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ANOMALOUS BUT CONSISTENT GRAVITY IN TWO DIMENSIONS*

Keke Li

Center for Theoretical Physics
Laboratory for Nuclear Science
and Department of Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139 U.S.A.

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ABSTRACT

Two-dimensional gravity coupled to chiral fermions possesses a Lorentz anomaly, but it can be made consistent by allowing the gauge degree of freedom to be physical at the quantum level. The theory has massless excitations or a ghost particle depending on the sign of a counterterm. The possibility of adding the Wess-Zumino term in order to cancel the anomaly is also discussed.

It has been shown that anomalous gauge theories, although gauge variant, may yield consistent quantum dynamics,^{1,2} because some gauge degrees of freedom, which decouple at the classical level, become coupled to the system after quantization, due to gauge anomalies.³ In this context, the covariant conservation of the gauge current is imposed as an additional field equation for the gauge fields, and the apparent inconsistency disappears. It is interesting to establish that this also happens in gravitationally anomalous theories.⁴

When there are gravitational anomalies, the energy-momentum tensor of the matter fields will no longer be covariantly conserved and symmetric. So, its coupling to Einstein gravity would seem to be inconsistent. However, as we shall show in the following for an example in two dimensions, it is possible to obtain a consistent quantum theory by allowing some gauge degrees of freedom to be physical at the quantum level.

Ordinary Einstein gravity cannot be formulated in two space-time dimensions, since the Einstein tensor $G_{\mu\nu}$ vanishes identically and the Hilbert-Einstein action $\int d^2x \sqrt{-g} R$ is a surface term. But a nontrivial invariant action can be obtained by dimensionally reducing the three-dimensional action.⁵ The Einstein equation is replaced by the constant curvature scalar equation [see Eq. (2) below].*

We shall consider the two-dimensional gravity coupled to a set of chiral fermions. Both the gravitational and fermionic actions are obtained from three-dimensional actions by dimensional reduction. We take the three-dimensional metric and dreibein to be

$$g_{UV} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & -N^2 \end{pmatrix},$$

$$e_U^A = \begin{pmatrix} e_\mu^a & 0 \\ 0 & N \end{pmatrix},$$

* This equation emerges in other formulations of two-dimensional gravity as well,⁶ but local and general covariant actions are not available there. See also Ref. 7.

where $U, V, A = 0, 1, 2$. $\mu, \nu, a = 0, 1$: and the Minkowski metric is $\eta_{AB} = \text{diag}(1, -1, -1)$. The γ -matrices are defined as: $\gamma^0 = \sigma_1, \gamma^1 = i\sigma_2, \gamma^2 = \gamma^5 = i\gamma^0\gamma^1 = -i\sigma_3, \Gamma^{AB} = \frac{1}{4}[\gamma^A, \gamma^B]$ and $\{\gamma^A, \gamma^B\} = 2\eta^{AB}$. The dependence on the third coordinate is ignored and the reduction for the curvature scalar and the Dirac operator is:

$$R^{(3)} \rightarrow R + \frac{2}{N} \nabla_\mu \nabla^\mu N$$

$$\begin{aligned} e_A^\mu \gamma^A D_\mu^{(3)} &\equiv e_A^\mu \gamma^A \left(\partial_\mu + \frac{1}{2} \omega_{\mu, BC} \Gamma^{BC} \right) \\ &\rightarrow e_a^\mu \gamma^a \left(\partial_\mu + \frac{1}{2} \omega_{\mu, bc} \Gamma^{bc} + \frac{1}{2} \partial_\mu \ln N \right) \\ &\equiv e_a^\mu \gamma^a \left(D_\mu + \frac{1}{2} \partial_\mu \ln N \right) \end{aligned}$$

where ∇_μ is the covariant derivative. The three-dimensional gravitational action

$$I_G^{(3)} = -\frac{1}{16\pi G^{(3)}} \int d^3x \sqrt{g^{(3)}} (R^{(3)} + 2\Lambda)$$

is reduced to

$$I_G = -\frac{1}{16\pi G} \int d^2x \cdot N \sqrt{-g} (R + 2\Lambda)$$

where a surface term has been dropped. We reduce the fermion field according to:

$$\psi^{(3)} \rightarrow \frac{1}{\sqrt{N}} \psi,$$

so that the fermionic action in three dimensions

$$I_F^{(3)} = \int d^3x \sqrt{g^{(3)}} \frac{1}{2} i e_A^\mu \bar{\psi}^{(3)} \gamma^A \overleftrightarrow{D}_\mu^{(3)} \psi^{(3)}$$

reduces to the usual two-dimensional one:

$$I_F = \int d^2x \sqrt{-g} \cdot \frac{1}{2} i e_a^\mu \bar{\psi} \gamma^a \overleftrightarrow{\partial}_\mu \psi$$

