalies and gravitational anomalies of the massless sector of this string theory can be cancelled.
by the Green Schwarz mechanism if \( \wedge \) is even and self dual. In any dimension other than ten, the gauge vector fields for \( G \) may not come from the string itself. However, this is irrelevant to the consideration of the anomalies of the massless sector.

In order to calculate the gauge anomaly of the string we will need the Chern character, \( \text{Ch}(F) \), of the gauge field for every level of the string. For the present we restrict our attention to the compactified, \( d \)-dimensional part of the left moving string. We will obtain a generating function \( P(q, F) \) whose expansion in \( q \) yields the Chern characters of every level. By conjugation inside the gauge group we can assume that \( F = -2\pi i y^a H_a \), where \( H_a \) are hermitian generators in the Cartan subalgebra, and \( y^a \) are 2-forms (c.f. [5]). The contribution to \( \text{Ch}(F) \) of a lattice vector \( \vec{b} = n_i \vec{e}_i \) on \( \wedge \) is simply

\[
\exp(y^a \beta_a) = \exp(2\pi i n_j x_j)
\]

where \( x_j = (\frac{1}{2\pi i} y^a e^a_j) \). The length squared of the vector \( \vec{b} \) is simply \( n_i A_{ij} n_j \). Define

\[
P(q, F) = q^{-\frac{d}{2}} \prod_{n=1}^{\infty} \left( 1 - q^{2n} \right)^{-n} \theta_\wedge(\omega^i | \tau)
\]

where \( q = e^{i\pi \tau} \) and

\[
\theta_\wedge(\omega^i | \tau) = \sum_{n_i \in \mathbb{Z}} q^{n_i A_{ij} n_j} e^{2\pi i n_i x_i}
\]

The purpose of the bosonic partition function in \( P(q, F) \) is to include the contributions from bosonic oscillator excitations from the compactified dimensions. Observe that \( \theta_\wedge(\omega | \tau) \) is the partition function of the lattice, \( \wedge \), and \( P(q, \xi) \) is the partition function of the compactified string. Note also that \( \theta_\wedge(\omega^i | \tau) \) is holomorphic for \( \text{Im}(\tau) > 0 \) [6].

The complete gauge and gravitational anomaly is the \( 8m + 4 \) form in the coefficient of \( q^0 \) in

\[
A(q, F, R) = \hat{A}(R) P(q, F) P_B(q, R)
\]

where \( \hat{A}(R) \) is the usual Dirac genus [5]. The partition function \( P_B \) is defined by [1]

\[
P_B(q, R) = q^{-\frac{d}{2}} \prod_{n=1}^{\infty} \left[ \text{det}(1 - q^{2n} e^{i\frac{R}{4}}) \right]^{-1}
\]
and it gives the contribution to the gravitational anomaly from the uncompactified part of the left moving string. In [1] it was shown how this may be rewritten in terms of Jacobi theta functions:

\[ P_B(q, R) = \lim_{n \to \infty} \frac{2 \sinh(\frac{i\pi}{2\pi}) \eta(\tau)}{\eta_1(\frac{2\pi n}{2\pi \tau})} \]

where \( \eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^{2n}) \) is the Dedekind \( \eta \)-function, and \( z_b \) are the 2-form skew eigenvalues of \( \frac{R}{2\pi} \).

Motivated by the results of [1] we study the modular properties of \( A(q, F, R) \). The modular group is generated by the transformations \( \tau \to \tau + 1 \) and \( \tau \to -\frac{1}{\tau} \). For invariance under \( \tau \to \tau + 1 \), \( A(q, F, R) \) must involve even powers of \( q \). This requires \( 8m + \ell = 24k \), and an even lattice.

To derive the transformation properties of \( \tilde{\theta}_\Lambda(z^i \mid \tau) \) under the map \( \tau \to -\frac{1}{\tau} \), consider the function

\[ \tilde{\theta}_\Lambda^*(z^i \mid \tau) = \sum_{n_i \in \mathbb{Z}} q^{(n_i + z_i) a_i^{-1} (n_i + z_i)} \]

This function is manifestly periodic under \( z_i \to z_i + 1 \), and can therefore be rewritten in a Fourier series [6]:

\[ \tilde{\theta}_\Lambda^*(z^i \mid \tau) = \sum_{m_i \in \mathbb{Z}} a_{m_i} e^{2\pi i m_i z_i} \]

where

\[ a_{m_i} = \sum_{n_i \in \mathbb{Z}} \left( \prod_{i=1}^{k} \int_{0}^{1} e^{-2\pi i m_i z_i} \tilde{\theta}_\Lambda^*(z^i \mid \tau) \right) \]

By standard Gaussian integration one obtains

\[ \tilde{\theta}_\Lambda^*(z^i \mid \tau) = |A|^{1/2} \left( -i\tau \right)^{\ell/2} \theta_\Lambda(z^i \mid -\frac{1}{\tau}) \]

Making a change of variable \( n_i = A_{ij} k_j \) in \( \tilde{\theta}_\Lambda^* \) one obtains

\[ \tilde{\theta}_\Lambda^*(z^i \mid \tau) = (e^{\pi i r z_i A_i^{-1} z_f}) \sum_{k_i} q^{k_i A_{ij} k_j 2\pi i r k_i z_i} \].
The sum over \( s_i \) will be a sum over all the integers if and only if \( A_{ij} \) and \( A_{ij}^{-1} \) are both integer matrices. This implies that \( \det(A) = 1 \) and hence the lattice is self dual. Therefore, for a self dual lattice we obtain the identity

\[
\theta_{\Lambda}(\frac{s}{r}, -\frac{1}{r}) = (-i\tau)^{\frac{1}{4}} \sum_{s, A_{ij}^{-1}s_j} \theta_{\Lambda}(s | r)
\]

