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BAKER-CAMPBELL-HAUSDORFF RELATIONS FOR THE SUPER-POINCARÉ GROUP

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ABSTRACT

We obtain Baker-Campbell-Hausdorff relations for the connected supergroup associated with the super-Poincaré algebra $\text{iosp}(1|4)$.

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I. INTRODUCTION

The super-Poincaré algebra in its simplest form is an extension of the Poincaré algebra that includes four anticommuting generators, called supersymmetry generators, in addition to the four translation, three rotation, and three boost generators. This superalgebra has been the basis for much research in the past fifteen years, especially in high-energy physics, where its representations are involved in the construction of globally supersymmetric field theories, supergravities, and superstring theories. A description of many of these applications may be found in Ref. 1.

Although the basic properties of superalgebras are well-established, the mathematically-rigorous definition of supergroups as abstract groups and as superanalytic supermanifolds² is relatively recent³. Since the original papers of Rogers, much progress has been made in elucidating the properties of these supermanifolds^{4,5} and supergroups^{6,7}. Explicit results for practical calculations exist, however, only for a few of the simpler supergroups. Thus, for example, the supergroup based on the super-Poincaré algebra with four anticommuting generators, called the super-Poincaré group, was first presented within Rogers' formalism in Ref. 3. However, in this formulation the supergroup has not been studied extensively, although there are applications to physics making use of Rogers' formalism⁸.

In this paper, we present Baker-Campbell-Hausdorff (BCH) relations^{7,9,10} for the super-Poincaré group. The basic forms of the relations are derived as solutions of a set of differential equations obtained by an appropriate method^{7,11}. The method has previously been applied⁷ to two other supergroups, CSQM(2) and CIOSP(1|2); however, the super-Poincaré group has

significantly more generators (14) than either of these cases (3 and 5, respectively). Thus, in addition to their intrinsic interest and practical application, the calculations presented here provide further evidence of the viability of the differential-equation approach¹¹ to BCH relations.

In Section II, we present our notation and conventions both for the super-Poincaré algebra and for the various canonical and non-canonical forms of supergroup elements that we use. The BCH relations for canonical coordinates in normal sequence are obtained in Section III, which also contains a description of the various techniques involved in the derivation. In Section IV, we extend these results to BCH relations for canonical coordinates in non-normal sequence. The results are further extended to non-canonical coordinates in Section V.

Note that our conventions are those of Refs. 6 and 7, which were based on those of Rogers^{2,3}. We remark, in particular, that we do not use the summation convention. Also, we do not repeat here basic results for supergroups developed elsewhere. To follow in detail the calculations, the reader will need some familiarity with Ref. 7. However, the BCH relations presented should be readily accessible for practical use.

II. THE SUPER-POINCARÉ ALGEBRA AND THE SUPER-POINCARÉ GROUP

II.1 The Super-Poincaré Algebra

The super-Poincaré algebra with four supersymmetry generators may be derived from the simple superalgebra $osp(1|4)$ by an Inönü-Wigner contraction¹² that leaves the usual Poincaré algebra as a subalgebra. The procedure is analogous to that presented in Ref. 7 for obtaining $iosp(1|2)$ from $osp(1|2)$. There exist also other inhomogeneous superalgebras, generically referred to as N-extended super-Poincaré algebras, that may be obtained by contraction from the simple superalgebra $osp(N|4)$. In the remainder of this paper, "super-Poincaré algebra" is taken to mean exclusively the case $N=1$.

In our notation, the nonvanishing graded commutation relations of the super-Poincaré algebra are:

$$\begin{aligned} \{Q_1, Q_3\} &= P_1, \\ \{Q_2, Q_3\} &= P_3, \\ \{Q_1, Q_4\} &= P_2, \\ \{Q_2, Q_4\} &= P_4, \end{aligned} \tag{2.1}$$

$$\begin{aligned} [J_1, Q_2] &= -Q_1, \\ [J_2, Q_1] &= 1/2 Q_1, \quad [J_2, Q_2] = -1/2 Q_2, \\ [J_3, Q_1] &= Q_2, \end{aligned} \tag{2.2}$$

$$\begin{aligned} [J_4, Q_4] &= Q_3, \\ [J_5, Q_3] &= -1/2 Q_3, \quad [J_5, Q_4] = 1/2 Q_4, \\ [J_6, Q_3] &= -Q_4, \end{aligned} \tag{2.3}$$

$$\begin{aligned} [J_1, P_3] &= -P_1, \quad [J_1, P_4] = -P_2, \\ [J_2, P_1] &= 1/2 P_1, \quad [J_2, P_3] = -1/2 P_3 \\ [J_2, P_2] &= 1/2 P_2, \quad [J_2, P_4] = -1/2 P_4 \\ [P_3, P_1] &= P_3, \quad [P_3, P_2] = P_4, \end{aligned} \tag{2.4}$$

$$\begin{aligned}
[J_4, P_2] &= P_1, & [J_4, P_4] &= P_3, \\
[J_5, P_1] &= -1/2 P_1, & [J_5, P_2] &= 1/2 P_2, \\
[J_5, P_3] &= -1/2 P_3, & [J_5, P_4] &= 1/2 P_4, \\
[J_6, P_1] &= -P_2, & [J_6, P_3] &= -P_4,
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
[J_1, J_2] &= -J_1, & [J_4, J_5] &= J_4, \\
[J_1, J_3] &= -2J_2, & [J_4, J_6] &= 2J_5, \\
[J_2, J_3] &= -J_3, & [J_5, J_6] &= J_6.
\end{aligned} \tag{2.6}$$

The Q_m , $m=1, \dots, 4$, generate supersymmetry transformations, the P_λ , $\lambda=1, \dots, 4$, generate translations, and the J_n , $n=1, \dots, 6$, generate rotations and boosts. These generators satisfy the following adjoint relations:¹

$$\begin{aligned}
Q_1^\dagger &= -Q_3, & Q_2^\dagger &= -Q_4, & Q_3^\dagger &= -Q_1, & Q_4^\dagger &= -Q_2, \\
P_1^\dagger &= P_1, & P_2^\dagger &= P_3, & P_3^\dagger &= P_2, & P_4^\dagger &= P_4, \\
J_1^\dagger &= J_4, & J_2^\dagger &= J_5, & J_3^\dagger &= J_6, \\
J_4^\dagger &= J_1, & J_5^\dagger &= J_2, & J_6^\dagger &= J_3,
\end{aligned} \tag{2.7}$$

which are essential for computing unitary supergroup operators.