Note that I_F is independent of the scalar factor N and the spin connection is absent from this Hermitian form for the action. In the following, we shall let ψ denote a set of n_L left-handed and n_R right-handed chiral fermions. Absorbing the dimensionless constant $(-\frac{1}{16\pi G})$ into N in I_G , we arrive at the action for the two-dimensional gravity coupled to the set of chiral fermions:

$$I = \int d^2x N \sqrt{-g} (R + 2\Lambda) + \int d^2x \sqrt{-g} \frac{1}{2} i e_a^\mu \bar{\psi} \gamma^a \overleftrightarrow{\partial}_\mu \psi \quad (1)$$

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The action I is general-coordinate invariant and local-Lorentz invariant. The classical equations of motion are

$$R + 2\Lambda = 0 \quad (2)$$

$$i\gamma^\mu \left(\partial_\mu + \frac{1}{2} \omega_{\mu, ab} \Gamma^{ab} \right) \psi = 0 \quad (3)$$

$$N_{\mu\nu} \equiv (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2) N + g_{\mu\nu} \Lambda N = 2T_{\mu\nu} \quad (4)$$

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{4} i \bar{\psi} \gamma_{(\mu} \overleftrightarrow{\partial}_{\nu)} \psi, \\ \gamma_\mu &\equiv e_\mu^a \gamma_a \end{aligned}$$

The tensor $N_{\mu\nu}$ is symmetric: $N_{\mu\nu} = N_{\nu\mu}$. Furthermore,

$$\nabla^\nu N_{\mu\nu} \equiv \frac{1}{2} (R + 2\Lambda) \nabla_\mu N = 0,$$

where Eq. (2) is used. Therefore, consistency requires that the energy-momentum tensor $T_{\mu\nu}$ should be symmetric and covariantly conserved.

This consistency is satisfied classically as a consequence of general-coordinate and local-Lorentz invariances of the classical action, and the gauge degrees of freedom corresponding to these invariances are absent from the theory. After quantization, the invariances are lost if $n_R \neq n_L$ because of anomalies. Correspondingly, the gauge degrees of freedom become coupled to the system. The loss of general-coordinate invariance would make the theory no longer geometrical. Fortunately, it is always possible to find a regularization with no general coordinate anomalies.⁸ In particular, Leutwyler has used a general covariant regularization to compute the exact functional determinant for the Weyl operator and has argued that Lorentz anomalies are more fundamental.⁹ We shall follow Leutwyler's approach.

In our case, the Weyl operator is:

$$\begin{aligned} e_a^\mu \gamma^a \left(\partial_\mu + \frac{1}{2} \omega_{\mu, ab} \Gamma^{ab} \right) \frac{1 \pm i\gamma_5}{2} &= e_a^\mu \gamma^a \left(\partial_\mu \pm \frac{1}{4} \omega_\mu \right) \frac{1 \pm i\gamma_5}{2} \\ \omega_\mu &\equiv \epsilon^{ab} \omega_{\mu, ab} = \epsilon^{ab} e_a^\mu \nabla_\mu \epsilon_{\nu b} \end{aligned}$$

where $\pm$ correspond to left- and right-handed fermions.

The effective action, obtained by functionally integrating out the fermions, is given by Leutwyler:

$$\begin{aligned} \exp \{ iW [e_\mu^a] \} &= \int D\bar{\psi} D\psi \cdot \exp(iI_F) \\ W[e_\mu^a] &= \frac{1}{192\pi} \int d^2x \sqrt{-g} R \cdot \frac{1}{\sqrt{-g} \nabla^2} \cdot [\alpha \sqrt{-g} R + \beta \partial_\mu \sqrt{-g} g^{\mu\nu} \omega_\nu] \end{aligned} \quad (5)$$

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where $\nabla^2 = \nabla_\mu \nabla^\mu$, $\alpha = n_R + n_L$ and $\beta = n_R - n_L$.