This is completely analogous to the transformation property for Jacobi theta functions [7]:

\[
\theta_1(\frac{\nu}{r}, -\frac{1}{r}) = (\sqrt{-i\tau})^{\frac{1}{2}} \theta_1(\nu | r)
\]

Finally, we note that

\[
\eta(-\frac{1}{r}) = (\sqrt{-i\tau})\eta(r)
\]

and recall that

\[
\Lambda(r) = \prod_{b=1}^{m} \frac{\zeta(b)}{\sinh(b\tau)}
\]

Therefore we find that

\[
A(q(-\frac{1}{r}), r^{-1}F, r^{-1}R) = r^{-4m} \exp \left[ \frac{i}{4\pi r} \left( \sum_b x_b^2 - \sum_a y_a^2 \right) \right] \cdot A(q(r), F, R)
\]

Observe that \( \sum_b x_b^2 = -\frac{1}{8\pi^2} \text{Tr}(R^2) \), whereas in the adjoint representation of \( G \),

\[
-\frac{1}{4\pi^2} \text{Tr}(R^2) = \sum_{\text{roots}, \mathfrak{r}} (y^a r_a)^2 = 2 \left( \frac{\text{dim}G}{\text{rank}G} - 1 \right) \sum_a (y^a)^2
\]

For a simply laced group, \( G \), the coefficient on the right hand side is simply the quadratic Casimir, \( C_A \), of the adjoint representation. Thus the exponential factor above is just

\[
\exp \left\{ \frac{i}{4\pi r} \left[ \frac{1}{2} \text{Tr} \left( \frac{iR}{2\pi} \right)^2 - \frac{1}{C_A} \text{Tr} \left( \frac{iF}{2\pi} \right)^2 \right] \right\}
\]

Consider the coefficient, \( C(q) \), of any \((8m+4)\)-form in \( F \) and \( R \) appearing in \( A(q, F, R) \). Because of the rescaling of \( F \) and \( R \) in the foregoing transformation, this coefficient would be a modular
function of weight two if one could ignore the exponential factor described above. Moreover, because \( \delta_A(z^2 | \tau) \) is holomorphic for \( \text{Im}(\tau) > 0 \), the only poles of \( C(q) \) occur at \( q = 0 \). By the same arguments as in [1] we could conclude (ignoring again the exponential factor) that the coefficient of \( q^0 \) in \( C(q) \) is zero. Thus the anomaly would cancel completely.

Therefore the anomalous "exponential factor" in the modular transformation of \( A(q, F, R) \) is the only obstruction to complete anomaly cancellation.

This exponential factor becomes trivial if \( \text{Tr}(F^2) = \frac{C_A}{2} \text{Tr}(R^2) \). Thus we can conclude that \( \text{Tr}(R^2) - \frac{C_A}{2} \text{Tr}(F^2) \) is a factor of the complete anomaly of the field theory. This anomaly can be cancelled by the simplest Green-Schwarz mechanism; that is, one needs a single anti-symmetric tensor field, \( B_{\mu \nu} \). Also note that the factor \( \frac{C_A}{2} \) in front of \( \text{Tr}(F^2) \) is the familiar factor of \( \frac{1}{36} \) for \( E_8 \) and \( SO(32) \).

Another way of seeing this result is by cancelling the "modular anomaly" in the partition function \( A(q, F, R) \). This can be achieved by defining

\[
\tilde{A}(q, F, R) = A(q, F, R) \cdot \exp \left\{ -G_2(\tau) \left[ \frac{1}{2} \text{Tr} \left( \frac{iR}{2\pi} \right)^2 - \frac{1}{C_A} \text{Tr} \left( \frac{F}{2\pi} \right)^2 \right] \right\}
\]

where

\[
G_2(q^2) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^{2n}
\]

is the first Eisenstein function, with anomalous modular transformation properties [6]:

\[
G_2 \left( q^2 \left( \frac{ar + b}{cr + d} \right) \right) = (cr + d)^2 G_2(q^2(\tau)) + \frac{i}{4\pi} c(r + d)
\]

The anomaly in the \( G_2 \) transformation cancels the anomalous exponential factor in the transformation of \( A(q, F, R) \). Hence

\[
\tilde{A}(q(-\frac{1}{\tau}), r^{-1} F, r^{-1} R) = r^{-4m} \tilde{A}(q(\tau), F, R)
\]

By our previous arguments, the \((8m + 4)\)-form coefficient of \( q^0 \) in \( \tilde{A}(q, F, R) \) is identically zero. This means that the same coefficient in \( A(q, F, R) \) must have \( \text{Tr}(R^2) - \frac{C_A}{2} \text{Tr}(F^2) \) as a factor.
It is interesting to note that the foregoing "modular anomaly" cancellation could also have been accomplished by adding to $G_2$ any modular function of weight two that is holomorphic for $\text{Im}(r) > 0$. Any such function can be written as the derivative of a polynomial in the absolute invariant $j(r)$ [16]. This suggests that the cancellation of the modular anomaly is related to the cohomology of modular forms.

Since we have obtained an almost modular invariant generating function, $A(q,F,R)$, for the gauge and gravitational anomalies, it seems likely that this function must be closely related to the one loop anomaly amplitude for the complete string. By the same token, we expect that the extra factor in $\tilde{A}(q,F,R)$ must be a direct result of the contribution of the anomaly cancelling field $B_{\mu\nu}$.

Acknowledgements:

One of us (A.N.S.) would like to thank the M.I.T. Mathematics Department for its hospitality.
References