Several different conventions and notations for the super-Poincaré algebra exist in the literature. In Eqs. (2.1)-(2.7), we have adopted the conventions of Ref. 1, chapter 3. To make formulae and calculations less cumbersome, however, our notation is different. The notation of Ref. 1 may be recovered by effectuating the following replacements:

$$\begin{aligned}
Q_1 &\longleftrightarrow Q_+, & P_1 &\longleftrightarrow P_{++}, & J_1 &\longleftrightarrow J_{++}, & J_4 &\longleftrightarrow J_{\dot{+}\dot{+}}, \\
Q_2 &\longleftrightarrow Q_-, & P_2 &\longleftrightarrow P_{+-}, & J_2 &\longleftrightarrow J_{+-}, & J_5 &\longleftrightarrow J_{\dot{+}\dot{-}}, \\
Q_3 &\longleftrightarrow Q_{\dot{+}}, & P_3 &\longleftrightarrow P_{-\dot{+}}, & J_3 &\longleftrightarrow J_{--}, & J_6 &\longleftrightarrow J_{\dot{-}\dot{-}}, \\
Q_4 &\longleftrightarrow Q_{\dot{-}}, & P_4 &\longleftrightarrow P_{--}.
\end{aligned} \tag{2.8}$$

The conventions of Ref. 1 are especially convenient for calculational purposes. Note, however, that the operators P_λ and J_n are not the usual momentum and angular-momentum physical observables. The P_λ are light-cone-type variables defined¹ as linear combinations of the standard four-momenta. Furthermore, as is usual practice, the hermitian generators of the Lorentz algebra $so(3,1)$ have been linearly combined to form the J_n . We remark that any BCH relation presented in this paper may readily be converted to an expression in terms of the standard four-momenta, using lemma 2 of Ref. 7. This is possible because the P_λ form an abelian subalgebra of $iosp(1|4)$.

II.2 The Super-Poincaré Group

Associated with the super-Poincaré algebra is a connected supergroup whose elements may be obtained (see theorem 2 of Ref. 6) by exponentiation of the superalgebra generators with Grassmann-valued variables. We refer to this supergroup as the super-Poincaré group.

The precise form of the supergroup elements depends upon the parametrization scheme adopted for the exponentiation. For supergroups, there exist three kinds of canonical coordinates^{6,7}. For elements g of the super-Poincaré group, canonical coordinates of the first kind take the form

$$g_I = \exp\left(\sum_{\lambda=1}^4 p^\lambda P_\lambda + \sum_{m=1}^4 q^m Q_m + \sum_{n=1}^6 j^n J_n\right), \quad (2.9)$$

where the p^λ and j^n are variables taking values in the even part 0B_L of a Grassmann algebra B_L over $\mathbb{C}^L$, and where the q^m are variables taking values in the odd part 1B_L . For a complete description of this construction and the discussion that follows for the case of a general supergroup, the reader

should consult Refs. 6 and 7. Note that a hermitian basis for the Grassmann algebra is established in Kostelecký and Rabin, Ref. 8.

Canonical coordinates of the second kind for a supergroup are constructed⁶ in terms of exponentials of the individual basis elements of the Lie algebra that is associated with the superalgebra in question. However, because each Q_m anticommutes with itself, it turns out that these canonical coordinates are identical to the canonical coordinates of the third kind (corollary to theorem 1 of Ref. 7). Therefore, in this paper we proceed directly to consideration of canonical coordinates of the third kind, which for elements g of the super-Poincaré group take the form

$$g_{III} = \prod_{\lambda=1}^4 \exp(\alpha^{\lambda} P_{\lambda}) \prod_{m=1}^4 \exp(\beta^m Q_m) \prod_{n=1}^6 \exp(\gamma^n J_n). \quad (2.10)$$

Here, $\alpha^{\lambda}, \gamma^n \in {}^0B_L$ and $\beta^m \in {}^1B_L$. Note that the order of appearance of the fourteen exponentials in Eq. (2.10) is important; by definition, we say that this sequence is "normal"⁷. Other non-normal sequences are possible and, indeed, will appear in the subsequent sections.

In addition to the canonical coordinate schemes, various non-canonical parametrizations may also be defined. In Section V, we shall consider extensions of the BCH relations to certain non-canonical coordinate schemes. For elements g of the super-Poincaré group, these non-canonical coordinates take the form

$$g_{N.C.} = \exp\left(\sum_{\lambda=1}^4 a^{\lambda} P_{\lambda}\right) \exp\left(\sum_{m=1}^4 b^m Q_m\right) \exp\left(\sum_{n=1}^6 c^n J_n\right), \quad (2.11)$$

where $a^{\lambda}, c^n \in {}^0B_L$ and $b^m \in {}^1B_L$. Again, this is defined as the normal sequence; several non-normal sequences may also be introduced.

To obtain unitary supergroup operators in terms of canonical coordinates of the first kind, Eq. (2.9), we require the condition

$$g_I^\dagger = g_I^{-1} \quad (2.12)$$

Computing the adjoint and the inverse of (2.9) and using the properties (2.7) of the generators, we obtain the conditions on the Grassmann parameters,

$$\begin{aligned} q^3 &= -(q^1)^*, & q^4 &= -(q^2)^*, \\ p^1 &= -(p^1)^*, & p^3 &= -(p^2)^*, & p^4 &= -(p^4)^*, \\ j^4 &= -(j^1)^*, & j^5 &= -(j^2)^*, & j^6 &= -(j^3)^*. \end{aligned} \quad (2.13)$$

Note that p^1 and p^4 are pure imaginary as a consequence of the fact that P_1 and P_4 are self-adjoint.

Thus, requiring that the parameters p^ℓ , q^m , j^n of Eq. (2.9) satisfy Eq. (2.13) ensures that the supergroup elements are unitary. In the remainder of the paper, the constraints (2.13) are not assumed. However, they may be imposed if desired.

III. BCH RELATIONS FOR CANONICAL COORDINATES IN NORMAL SEQUENCE

Here, we construct explicitly the BCH relations between canonical coordinates of the first and third kinds using the differential equation method expounded in Ref. 7. Only canonical coordinates of the third kind in normal sequence are considered in this section.

Applying the general method of Ref. 7, we introduce a real parameter t and write

$$\begin{aligned} & \exp\left[t\left(\sum_{\lambda=1}^4 p^{\lambda} p_{\lambda} + \sum_{m=1}^4 q^m Q_m + \sum_{n=1}^6 j^n J_n\right)\right] \\ &= \prod_{\lambda'=1}^4 \exp(\alpha^{\lambda'} p_{\lambda'}) \prod_{m'=1}^4 \exp(\beta^{m'} Q_{m'}) \prod_{n'=1}^6 \exp(\gamma^{n'} J_{n'}), \end{aligned} \quad (3.1)$$

where $\alpha^{\lambda'}$, $\beta^{m'}$, $\gamma^{n'}$ are taken as unknown functions of p^{λ} , q^m , j^n , and t :

$$\begin{aligned} \alpha^{\lambda'} &= \alpha^{\lambda'}(p^{\lambda}, q^m, j^n, t) \in {}^0B_L, \\ \beta^{m'} &= \beta^{m'}(p^{\lambda}, q^m, j^n, t) \in {}^1B_L, \\ \gamma^{n'} &= \gamma^{n'}(p^{\lambda}, q^m, j^n, t) \in {}^0B_L, \\ p^{\lambda}, j^n &\in {}^0B_L, q^m \in {}^1B_L, t \in \mathbb{R}. \end{aligned} \quad (3.2)$$

For $t=1$, Eq. (3.1) and knowledge of the functions in Eq. (3.2) together form the desired BCH relation.