$W[e_\mu^a]$ is defined up to local counterterms. There are only three possible general covariant local counterterms: $\alpha_1 \sqrt{-g} R$, $\mu \sqrt{-g}$ and $\frac{a}{2} \sqrt{-g} g^{\mu\nu} \omega_\mu \omega_\nu$. The first term is a surface term and thus does not contribute. In general, the complete action after integrating out the fermions reads

$$I_e = \int d^2x \left[N \sqrt{-g} (R + 2\Lambda) + \frac{1}{192\pi} \sqrt{-g} R \right. \\ \left. \cdot \frac{1}{\sqrt{-g} \nabla^2} \cdot (\alpha \sqrt{-g} R + \beta \partial_\mu \sqrt{-g} g^{\mu\nu} \omega_\nu) + \mu \sqrt{-g} + \frac{a}{2} \sqrt{-g} g^{\mu\nu} \omega_\mu \omega_\nu \right], \quad (6)$$

where μ and a are arbitrary constants. Note that for $\beta = n_R - n_L \neq 0$, no choice of a can make I_e invariant under local Lorentz transformation:

$$\omega_\mu \rightarrow \omega_\mu + \partial_\mu \tilde{\theta}$$

This is the Lorentz anomaly. Therefore, the gauge degree of freedom corresponding to local Lorentz rotation, called Lorentz variable in the following, is always present.

In order to study the consistency of the theory in Hamiltonian formalism, we need to localize the effective action I_e . This can be done by introducing a scalar field φ :

$$\exp\{iI_e[N, e_\mu^a]\} = \int D\varphi \exp\{iI_b[N, e_\mu^a, \varphi]\} \\ I_b = \int d^2x \left\{ N \sqrt{-g} (R + 2\Lambda) + \frac{1}{2} \sqrt{-g} g^{\mu\nu} (\partial_\mu \varphi) (\partial_\nu \varphi) \right. \\ \left. + \varphi (c \sqrt{-g} R + b \partial_\mu \sqrt{-g} g^{\mu\nu} \omega_\nu) + \mu \sqrt{-g} + \frac{a'}{2} \sqrt{-g} g^{\mu\nu} \omega_\mu \omega_\nu \right\} \quad (7)$$

where b , c and a' are constants and determined by performing the functional integration as shown below. Let $\varphi = \tilde{\varphi} + \varphi_c$, where φ_c is the classical solution of the scalar field:

$$\varphi_c = \frac{1}{\sqrt{-g} \nabla^2} (c \sqrt{-g} R + b \partial_\mu \sqrt{-g} g^{\mu\nu} \omega_\nu)$$

The action I_b becomes:

$$I_b = \int d^2x \left\{ N \sqrt{-g} (R + 2\Lambda) + \frac{1}{2} \sqrt{-g} g^{\mu\nu} (\partial_\mu \tilde{\varphi}) (\partial_\nu \tilde{\varphi}) \right. \\ \left. + \frac{1}{2} (c \sqrt{-g} R + b \partial_\mu \sqrt{-g} g^{\mu\nu} \omega_\nu) \frac{1}{\sqrt{-g} \nabla^2} (c \sqrt{-g} R + b \partial_\mu \sqrt{-g} g^{\mu\nu} \omega_\nu) \right. \\ \left. + \mu \sqrt{-g} + \frac{a'}{2} \sqrt{-g} g^{\mu\nu} \omega_\mu \omega_\nu \right\}$$

Now the functional integration over the scalar field can be done and the result is well-known:¹⁰

$$\int D\tilde{\varphi} \exp \left[i \int d^2x \frac{1}{2} \sqrt{-g} g^{\mu\nu} (\partial_\mu \tilde{\varphi}) (\partial_\nu \tilde{\varphi}) \right] \\ \exp \left[\frac{i}{96\pi} \int d^2x \sqrt{-g} R \cdot \frac{1}{\sqrt{-g} \nabla^2} \sqrt{-g} R + \text{const.} \cdot i \int d^2x \sqrt{-g} \right] \quad (8)$$

Therefore, the constants b , c and a' must satisfy

$$\frac{b^2 + c^2}{2} + \frac{1}{96\pi} = \frac{1}{192\pi} (n_R + n_L) \\ bc = \frac{1}{192\pi} (n_R - n_L) \\ a' = a + b^2$$

The constant in (8) is absorbed into the arbitrary constant μ in (6).

Different solutions for b , c , and a' can be shown to be equivalent by redefining the scalar field φ . We can take

$$b = \frac{1}{8\sqrt{3\pi}} (\sqrt{n_R - 1} - \sqrt{n_L - 1}) \\ c = \frac{1}{8\sqrt{3\pi}} (\sqrt{n_R - 1} + \sqrt{n_L - 1})$$

Note that b and c are real for $n_R, n_L \geq 1$.

