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CROSS-SECTIONS FOR THE PRODUCTION OF
AN ARBITRARY NUMBER OF PHOTONS IN
ELECTRON-POSITRON ANNIHILATION

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ABSTRACT

Spinor techniques are used to derive an expression for the cross-section for the process $e^+e^- \rightarrow N\gamma$, where N is arbitrary. The result is convenient for numerical evaluation. We present results for typical cross-sections for the production of up to seven photons.

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1. - INTRODUCTION

The calculation of cross-sections for the production of many particles (leptons, quarks, gluons, etc.) in high energy collisions has traditionally been restricted by the technical difficulties associated with the evaluation of the corresponding multiparticle Feynman diagrams. Significant progress has however been made in recent years with the development of new techniques for evaluating tree-level helicity amplitudes. The original work of the CALKUL group¹⁾, which concentrated mainly on QED processes, has now been extended by others to cover a wide variety of processes. These include cross-sections for the production of W or Z with up to two parton jets²⁾, cross-sections for the production and decay of Higgs bosons in very high energy hadron collisions with corresponding backgrounds³⁾, and the complete calculation of all $2 \rightarrow 4$ scattering subprocesses involving quarks and gluons⁴⁾. Typically these new "spinor techniques" have given access to one or two higher orders in complexity compared to the usual trace techniques for squares of amplitudes.

In this paper we further demonstrate the power of spinor techniques by calculating exactly a physical cross-section to arbitrary order in (tree level) perturbation theory. In practice, the only restriction on the order is determined by the amount of computer time available for numerical evaluation.

We consider the process $e^+e^- \rightarrow N$ photons. For $N = 2$ and 3 simple analytic expressions for the spin-summed amplitude squared can be found in the literature¹⁾. The result for $N = 4$ is also known⁵⁾ in the form of an expression for the various helicity amplitudes which can be summed and squared numerically. Before going further, we should point out that we are not primarily interested in the phenomenology of multiphoton production, since the cross-sections rapidly become very small as N increases and $N \leq 4$ appears sufficient for current experiments. However, there are many interesting questions which can be addressed. We can study the cross-sections σ_N as a function of N and $\sqrt{s}$, we can see what limits are imposed on N by current computing resources, we can address the question of whether there may be simple underlying analytic expressions for amplitudes, as suggested by low order analyses, we can test the validity of previous "soft photon" approximations to these cross-sections and, finally, we can assess the difficulties of generalizing our results to QCD processes, e.g., $q\bar{q} \rightarrow Ng$, which are certainly of phenomenological relevance even for $N > 4$.

Some of these issues will be addressed in Section 3. In the next section we outline the method of calculation.

2. - THE CALCULATION

Consider the process

$$e^+(p_a) + e^-(p_b) \rightarrow \gamma(k_1) + \gamma(k_2) + \dots + \gamma_N(k_N) . \quad (1)$$

We shall initially calculate for massless fermions only and generalize to the massive case later. In the massless limit, helicity is conserved at each electron-photon vertex and so the helicities of the electron and positron will be the same, i.e., $\lambda_{e-} = \lambda_{e+}$. Summing over the photon spins will give the same result for the two different possibilities (++) and (--) and so we need only consider one. There are then 2^N distinct spin amplitudes (two polarizations for each photon). The Feynman diagrams are obtained simply by joining the N photons to the fermion line in all possible ways. Labelling the distinct polarization states by S (= {++,...,+}, {+-,...,+}, {--,...,+}, etc.) and the diagrams by D gives an expression for the cross-section:

$$d\sigma_N = \frac{1}{F} (d\Phi_N) \cdot \frac{1}{N!} \cdot \frac{1}{4} \sum_S \left| \sum_D M(D,S) \right|^2, \quad (2)$$

where the terms on the right-hand side are the flux factor, the phase space volume, the symmetry factor and the spin summed/averaged amplitude squared respectively.