We can obtain the explicit form of Eqs. (3.2) by solving a set of fourteen coupled linear first-order differential equations in the real parameter t . These equations are obtained by differentiating Eq. (3.1), simplifying, and equating coefficients of the superalgebra generators. The appropriate boundary conditions⁷ are $\alpha^{\lambda}(0) = \beta^m(0) = \gamma^n(0) = 0$. The details of the process for obtaining the differential equations have been discussed in Ref. 7; the present case is a straightforward application of those techniques.

The methods yield the following fourteen differential equations, where a dot above a symbol means differentiation with respect to t:

$$\dot{\gamma}^6 e^{\gamma^5} = j^6 \quad (3.3)$$

$$\dot{\gamma}^5 + 2\gamma^4 \dot{\gamma}^6 e^{\gamma^5} = j^5 \quad (3.4)$$

$$\dot{\gamma}^4 + \gamma^4 \dot{\gamma}^5 + (\gamma^4)^2 \dot{\gamma}^6 e^{\gamma^5} = j^4 \quad (3.5)$$

$$\dot{\gamma}^3 e^{-\gamma^2} = j^3 \quad (3.6)$$

$$\dot{\gamma}^2 - 2\gamma^1 \dot{\gamma}^3 e^{-\gamma^2} = j^2 \quad (3.7)$$

$$\dot{\gamma}^1 - \gamma^1 \dot{\gamma}^2 + (\gamma^1)^2 \dot{\gamma}^3 e^{-\gamma^2} = j^1 \quad (3.8)$$

$$\dot{\beta}^4 - 1/2 \beta^4 \dot{\gamma}^5 + \dot{\gamma}^6 e^{\gamma^5} (\beta^3 - \gamma^4 \beta^4) = q^4 \quad (3.9)$$

$$\dot{\beta}^3 + \dot{\gamma}^5 (1/2 \beta^3 - \gamma^4 \beta^4) + \dot{\gamma}^6 e^{\gamma^5} \gamma^4 (\beta^3 - \gamma^4 \beta^4) - \beta^4 \dot{\gamma}^4 = q^3 \quad (3.10)$$

$$\dot{\beta}^2 + 1/2 \dot{\gamma}^2 \beta^2 - \dot{\gamma}^3 e^{-\gamma^2} (\gamma^1 \beta^2 + \beta^1) = q^2 \quad (3.11)$$

$$\dot{\beta}^1 + \dot{\gamma}^1 \beta^2 - \dot{\gamma}^2 (\gamma^1 \beta^2 + 1/2 \beta^1) + \dot{\gamma}^3 e^{-\gamma^2} \gamma^1 (\beta^1 + \gamma^1 \beta^2) = q^1 \quad (3.12)$$

$$\begin{aligned} \dot{\alpha}^4 - \beta^2 \dot{\beta}^4 + 1/2 \dot{\gamma}^2 \alpha^4 - \dot{\gamma}^3 e^{-\gamma^2} (\alpha^2 + \gamma^1 \alpha^4) + 1/2 \dot{\gamma}^5 (\beta^2 \beta^4 - \alpha^4) \\ + \dot{\gamma}^6 e^{\gamma^5} [\alpha^3 - \beta^2 \beta^3 + \gamma^4 (\beta^2 \beta^4 - \alpha^4)] = p^4 \end{aligned} \quad (3.13)$$

$$\begin{aligned} \dot{\alpha}^3 - \beta^2 \dot{\beta}^3 + 1/2 \dot{\gamma}^2 \alpha^3 - \dot{\gamma}^3 e^{-\gamma^2} (\alpha^1 + \gamma^1 \alpha^3) + \dot{\gamma}^4 (\beta^2 \beta^4 - \alpha^4) \\ + 1/2 \dot{\gamma}^5 [\alpha^3 - \beta^2 \beta^3 + 2\gamma^4 (\beta^2 \beta^4 - \alpha^4)] \\ + \dot{\gamma}^6 e^{\gamma^5} \gamma^4 [\alpha^3 - \beta^2 \beta^3 + \gamma^4 (\beta^2 \beta^4 - \alpha^4)] = p^3 \end{aligned} \quad (3.14)$$

$$\begin{aligned} \dot{\alpha}^2 - \beta^1 \dot{\beta}^4 + \alpha^4 \dot{\gamma}^1 - \dot{\gamma}^2 (1/2 \alpha^2 + \gamma^1 \alpha^4) + \dot{\gamma}^3 \gamma^1 e^{-\gamma^2} (\alpha^2 + \gamma^1 \alpha^4) \\ + 1/2 \dot{\gamma}^5 (\beta^1 \beta^4 - \alpha^2) + \dot{\gamma}^6 e^{\gamma^5} [\alpha^1 - \beta^1 \beta^3 + \gamma^4 (\beta^1 \beta^4 - \alpha^2)] = p^2 \end{aligned} \quad (3.15)$$

$$\begin{aligned} \dot{\alpha}^1 - \beta^1 \dot{\beta}^3 + \alpha^3 \dot{\gamma}^1 - \dot{\gamma}^2 (1/2 \alpha^1 + \gamma^1 \alpha^3) + \dot{\gamma}^3 \gamma^1 e^{-\gamma^2} (\alpha^1 + \gamma^1 \alpha^3) \\ + \dot{\gamma}^4 (\beta^1 \beta^4 - \alpha^2) + 1/2 \dot{\gamma}^5 [\alpha^1 - \beta^1 \beta^3 + 2\gamma^4 (\beta^1 \beta^4 - \alpha^2)] \\ + \dot{\gamma}^6 e^{\gamma^5} \gamma^4 [\alpha^1 - \beta^1 \beta^3 + \gamma^4 (\beta^1 \beta^4 - \alpha^2)] = p^1 \end{aligned} \quad (3.16)$$

The remainder of this section is concerned with solving these equations.

In fact, each of the ordered sets of Eqs. {(3.3)-(3.5)}, {(3.6)-(3.8)}, {(3.9)-(3.10)}, and {(3.11)-(3.12)} is self-coupled and can be solved using

only the solutions to the sets occurring earlier in the ordered sequence, as we show below. This occurs because the J_n form a direct-product Lie subalgebra, because the Q_m commute with the P_ℓ , and because Q_1, Q_3 anticommute with Q_2, Q_4 , respectively.

The differential equations thus admit a natural sequence for solution. We outline first the solution of Eqs. (3.3)-(3.5). Substitution of (3.3) and (3.4) into (3.5) yields a Riccati equation¹³ for γ^4 :

$$\dot{\gamma}^4 + j^5 \gamma^4 + j^6 (\gamma^4)^2 = j^4. \quad (3.17)$$

The solution may be shown by standard methods¹¹ to be

$$\gamma^4 = j^4 \sinh(K_2 t) / [K_2 \cosh(K_2 t) + 1/2 j^5 \sinh(K_2 t)], \quad (3.18)$$

where

$$\begin{aligned} \Delta_2^2 &= \left| 1/4 (j^5)^2 - j^4 j^6 \right|, \\ \sigma_2 &= \text{sign}[1/4 (j^5)^2 - j^4 j^6], \quad \hat{\sigma}_2 = +\sqrt{\sigma_2}, \\ K_2 &= \hat{\sigma}_2 \Delta_2, \end{aligned} \quad (3.19)$$

and where we used the initial condition $\gamma^4(0) = 0$. Then, the solutions for γ^5 and γ^6 can be obtained by straightforward integration of Eq. (3.4) followed by (3.3), again using the initial conditions:

$$\gamma^5 = 2 \ln[\cosh(K_2 t) + 1/2 j^5 K_2^{-1} \sinh(K_2 t)], \quad (3.20)$$

$$\gamma^6 = j^6 \sinh(K_2 t) / [K_2 \cosh(K_2 t) + 1/2 j^5 \sinh(K_2 t)]. \quad (3.21)$$

By inspection, the corresponding solutions $\gamma^1, \gamma^2, \gamma^3$ to Eqs. (3.6)-(3.8) are obtained by substituting $j^6 \leftrightarrow j^3, j^5 \leftrightarrow -j^2, j^4 \leftrightarrow j^1, \gamma^6 \leftrightarrow \gamma^3, \gamma^5 \leftrightarrow -\gamma^2, \gamma^4 \leftrightarrow \gamma^1$. Explicitly, this yields

$$\gamma^1 = j^1 \sinh(K_1 t) / [K_1 \cosh(K_1 t) - 1/2 j^2 \sinh(K_1 t)], \quad (3.22)$$

$$\gamma^2 = -2\lambda n [\cosh(K_1 t) - 1/2 j^2 K_1^{-1} \sinh(K_1 t)], \quad (3.23)$$

$$\gamma^3 = j^3 \sinh(K_1 t) / [K_1 \cosh(K_1 t) - 1/2 j^2 \sinh(K_1 t)], \quad (3.24)$$

where

$$\begin{aligned} \Delta_1^2 &= \left| 1/4 (j^2)^2 - j^1 j^3 \right| \\ \sigma_1 &= \text{sign}[1/4 (j^2)^2 - j^1 j^3], \quad \hat{\sigma}_1 = +\sqrt{\sigma_1}, \\ K_1 &= \hat{\sigma}_1 \Delta_1. \end{aligned} \quad (3.25)$$

The results (3.18) and (3.20)-(3.24) are the same as those yielded for a BCH relation of the direct-product Lie algebra $\text{su}(1,1) \otimes \text{su}(1,1)$ (see, for example, Ref. 11). This occurs because the J_m generate this Lie subalgebra of the super-Poincaré algebra.

Next, we turn to the solution of Eqs. (3.9) and (3.10). By substitution for $\dot{\gamma}^4, \dot{\gamma}^5, \dot{\gamma}^6$ from Eqs. (3.3)-(3.5), we obtain

$$\dot{\beta}^4 - 1/2 j^5 \beta^4 + j^6 \beta^3 = q^4, \quad (3.26)$$

$$\dot{\beta}^3 + 1/2 j^5 \beta^3 - j^4 \beta^4 = q^3. \quad (3.27)$$

Solving Eq. (3.26) for β^3 and using the result together with its t -derivative in Eq. (3.27) yields a second-order equation for $\ddot{\beta}^4$. The solution is straightforward, as is the subsequent solution of (3.26). Fixing the four integration constants by using the initial conditions and by requiring consistency of the solutions with Eqs. (3.26) and (3.27), we find

$$\beta^4 = q^4 K_2^{-1} \sinh(K_2 t) + K_2^{-2} (1/2 j^5 q^4 - j^6 q^3) (\cosh(K_2 t) - 1), \quad (3.28)$$

$$\beta^3 = q^3 K_2^{-1} \sinh(K_2 t) - K_2^{-2} (1/2 j^5 q^3 - j^4 q^4) (\cosh(K_2 t) - 1). \quad (3.29)$$

In a similar manner, Eqs. (3.11) and (3.12) may be re-expressed as

$$\dot{\beta}^2 + 1/2 j^2 \beta^2 - j^3 \beta^1 = q^2, \quad (3.30)$$

$$\dot{\beta}^1 - 1/2 j^2 \beta^1 + j^1 \beta^2 = q^1. \quad (3.31)$$

Making the substitutions $j^6 \leftrightarrow j^3$, $j^5 \leftrightarrow -j^2$, $j^4 \leftrightarrow j^1$, $\beta^4 \leftrightarrow \beta^2$, $\beta^3 \leftrightarrow -\beta^1$, $q^4 \leftrightarrow q^2$, $q^3 \leftrightarrow -q^1$ enables us to write the solutions by inspection:

$$\beta^2 = q^2 K_1^{-1} \sinh(K_1 t) - K_1^{-2} (1/2 j^2 q^2 - j^3 q^1) (\cosh(K_1 t) - 1), \quad (3.32)$$

$$\beta^1 = q^1 K_1^{-1} \sinh(K_1 t) + K_1^{-2} (1/2 j^2 q^1 - j^1 q^2) (\cosh(K_1 t) - 1). \quad (3.33)$$

We are left with the four coupled differential Eqs. (3.13)-(3.16) for $\dot{\alpha}^1, \dot{\alpha}^2, \dot{\alpha}^3, \dot{\alpha}^4$. As for the other variables, these equations may be simplified by substitution of Eqs. (3.3)-(3.12) for $\dot{\gamma}^n$ and $\dot{\beta}^m$. Introducing the four-component column vector

$$\mathcal{X} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}, \quad (3.34)$$

these four equations may be written in abbreviated form as

$$\dot{\mathcal{X}} = \mathbb{P} \mathcal{X} + \mathcal{X}(t). \quad (3.35)$$

Here, $\mathbb{P}$ is the constant 4×4 matrix

$$\mathbb{P} = \begin{pmatrix} j^- & j^4 & -j^1 & 0 \\ -j^6 & j^+ & 0 & -j^1 \\ j^3 & 0 & -j^+ & j^4 \\ 0 & j^3 & -j^6 & -j^- \end{pmatrix}, \quad (3.36)$$

where

$$j^{\pm} = 1/2(j^2 \pm j^5), \quad (3.37)$$

and where $\underline{r}(t)$ is the four-component vector given by

$$\begin{aligned} r^1 &= \beta^1 q^3 + p^1, \\ r^2 &= \beta^1 q^4 + p^2, \\ r^3 &= \beta^2 q^3 + p^3, \\ r^4 &= \beta^2 q^4 + p^4. \end{aligned} \quad (3.38)$$

Equation (3.35) has as general solution¹⁴

$$\underline{\alpha} = \Psi(t)\underline{c} + \Psi(t) \int_0^t \Psi^{-1}(s)\chi(s)ds. \quad (3.39)$$

The constant vector $\underline{c}$ is to be determined by the initial conditions. Also, $\Psi(t)$ is a fundamental 4×4 matrix solution¹⁴, i.e., $\Psi(t)$ is a 4×4 matrix whose columns are four linearly-independent solutions to the homogeneous equations $\dot{\underline{\alpha}} = \mathbb{P}\underline{\alpha}$. Since $\Psi(0) \neq 0$, as we argue below, the initial condition $\underline{\alpha}(0) = 0$ implies $\underline{c} = 0$.