To calculate a particular amplitude $M(D,S)$, we first relabel the momenta according to the order in which the photons join on to the fermion line. Thus, for the example shown in Fig. 1, we have

$$\begin{aligned} \hat{k}_1 &= k_3, & \lambda_1 &= - , \\ \hat{k}_2 &= k_1, & \lambda_2 &= + , \\ \hat{k}_3 &= k_2, & \lambda_3 &= + . \end{aligned} \quad (3)$$

The amplitude is then

$$M = e^N \bar{u}_+(p_a) \not{\epsilon}_{\lambda_1}(\hat{k}_1) \frac{\not{q}_1}{q_1^2} \not{\epsilon}_{\lambda_2}(\hat{k}_2) \frac{\not{q}_2}{q_2^2} \dots \not{\epsilon}_{\lambda_N}(\hat{k}_N) u_+(p_b), \quad (4)$$

$$q_i = \sum_{j=1}^i \hat{k}_j - p_a, \quad i = 1, \dots, N, \quad (q_N = p_b).$$

We can evaluate this amplitude using the spinor techniques of Ref. 6). Thus, we introduce a fundamental negative helicity spinor $u_-(k_0)$ and define the corresponding positive helicity spinor by

$$u_+(k_0) \equiv \not{k}_0' u_-(k_0), \quad (5)$$

where the four vectors k_0, k_0' have the following properties

$$k_0 \cdot k_0 = 0, \quad k_0' \cdot k_0' = -1, \quad k_0 \cdot k_0' = 0. \quad (6)$$

These spinors represent the helicity states of a massless fermion of momentum k_0 , i.e.,

$$u_\lambda(k_0) \bar{u}_\lambda(k_0) = \frac{1}{2} (1 + \lambda \gamma^5) \not{k}_0. \quad (7)$$

From these, we can construct spinors for any other light-like momentum p :

$$u_\lambda(p) = \not{p} u_\lambda(k_0) / \sqrt{2p \cdot k_0}, \quad (8)$$

which again satisfy (7) with k_0 replaced by p . For photons we can define polarization vectors corresponding to states of definite helicity, denoted by $\epsilon_\pm^\mu(k)$

$$\epsilon_\lambda^\mu(k) = \bar{u}_\lambda(k) \gamma^\mu u_\lambda(p) / \sqrt{4p \cdot k}, \quad (9)$$

where p is any light-like vector not collinear to k or k_0 ; different choices of p give polarization vectors which differ by at most a gauge transformation and a complex phase⁶⁾. It can then be shown that

$$\not{\epsilon}_\lambda(k) = \{ u_\lambda(p) \bar{u}_\lambda(k) + u_{-\lambda}(k) \bar{u}_{-\lambda}(p) \} / \sqrt{p \cdot k} . \quad (10)$$

Finally, we define the spinor products

$$\begin{aligned} s(p_1, p_2) &\equiv \bar{u}_+(p_1) u_-(p_2) = -s(p_2, p_1) , \\ t(p_1, p_2) &\equiv \bar{u}_-(p_1) u_+(p_2) = [s(p_2, p_1)]^* , \\ |s(p_1, p_2)|^2 &= 2 p_1 \cdot p_2 , \quad p_i^2 = 0 . \end{aligned} \quad (11)$$

Using the definition (8), these spinor products can be expressed in terms of the four vectors p_1, p_2, k_0, k_0' .

A useful choice in practice is to choose

$$\begin{aligned} p_i^\mu &= (p_i^0, p_i^x, p_i^y, p_i^z) , \\ k_0^\mu &= (1, 1, 0, 0) , \\ k_0'^\mu &= (0, 0, 1, 0) , \end{aligned} \quad (12)$$

which results in

$$s(p_1, p_2) = (p_1^y + i p_1^z) \left[\frac{p_2^0 - p_2^x}{p_1^0 - p_1^x} \right]^{1/2} - (p_2^y + i p_2^z) \left[\frac{p_1^0 - p_1^x}{p_2^0 - p_2^x} \right]^{1/2} . \quad (13)$$

Further discussion on the use of such spinor products can be found in Ref. 6).

The basic idea is to reduce the amplitude M in Eq. (4) to an expression involving only spinor products. To this end, we rewrite the internal momenta q_i as linear combinations of light-like momenta, i.e., we define

$$\begin{aligned}\hat{q}_i^\mu &= \zeta_i \left(q_i^\mu - \frac{q_i^2}{2p \cdot q_i} p_i^\mu \right) , \quad \hat{q}_i^2 = 0 , \\ \zeta_i &= \text{sign} \left(q_i^0 - \frac{q_i^2}{2p \cdot q_i} p_i^0 \right) ,\end{aligned}\tag{14}$$

where p is an arbitrary light-like four vector. In practice, we shall use for p the same vector used in (9) to define the polarization vectors. We can then write

$$\begin{aligned}\frac{\not{q}_i}{q_i^2} &= \zeta_i \frac{\not{\hat{q}}_i}{q_i^2} + \frac{\not{p}}{2p \cdot q_i} \\ &= \sum_\lambda \left\{ \frac{\zeta_i}{q_i^2} u_\lambda(\hat{q}_i) \bar{u}_\lambda(\hat{q}_i) + \frac{1}{2p \cdot q_i} u_\lambda(p) \bar{u}_\lambda(p) \right\} .\end{aligned}\tag{15}$$

By substituting Eqs. (10) and (15) into Eq. (4), we obtain an expression for M in terms of products of massless spinors. The final result can be written very compactly

$$\begin{aligned}M &= e^N \left(\prod_{j=1}^N p \cdot k_j \right)^{-1/2} s(p_a, a_1) t(b_N, p_b) \times \\ &\quad \prod_{i=1}^{N-1} \left\{ \frac{\zeta_i t(b_i, \hat{q}_i) s(\hat{q}_i, a_{i+1})}{q_i^2} + \frac{t(b_i, p) s(p, a_{i+1})}{2p \cdot q_i} \right\} ,\end{aligned}\tag{16}$$

where

$$\begin{aligned}a_i &= p , \quad b_i = k_i \quad \text{if } \lambda_i = + , \\ a_i &= k , \quad b_i = p \quad \text{if } \lambda_i = - .\end{aligned}$$

[Notice that since $s(p, p) = 0$, the second term inside the $\{ \}$ in Eq. (16)

vanishes if $\lambda_i = -$ or $\lambda_{i+1} = +$.] This defines the amplitude for a given diagram (equivalently, a given permutation of the k_i) and a given set of photon polarizations. The calculation is completed by summing the M as specified in Eq. (2).

3. - DISCUSSION

It is straightforward to implement the above result into a numerical evaluation of the cross-section. The fact that the result is independent of the choice of the "dummy" four-vector p provides a useful check on the computation. In principle, the direction of p should not coincide with that of any of the photons [recall the $[p \cdot k_i]^{-\frac{1}{2}}$ singularities in Eq. (16)]. In practice, however, a random generation of a finite number of Monte Carlo events is extremely unlikely to generate a k vector which is sufficiently close to a fixed vector p to cause numerical problems. There are, in addition, two special choices for p . If $p = p_a$, then the amplitude defined in Eq. (16) vanishes for $\lambda_1 = +$. Likewise the choice $p = p_b$ gives a vanishing amplitude for $\lambda_N = -$. In each case, therefore, half of the $2^N N!$ amplitudes are identically zero.

The generalization to massive fermions is conceptually rather straightforward. The key is to choose appropriate massive spinors such that the overall structure of the calculation remains the same.

Since none of the applications of our result depend critically on whether the fermion mass is non-zero, we have relegated the details to an Appendix.

It is interesting to speculate whether the N -photon amplitude squared can be written as a simple analytic expression in terms of the dot products of the four-vectors. Consider the expression

$$A_N = e^{2N} s^{N-2} \left[\sum_{i=1}^N x_i y_i (x_i^2 + y_i^2) \right] \prod_{j=1}^N (x_j y_j)^{-1},$$

$$s = (p_a + p_b)^2, \tag{17}$$

$$x_i = p_a \cdot k_i, \quad y_i = p_b \cdot k_i.$$

For $N = 2$ and $N = 3$, this describes exactly the spin summed/averaged amplitude squared¹⁾. By explicit numerical evaluation, we can show that it does not in

general give correct results for $N > 4$. (This is not surprising since the structure of A_N is not sufficient to describe the internal propagators which appear for $N > 4$.) However, it does give the correct result in the asymptotic limit where $N-2$ photons become soft and each contributes a "soft form factor" $e^2 s/(xy)$.

We have not yet succeeded in finding a generalization of Eq. (17) which is valid for arbitrary N . If such an expression does exist, it would of course give an enormous improvement in the speed of the numerical evaluation especially for large N .