A fundamental matrix solution to the homogeneous equation $\dot{\underline{\alpha}} = \mathbb{P}\underline{\alpha}$ may be constructed by applying the trial solution $\underline{\alpha} = \underline{A}e^{\lambda t}$ and diagonalizing the resulting matrix of coefficients for $\underline{A}$. The method is described in detail in Ref. 14. We obtain

$$\Psi(t) = \begin{pmatrix} j^1 j^4 e^{\lambda_1 t} & j^1 j^4 e^{\lambda_2 t} & j^1 j^4 e^{\lambda_3 t} & j^1 j^4 e^{\lambda_4 t} \\ j^1 \phi_+^5 e^{\lambda_1 t} & -j^1 \phi_-^5 e^{\lambda_2 t} & -j^1 \phi_-^5 e^{\lambda_3 t} & j^1 \phi_+^5 e^{\lambda_4 t} \\ -j^4 \phi_-^2 e^{\lambda_1 t} & j^4 \phi_+^2 e^{\lambda_2 t} & -j^4 \phi_-^2 e^{\lambda_3 t} & j^4 \phi_+^2 e^{\lambda_4 t} \\ -\phi_-^2 \phi_+^5 e^{\lambda_1 t} & -\phi_+^2 \phi_-^5 e^{\lambda_2 t} & \phi_-^2 \phi_-^5 e^{\lambda_3 t} & \phi_+^2 \phi_+^5 e^{\lambda_4 t} \end{pmatrix}, \quad (3.40)$$

where

$$\begin{aligned}\lambda_1 &= -\lambda_2 = K_1 + K_2, \\ \lambda_3 &= -\lambda_4 = K_1 - K_2,\end{aligned}\tag{3.41}$$

and

$$\begin{aligned}\phi_{\pm}^2 &= K_1 \pm 1/2 j^2, \\ \phi_{\pm}^5 &= K_2 \pm 1/2 j^5.\end{aligned}\tag{3.42}$$

The inverse fundamental matrix may be calculated as

$$\Psi^{-1}(t) = (4K_1 K_2 j^1 j^4)^{-1} \begin{pmatrix} \phi_+^2 \phi_-^5 e^{-\lambda_1 t} & j^4 \phi_+^2 e^{-\lambda_1 t} & -j^1 \phi_-^5 e^{-\lambda_1 t} & -j^1 j^4 e^{-\lambda_1 t} \\ \phi_-^2 \phi_+^5 e^{-\lambda_2 t} & -j^4 \phi_-^2 e^{-\lambda_2 t} & j^1 \phi_+^5 e^{-\lambda_2 t} & -j^1 j^4 e^{-\lambda_2 t} \\ \phi_+^2 \phi_+^5 e^{-\lambda_3 t} & -j^4 \phi_+^2 e^{-\lambda_3 t} & -j^1 \phi_+^5 e^{-\lambda_3 t} & j^1 j^4 e^{-\lambda_3 t} \\ \phi_-^2 \phi_-^5 e^{-\lambda_4 t} & j^4 \phi_-^2 e^{-\lambda_4 t} & j^1 \phi_-^5 e^{-\lambda_4 t} & j^1 j^4 e^{-\lambda_4 t} \end{pmatrix}.\tag{3.43}$$

With expressions (3.38), (3.40), and (3.43), the general solution for α may now be obtained from Eq. (3.39). The integrations are straightforward. After some algebra, the expression for α^1 takes the form

$$\begin{aligned}\alpha^1 &= 2K_1 k^{-2} q^1 C^2 \sinh(K_1 t) \\ &+ k^{-2} C^1 C^2 \cosh(K_1 t) \\ &+ k^{-2} K_+^{-1} [K_1 \cosh(K_+ t) + K_2] [q^1 q^3 k - 1/2 C^1 C^2] \\ &+ k^{-2} K_-^{-1} [K_1 \cosh(K_- t) + K_2] [-q^1 q^3 k - 1/2 C^1 C^2] \\ &+ k^{-2} K_1 K_+^{-1} \sinh(K_+ t) [K_2 C^1 q^3 - K_1 q^1 C^2] \\ &- k^{-2} K_1 K_-^{-1} \sinh(K_- t) [K_2 C^1 q^3 + K_1 q^1 C^2] \\ &+ 1/2 k^{-1} K_+^{-1} [1 - \cosh(K_+ t)] [K_1 D^1 - K_2 D^2] \\ &- 1/2 k^{-1} K_-^{-1} [1 - \cosh(K_- t)] [K_1 D^1 + K_2 D^2] \\ &+ 1/2 k^{-1} K_+^{-1} \sinh(K_+ t) [p^1 k - 1/2 (D^3 j^5 - 2D^4 j^4)] \\ &+ 1/2 k^{-1} K_-^{-1} \sinh(K_- t) [p^1 k + 1/2 (D^3 j^5 - 2D^4 j^4)].\end{aligned}\tag{3.44}$$

In this expression, we have defined

$$\begin{aligned} k &= 2K_1K_2, \\ K_{\pm} &= K_1 \pm K_2, \end{aligned} \quad (3.45)$$

and

$$\begin{aligned} C^1 &= q^1j^2 - 2q^2j^1, \\ C^2 &= q^3j^5 - 2q^4j^4, \\ D^1 &= p^1j^5 - 2p^2j^4, \\ D^2 &= p^1j^2 - 2p^3j^1, \\ D^3 &= p^1j^2 - 2p^3j^1, \\ D^4 &= p^2j^2 - 2p^4j^1. \end{aligned} \quad (3.46)$$

The reader should note that C^1 and C^2 , like the q^m , are odd Grassmann variables.