For cross-section calculations, we need a phase space generator. For N massless particle production, the phase space routine RAMBO⁷⁾ is ideally suited, since it is optimally efficient, i.e., each event has the same weight. We must in addition impose cuts on the photons which avoid the singularities of the matrix element. It is straightforward to show that the cross-section diverges either when a photon becomes soft (double pole) or when it becomes collinear with either of the initial state fermions (single pole). Note that the latter are regulated by including a non-zero fermion mass. We therefore define a set of minimal cuts

$$k_i^0 > E_{\min} , \quad |k_i^z/k_i^0| < C_{\max} , \quad i=1, \dots, N .$$

For the sake of illustration, we shall use the values

$$\sqrt{s} = 100 \text{ GeV} , \quad E_{\min} = 5 \text{ GeV} , \quad C_{\max} = 0.9 , \quad (18)$$

in deriving the results which follow. With this choice, the cross-sections σ_N , $N > 100/5 = 20$, are of course zero, but this critical number is far beyond the upper limit imposed by the limitations on computer time. On an IBM 3090 we have been able to calculate up to and including $N = 7$ photons with reasonable accuracy: for this number of photons, the evaluation of each event takes approximately 1 minute of CPU time. The table shows the cross-sections σ_N and for $N = 2, \dots, 7$. As might be expected, the cross-sections decrease roughly geometrically. The table also shows the cross-sections obtained by using the approximate analytic result A_N in place of the exact matrix element. It is interesting that the discrepancy between the two appears to be always no bigger than about 30% independent of N .

APPENDIX

In this appendix we generalize the analysis of Section 2 to allow for a non-zero fermion mass. For a fermion of mass m and momentum p , we introduce the spinor [cf. Eq. (8)]

$$u_{\lambda}(p) = (\not{p} + m) u_{\lambda}(k_0) / \sqrt{2p \cdot k_0} , \quad \lambda = \pm 1 , \quad (A1)$$

with $m > 0$ for a particle and $m < 0$ for an antiparticle. There is a unique polarization vector

$$\sigma^{\mu} = \frac{1}{m} (p^{\mu} - \frac{m^2}{p \cdot k_0} k_0^{\mu}) , \quad (A2)$$

with $\sigma \cdot \sigma = -1$ and $\sigma \cdot p = 0$. The two-spin states are given by $\lambda = \pm 1$:

$$(1 - \lambda \gamma^5 \not{\sigma}) u_{\lambda}(p) = 0 , \quad (A3)$$

and the spinors are normalized in the usual way

$$\bar{u}_{\lambda}(p) u_{\lambda}(p) = 2m . \quad (A4)$$

There are now three different spinor products which, with k_0 defined in Eq. (12), are given by

$$\begin{aligned} s_{+-}(p_1, p_2) &= \bar{u}_+(p_1) u_-(p_2) \\ &= (p_1^y + i p_1^z) \eta_2 / \eta_1 - (p_2^y + i p_2^z) \eta_1 / \eta_2 , \\ s_{-+}(p_1, p_2) &= [s_{+-}(p_2, p_1)]^* , \\ s_{\pm\pm}(p_1, p_2) &= m_1 \eta_2 / \eta_1 + m_2 \eta_1 / \eta_2 , \\ \eta_i &= [2p_i^0 - 2p_i^x]^{1/2} . \end{aligned} \quad (A5)$$

Although not the only way of introducing a non-zero mass⁶⁾, the above definition has the advantage that the limit $m \rightarrow 0$ is smooth. In this limit

$$\begin{aligned} S_{+-}(p_1, p_2) &\rightarrow S(p_1, p_2), \\ S_{-+}(p_1, p_2) &\rightarrow t(p_1, p_2), \\ S_{\pm\pm}(p_1, p_2) &\rightarrow 0, \end{aligned} \quad (A6)$$

with s and t as defined before.

The generalization of Eq. (4) is

$$M_{\lambda\lambda'} = e^N \bar{u}_\lambda(p_a) \not{\epsilon}_{\lambda_1}(\hat{k}_1) \frac{\not{A}_1 + m}{q_1^2 - m^2} \not{\epsilon}_{\lambda_2}(\hat{k}_2) \dots \not{\epsilon}_{\lambda_N}(\hat{k}_N) u_{\lambda'}(p_b). \quad (A7)$$

Again, we define

$$\begin{aligned} \hat{q}_i^\mu &= \zeta_i \left(q_i^\mu - \frac{q_i^2 - m^2}{2p \cdot q_i} p^\mu \right), \quad \hat{q}_i^2 = m^2, \\ \zeta_i &= \text{sign} \left(q_i^0 - \frac{q_i^2 - m^2}{2p \cdot q_i} p^0 \right), \end{aligned} \quad (A8)$$

such that

$$\begin{aligned} \frac{\not{A}_i + m}{q_i^2 - m^2} &= \frac{\zeta_i}{q_i^2 - m^2} (\hat{A}_i + \zeta_i m) + \frac{\not{p}}{2p \cdot q_i} \\ &= \sum_{\lambda} \left\{ \frac{\zeta_i u_\lambda(\hat{q}_i) \bar{u}_\lambda(\hat{q}_i)}{q_i^2 - m^2} + \frac{u_\lambda(p) \bar{u}_\lambda(p)}{2p \cdot q_i} \right\}, \end{aligned} \quad (A9)$$

with the q_i treated as particles or antiparticles for $\zeta_i = +, -$ respectively. Proceeding as before, we eventually arrive at the final result [cf. Eq. (16)]:

$$\begin{aligned} \underline{M} &= e^N \left(\prod_{j=1}^N p \cdot k_j \right)^{-1/2} \underline{M}_a \prod_{i=1}^{N-1} \left\{ \frac{\zeta_i}{q_i^2 - m^2} \underline{M}_i(\hat{q}_i) + \frac{1}{2p \cdot q_i} \underline{M}_i(p) \right\} \underline{M}_b, \\ \underline{M} &\equiv \begin{pmatrix} M_{++} & M_{+-} \\ M_{-+} & M_{--} \end{pmatrix}, \end{aligned} \quad (A10)$$

$$\underline{M}_a = \begin{pmatrix} s_{++}(p_a, a_1) & s_{+-}(p_a, b_1) \\ s_{-+}(p_a, a_1) & s_{--}(p_a, b_1) \end{pmatrix},$$

$$\underline{M}_i(\hat{q}_i) = \begin{pmatrix} s_{++}(b_i, \hat{q}_i) & s_{+-}(b_i, \hat{q}_i) \\ s_{-+}(a_i, \hat{q}_i) & s_{--}(b_i, \hat{q}_i) \end{pmatrix} \begin{pmatrix} s_{++}(\hat{q}_i, a_{i+1}) & s_{+-}(\hat{q}_i, b_{i+1}) \\ s_{-+}(\hat{q}_i, a_{i+1}) & s_{--}(\hat{q}_i, b_{i+1}) \end{pmatrix},$$

$$\underline{M}_b = \begin{pmatrix} s_{++}(b_N, p_b) & s_{+-}(b_N, p_b) \\ s_{-+}(a_N, p_b) & s_{--}(a_N, p_b) \end{pmatrix}. \quad \begin{matrix} \text{(A10)} \\ \text{cont.} \end{matrix}$$

Notice that the complex multiplication of Eq. (16) has been replaced by complex 2×2 matrix multiplication, but otherwise the structure remains the same. It is straightforward to show that in the limit $m \rightarrow 0$ the off diagonal elements of M vanish and the result of Section 2 (and the corresponding helicity reversed result) obtains.

TABLE

Cross-sections for $\sigma_N(e^+e^- \rightarrow N\gamma)$ for different N , with cuts as described in the text. The statistical errors are from the Monte Carlo phase space integration. Also shown is the cross-section $\tilde{\sigma}_N$ obtained using the approximate (for $N > 4$) analytic expression for the matrix element squared given in Eq. (17).

N	σ_N (nb)	$\tilde{\sigma}_N$
2	0.27×10^{-1}	0.27×10^{-1}
3	0.63×10^{-3}	0.63×10^{-3}
4	$(0.71 \pm 0.03) \times 10^{-5}$	0.67×10^{-5}
5	$(0.47 \pm 0.03) \times 10^{-7}$	0.41×10^{-7}
6	$(0.19 \pm 0.03) \times 10^{-9}$	0.15×10^{-9}
7	$(0.36 \pm 0.13) \times 10^{-12}$	0.36×10^{-12}

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FIGURE CAPTION

A Feynman diagram for $N = 3$ showing the different moment labellings as described in the text.