Similarly, the solutions for α^2 , α^3 , and α^4 may be obtained. Due to the structure of the differential equations (3.35), these solutions can be expressed in the form of Eq. (3.44) with simple parameter substitutions. If we write explicitly the functional dependence of α^1 in Eq. (3.44) as

$$\alpha^1 = \alpha^1(p^1, p^2, p^3, p^4; q^1, q^2, q^3, q^4; j^1, j^2, j^3, j^4, j^5, j^6), \quad (3.47)$$

then the solutions α^2 , α^3 , α^4 may be written

$$\alpha^2 = \alpha^1(p^2, p^1, p^4, p^3; q^1, q^2, q^4, q^3; j^1, j^2, j^3, -j^6, -j^5, -j^4), \quad (3.48)$$

$$\alpha^3 = \alpha^1(p^3, p^4, p^1, p^2; q^2, q^1, q^3, q^4; -j^3, -j^2, -j^1, j^4, j^5, j^6), \quad (3.49)$$

$$\alpha^4 = \alpha^1(p^4, p^3, p^2, p^1; q^2, q^1, q^4, q^3; -j^3, -j^2, -j^1, -j^6, -j^5, -j^4). \quad (3.50)$$

Note in particular that under any of these parameter changes K_1 and K_2 remain unaffected.

In summary, we have obtained in this section the BCH relation between canonical coordinates of the first and third kinds, in normal sequence. The relation is of the form of Eqs. (3.1) and (3.2) with $t = 1$, where the fourteen equations in (3.2) are given explicitly by Eqs. (3.18), (3.20)-(3.24), (3.28), (3.29), (3.32), (3.33), (3.44), and (3.48)-(3.50). We remark that the results may also be applied in the limit in which any of the parameters p^ℓ , q^m , j^n are taken to zero.

IV. BCH RELATIONS FOR CANONICAL COORDINATES IN NON-NORMAL SEQUENCE

In this section, we construct BCH relations between canonical coordinates of the first and third kinds for certain non-normal sequences. Although we could proceed as in Section III, obtaining and solving sets of fourteen differential equations for each BCH relation, it is simpler to proceed by relating the various non-normal sequences for canonical coordinates of the third kind to the normal sequence. Combined with the BCH relation already obtained, these results will yield BCH relations of the desired type.

We begin by recalling that the P_λ commute among themselves and with the Q_m . Furthermore, Q_1, Q_3 anticommute with Q_2, Q_4 , respectively. Therefore, by Lemma 2 of Ref. 7, the explicit solutions of the form (3.2) that we have obtained in Section III will be valid for a BCH relation of the form (3.1) but with the exponentials involving P_λ and Q_m , taken in any order, provided all exponentials with P_λ and Q_m are to the left of those with J_n , and provided the exponentials with Q_1, Q_2 appear to the left of those with Q_3 and Q_4 . In particular, this means that the BCH relation

$$\begin{aligned} & \exp\left(\sum_{\lambda=1}^4 p^\lambda P_\lambda + \sum_{m=1}^4 q^m Q_m + \sum_{n=1}^6 j^n J_n\right) \\ &= \prod_{m'=1}^4 \exp(\beta^{m'} Q_{m'}) \prod_{\lambda'=1}^4 \exp(\alpha^{\lambda'} P_{\lambda'}) \prod_{n'=1}^6 \exp(\gamma^{n'} J_{n'}) \end{aligned} \quad (4.1)$$

is obtained with the same explicit solutions of the form (3.2) as we found in Section III.

Next, we establish a link of the following kind:

$$\begin{aligned} & \prod_{\ell=1}^4 \exp(\alpha^{\ell} p_{\ell}) \prod_{m=1}^4 \exp(\beta^m Q_m) \prod_{n=1}^6 \exp(\gamma^n J_n) \\ &= \prod_{\ell'=1}^4 \exp(\alpha^{\ell'} p_{\ell'}) \prod_{n'=1}^6 \exp(\gamma^{n'} J_{n'}) \prod_{m'=1}^4 \exp(\sigma^{m'} Q_{m'}), \end{aligned} \quad (4.2)$$

where the quantities $\sigma^{m'}$ are functions of β^m and γ^n . We proceed by inserting the identity I, in the form

$$I = \prod_{k=1}^6 \exp(\gamma^k J_k) \prod_{k'=6}^1 \exp(-\gamma^{k'} J_{k'}), \quad (4.3)$$

between the P_{ℓ} and Q_m exponentiations on the left-hand side of Eq. (4.2).

Then, the piece

$$\prod_{k'=6}^1 \exp(-\gamma^{k'} J_{k'}) \prod_{m=1}^4 \exp(\beta^m Q_m) \prod_{n=1}^6 \exp(\gamma^n J_n) \quad (4.4)$$

of the resulting expression can be simplified by repeated use of theorem 2, theorem 3, and lemma 7 of Ref. 7, in a fashion analogous to their use in obtaining the differential equations (3.3)-(3.16).

Thus, for example, we find

$$\begin{aligned} & \exp(-\gamma^1 J_1) \prod_{m=1}^4 \exp(\beta^m Q_m) \exp(\gamma^1 J_1) \\ &= e^{-\gamma^1 J_1} e^{\beta^1 Q_1} e^{\gamma^1 J_1} \cdot e^{-\gamma^1 J_1} e^{\beta^2 Q_2} e^{\gamma^1 J_1} \cdot e^{-\gamma^1 J_1} e^{\beta^3 Q_3} e^{\gamma^1 J_1} \cdot e^{-\gamma^1 J_1} e^{\beta^4 Q_4} e^{\gamma^1 J_1} \\ &= (1 + \beta^1 Q_1) [1 + \beta^2 (Q_2 + \gamma^1 Q_1)] (1 + \beta^3 Q_3) (1 + \beta^4 Q_4) \\ &= e^{(\beta^1 + \beta^2 \gamma^1) Q_1} e^{\beta^2 Q_2} e^{\beta^3 Q_3} e^{\beta^4 Q_4}. \end{aligned} \quad (4.5)$$

Proceeding in this manner ultimately yields expressions for the $\sigma^{m'}$ of Eq. (4.2) in terms of β^m and γ^n :

$$\begin{aligned}
 \sigma^1 &= (\beta^1 + \beta^2 \gamma^1) e^{-1/2 \gamma^2}, \\
 \sigma^2 &= \beta^2 e^{1/2 \gamma^2} - (\beta^1 + \beta^2 \gamma^4) \gamma^3 e^{-1/2 \gamma^2}, \\
 \sigma^3 &= (\beta^3 - \beta^4 \gamma^4) e^{1/2 \gamma^5}, \\
 \sigma^4 &= \beta^4 e^{-1/2 \gamma^5} + (\beta^3 - \beta^4 \gamma^4) \gamma^6 e^{1/2 \gamma^5}.
 \end{aligned} \tag{4.6}$$

Substitution of the solutions for β^m , γ^n obtained in Section III and using the resulting form of Eq. (4.2) yields the BCH relation of the form

$$\begin{aligned}
 &\exp\left(\sum_{\ell=1}^4 p^\ell P_\ell + \sum_{m=1}^4 q^m Q_m + \sum_{n=1}^6 j^n J_n\right) \\
 &= \prod_{\ell'=1}^4 \exp(\alpha^{\ell'} P_{\ell'}) \prod_{n'=1}^6 \exp(\gamma^{n'} J_{n'}) \prod_{m'=1}^4 \exp(\sigma^{m'} Q_{m'}).
 \end{aligned} \tag{4.7}$$

By an analogous method, we can establish the BCH relation of the type

$$\begin{aligned}
 &\exp\left(\sum_{\ell=1}^4 p^\ell P_\ell + \sum_{m=1}^4 q^m Q_m + \sum_{n=1}^6 j^n J_n\right) \\
 &= \prod_{m'=1}^4 \exp(\beta^{m'} Q_{m'}) \prod_{n'=1}^6 \exp(\gamma^{n'} J_{n'}) \prod_{\ell'=1}^4 \exp(\rho^{\ell'} P_{\ell'}).
 \end{aligned} \tag{4.8}$$

In this case, the $\rho^{\ell'}$ are functions of the α^{ℓ} and γ^n of Section III, given by

$$\begin{aligned}
 \rho^1 &= (\alpha^1 - \alpha^2 \gamma^4 - \alpha^3 \gamma^1 - \alpha^4 \gamma^1 \gamma^4) e^{-1/2 \gamma^-}, \\
 \rho^2 &= (\alpha^1 - \alpha^2 \gamma^4 + \alpha^3 \gamma^1 - \alpha^4 \gamma^1 \gamma^4) \gamma^6 e^{-1/2 \gamma^-} + (\alpha^2 + \alpha^4 \gamma^1) e^{-1/2 \gamma^+}, \\
 \rho^3 &= -(\alpha^1 - \alpha^2 \gamma^4 + \alpha^3 \gamma^1 - \alpha^4 \gamma^1 \gamma^4) \gamma^3 e^{-1/2 \gamma^-} + (\alpha^3 - \alpha^4 \gamma^4) e^{1/2 \gamma^+}, \\
 \rho^4 &= -(\alpha^1 - \alpha^2 \gamma^4 + \alpha^3 \gamma^1 - \alpha^4 \gamma^1 \gamma^4) \gamma^3 \gamma^6 e^{-1/2 \gamma^-} - (\alpha^2 + \alpha^4 \gamma^1) \gamma^3 e^{-1/2 \gamma^+} \\
 &\quad + (\alpha^3 - \alpha^4 \gamma^4) \gamma^6 e^{1/2 \gamma^+} + \alpha^4 e^{1/2 \gamma^-}.
 \end{aligned} \tag{4.9}$$

In these equations, $\gamma^{\pm} = \gamma^2 \pm \gamma^5$. Again, substitution of the solutions of Section III for α^{ℓ} and γ^n yields the desired BCH relation.

Finally, consider the BCH relations of the type

$$\begin{aligned} & \exp\left(\sum_{\ell=1}^4 p^{\ell} p_{\ell} + \sum_{m=1}^4 q^m Q_m + \sum_{n=1}^6 j^n J_n\right) \\ &= \prod_{n'=1}^6 \exp(\gamma^{n'} J_{n'}) \prod_{\ell'=1}^4 \exp(\rho^{\ell'} p_{\ell'}) \prod_{m'=1}^4 \exp(\sigma^{m'} Q_{m'}). \end{aligned} \quad (4.10)$$

This is in fact a class of relations of the same size as that of the coordinates in normal sequence, for the same reasons (see Eq. (4.1) and the associated discussion). Furthermore, the solution is already known, since Eq. (4.10) can be obtained from Eq. (4.7) by applying a similar derivation to that of Eq. (4.8). Thus, the solutions for $\rho^{\ell'}$ and $\sigma^{m'}$ in equation (4.10) are just those of Eqs. (4.6) and (4.9). As before, substitution of the solutions for α^{ℓ} , β^m , and γ^n from Section III then yields the desired BCH relation.

In summary, this section contains BCH relations for canonical coordinates in non-normal sequence. Schematically, the expressions obtained relate $e^{\Sigma(P+Q+J)}$ to $\Pi e^Q \Pi e^P \Pi e^J$ and certain permutations [Eq. (4.1)], to $\Pi e^P \Pi e^J \Pi e^Q$ [Eqs. (4.6) and (4.7)], to $\Pi e^Q \Pi e^J \Pi e^P$ [Eqs. (4.8) and (4.9)], and to $\Pi e^J \Pi e^P \Pi e^Q$ and certain permutations [Eq. (4.10)]. Note that all expressions remain valid in the limit in which any of the parameters p^{ℓ} , q^m , j^n are taken to zero.

V. BCH RELATIONS FOR NON-CANONICAL COORDINATES

It is also possible to obtain BCH relations between canonical coordinates of the first kind and certain non-canonical coordinate schemes. We present in this section some of the methods for obtaining such relations.

First, consider non-canonical coordinates as defined in Eq. (2.10). A relationship is readily found between canonical coordinates of the third kind in normal sequence, Eq. (2.9), and the non-canonical coordinates of Eq. (2.10). From this, BCH relation between Eqs. (2.8) and (2.10) can be established.

To begin, we note that since the J_n form a subalgebra of the Poincaré subalgebra, there exists a BCH relation of the form

$$\prod_{n'=1}^6 \exp(\gamma^{n'} J_{n'}) = \exp\left(\sum_{n=1}^6 j^n J_n\right), \quad (5.1)$$

where the $\gamma^{n'}$ are given as functions of the j^n by Eqs. (3.18) and (3.20)-(3.24). Next, recalling that the P_λ commute among themselves, we have

$$\prod_{\lambda=1}^4 \exp(\alpha^\lambda P_\lambda) = \exp\left(\sum_{\lambda=1}^4 \alpha^\lambda P_\lambda\right). \quad (5.2)$$

Finally, using the anticommutators for the Q_m in Eq. (2.1) and also lemma 2 of Ref. 7, we have

$$\begin{aligned} \prod_{m=1}^4 \exp(\beta^m Q_m) &= \exp(\beta^1 Q_1 + \beta^2 Q_2) \exp(\beta^3 Q_3 + \beta^4 Q_4) \\ &= \exp\left(\sum_{m=1}^4 \beta^m Q_m + \frac{1}{2}[\beta^1 Q_1 + \beta^2 Q_2, \beta^3 Q_3 + \beta^4 Q_4]\right) \\ &= \exp\left(\sum_{m=1}^4 \beta^m Q_m\right) \exp\left(-\frac{1}{2}[\beta^1 \beta^3 P_1 + \beta^1 \beta^4 P_2 + \beta^2 \beta^3 P_3 + \beta^2 \beta^4 P_4]\right). \end{aligned} \quad (5.3)$$

Combining Eqs. (5.1)-(5.3) and the results of Section III yields the desired BCH relation as

$$\begin{aligned} & \exp\left(\sum_{\ell=1}^4 p^{\ell} p_{\ell} + \sum_{m=1}^4 q^m Q_m + \sum_{n=1}^6 j^n J_n\right) \\ &= \exp\left(\sum_{\ell'=1}^4 a^{\ell'} P_{\ell'}\right) \exp\left(\sum_{m'=1}^4 q^{m'} Q_{m'}\right) \exp\left(\sum_{n'=1}^6 j^{n'} J_{n'}\right), \end{aligned} \quad (5.4)$$

where

$$\begin{aligned} a^1 &= p^1 - 1/2 q^1 q^3, \\ a^2 &= p^2 - 1/2 q^1 q^4, \\ a^3 &= p^3 - 1/2 q^2 q^3, \\ a^4 &= p^4 - 1/2 q^2 q^4. \end{aligned} \quad (5.5)$$

Since the P_{ℓ} commute with the Q_m , we can interchange the order of the $P_{\ell'}$ and $Q_{m'}$ exponentials on the right-hand side of Eq. (5.4) without changing the solutions (5.5). This immediately gives another BCH relation with the non-canonical coordinates in non-normal sequence:

$$\begin{aligned} & \exp\left(\sum_{\ell=1}^4 p^{\ell} p_{\ell} + \sum_{m=1}^4 q^m Q_m + \sum_{n=1}^6 j^n J_n\right) \\ &= \exp\left(\sum_{m'=1}^4 q^{m'} Q_{m'}\right) \exp\left(\sum_{\ell'=1}^4 a^{\ell'} P_{\ell'}\right) \exp\left(\sum_{n'=1}^6 j^{n'} J_{n'}\right). \end{aligned} \quad (5.6)$$

Other non-normal sequences are possible for this type of non-canonical coordinates. For instance, a BCH relation of the form

$$\begin{aligned} & \exp\left(\sum_{\ell=1}^4 p^{\ell} p_{\ell} + \sum_{m=1}^4 q^m Q_m + \sum_{n=1}^6 j^n J_n\right) \\ &= \exp\left(\sum_{m'=1}^4 q^{m'} Q_{m'}\right) \exp\left(\sum_{n'=1}^6 j^{n'} J_{n'}\right) \exp\left(\sum_{\ell'=1}^4 x^{\ell'} P_{\ell'}\right) \end{aligned} \quad (5.7)$$

may be obtained by insertion of the identity I,

$$\begin{aligned} I &= \exp\left(\sum_{k=1}^6 j^k J_k\right) \exp\left(-\sum_{k'=1}^6 j^{k'} J_{k'}\right) \\ &\equiv \prod_{k=1}^6 \exp(\gamma^k J_k) \prod_{k'=6}^1 \exp(-\gamma^{k'} J_{k'}), \end{aligned} \quad (5.8)$$

between the Q_m and P_λ exponentials on the right-hand side of Eq. (5.6). From the similarity to the analysis leading to Eq. (4.8), we find immediately

$$\underline{X} = \underline{\rho}(\alpha^\lambda, \gamma^n), \quad (5.9)$$

where $\underline{\rho}(\alpha^\lambda, \gamma^n)$ are given by Eqs. (4.9). This establishes the explicit form of the BCH relation (5.7).

Next, consider the BCH relation of the type

$$\begin{aligned} &\exp\left(\sum_{\lambda=1}^4 p^\lambda P_\lambda + \sum_{m=1}^4 q^m Q_m + \sum_{n=1}^6 j^n J_n\right) \\ &= \exp\left(\sum_{\lambda'=1}^4 a^{\lambda'} P_{\lambda'}\right) \exp\left(\sum_{n'=1}^6 j^{n'} J_{n'}\right) \exp\left(\sum_{m'=1}^4 b^{m'} Q_{m'}\right). \end{aligned} \quad (5.10)$$

We obtain this form by inserting the identity I of Eq. (5.8) between the P_λ and Q_m exponentials of Eq. (5.4) and using

$$\begin{aligned} &\prod_{k'=6}^1 \exp(-\gamma^{k'} J_{k'}) \exp\left(\sum_{m'=1}^4 q^{m'} Q_{m'}\right) \prod_{n'=1}^6 \exp(\gamma^{n'} J_{n'}) \\ &= \exp\left[\prod_{k'=6}^1 \exp(-\gamma^{k'} J_{k'}) \left(\sum_{m'=1}^4 q^{m'} Q_{m'}\right) \prod_{n'=1}^6 \exp(\gamma^{n'} J_{n'})\right], \end{aligned} \quad (5.11)$$

which follows from theorem 3 of Ref. 7. Thus, the $b^{m'}$ of Eq. (5.10) must be functions of q^m and γ^n . By comparison with the calculation leading to Eq. (4.7), we find

$$b = g(q^m, \gamma^n), \quad (5.12)$$

where $g(\beta^m, \gamma^n)$ are the solutions (4.6). Thus, the BCH relation (5.10) is also explicitly obtained.

Finally, we may readily determine the BCH relations

$$\begin{aligned} & \exp\left(\sum_{\lambda=1}^4 p^{\lambda} P_{\lambda} + \sum_{m=1}^4 q^m Q_m + \sum_{n=1}^6 j^n J_n\right) \\ &= \exp\left(\sum_{n'=1}^6 j^{n'} J_{n'}\right) \exp\left(\sum_{\lambda'=1}^4 x^{\lambda'} P_{\lambda'}\right) \exp\left(\sum_{m'=1}^4 b^{m'} Q_{m'}\right), \\ &\equiv \exp\left(\sum_{n'=1}^6 j^{n'} J_{n'}\right) \exp\left(\sum_{m'=1}^4 b^{m'} Q_{m'}\right) \exp\left(\sum_{\lambda'=1}^4 x^{\lambda'} P_{\lambda'}\right). \end{aligned} \quad (5.13)$$

The identity (5.8) is inserted in front of the P_{λ} exponential of Eq. (5.10), and the steps leading to Eq. (5.9) are repeated. Thus, the expressions for $b^{m'}$ and $x^{\lambda'}$ in Eq. (5.13) are precisely those of Eqs. (5.9) and (5.12).

In this section, we have obtained BCH relations for certain non-canonical coordinates. In schematic form, the results may be summarized as relating $e^{\Sigma(P+Q+J)}$ to $e^{\Sigma P} e^{\Sigma Q} e^{\Sigma J}$ [Eqs. (5.4) and (5.5)], to $e^{\Sigma Q} e^{\Sigma P} e^{\Sigma J}$ [Eq. (5.6)], to $e^{\Sigma Q} e^{\Sigma J} e^{\Sigma P}$ [Eqs. (5.7) and (5.9)], to $e^{\Sigma P} e^{\Sigma J} e^{\Sigma Q}$ [Eqs. (5.10) and (5.12)], and to $e^{\Sigma J} e^{\Sigma P} e^{\Sigma Q}$ or $e^{\Sigma J} e^{\Sigma Q} e^{\Sigma P}$ [Eq. (5.13)]. As for other BCH relations in this paper, the results remain valid in the limit in which any of the parameters p^{λ} , q^m , j^n are taken to zero.

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