Next, we construct a Hamiltonian formalism for the general-coordinate invariant local action I_b , which is equivalent to the original effective theory, quantum mechanically. The metric tensor is decomposed in the standard way:

$$g_{\mu\nu} = e^\phi \begin{pmatrix} \eta_0^2 - \eta_1^2 & -\eta_1 \\ -\eta_1 & -1 \end{pmatrix}_{\mu, \nu} \quad (9)$$

The dreibein e_μ^a depends on the Lorentz variable F also:

$$e_\mu^a = \begin{pmatrix} \cosh \frac{1}{2} F & -\sinh \frac{1}{2} F \\ -\sinh \frac{1}{2} F & \cosh \frac{1}{2} F \end{pmatrix}_{a, b} e_\mu^b \quad (10)$$

The spin connection ω_μ is related to F in a simple way:

$$\omega_\mu = \epsilon^{ab} e_\mu^a \nabla_\mu e_{\nu b} = \partial_\mu F + \epsilon^{ab} \partial_\mu^\nu \nabla_\mu \tilde{e}_{\nu b} \\ \equiv \partial_\mu F + \tilde{\omega}_\mu, \\ \tilde{\omega}_1 = \frac{1}{\eta_0} (\dot{\phi} - 2\eta_1' - \eta_1 \phi'), \\ \tilde{\omega}_0 = \eta_1 \tilde{\omega}_1 + \phi' \eta_0 + 2\eta_0',$$

where the dots and the primes denote the derivatives with respect to the time and space coordinates, respectively. By dropping surface terms, the action I_0 can be written in the explicit form

$$I_0 = \int d^2x \left\{ 2\Delta\eta_0 e^\phi (\dot{\zeta} - c\chi + bcF) + \mu\eta_0 e^\phi \right. \\ + (\dot{\phi}\eta_0 + 2\eta'_0)\dot{\zeta}' - \frac{1}{\eta_0}(\dot{\phi} - 2\eta'_1 - \eta_1\dot{\phi}')(\dot{\zeta} - \eta_1\dot{\zeta}') \\ + \frac{a}{2\eta_0} \left[(\eta_0\dot{\phi}' + 2\eta'_0 + \dot{F} - \eta_1 F')^2 - (\dot{\phi} - 2\eta'_1 - \eta_1\dot{\phi}' + \eta_0 F')^2 \right] \\ \left. + \frac{1}{2\eta_0} \left[(\dot{\chi} - \eta_1\dot{\chi}' + b\eta_0\dot{\phi}' + 2b\eta'_0)^2 - (b\dot{\phi} - 2b\eta'_1 - b\eta_1\dot{\phi}' + \eta_0\dot{\chi}')^2 \right] \right\} \quad (11)$$

where, for later convenience, new variables ζ and χ are used:

$$\zeta \equiv N + c\varphi \\ \chi \equiv \varphi + bF$$

Note that the time derivatives of η_0 and η_1 do not appear in I_0 . So, their conjugate momenta are constrained to vanish and η_0 and η_1 are not determined by the dynamics of the system. Considering ϕ , ζ , χ and F as dynamical variables, we can put the action I_0 into the canonical form:

$$I_0 = \int d^2x \left[\dot{\phi}P_\phi + \dot{\zeta}P_\zeta + \dot{\chi}P_\chi + \dot{F}P_F - \eta_0 H_0 - \eta_1 H_1 \right] \\ H_1 = -2P'_\phi + \phi'P_\phi + \zeta'P_\zeta + \chi'P_\chi + F'P_F \quad (12)$$

$$H_0 = 2\zeta'' + 2P'_F + 2bP'_X - \zeta'\phi' - \phi'P_F - b\phi'P_X - P_\zeta P_\phi - \mu e^\phi \\ - 2\Delta e^\phi (\zeta - c\chi + bcF) + \left[\frac{1}{2}P_X^2 + \frac{1}{2}(\chi' - P_\zeta)^2 + \frac{1}{2a}P_a^2 + \frac{a}{2}(P'_\zeta - P'_\zeta)^2 \right], \quad (13)$$

where P_ϕ , P_ζ , P_χ and P_F are the conjugate momenta.

The equations of motion impose the constraints $H_0 = H_1 = 0$. It is easy to check that H_0 and H_1 satisfy closed algebras:

$$\{H_0(x), H_0(y)\}_{P.B.} = \partial_x \delta(x-y) [H_1(x) + H_1(y)] \\ \{H_1(x), H_0(y)\}_{P.B.} = \partial_x \delta(x-y) [H_0(x) + H_0(y)] \\ \{H_1(x), H_1(y)\}_{P.B.} = \partial_x \delta(x-y) [H_1(x) + H_1(y)]$$

This simply reflects the general covariance of the localized action I_0 . From the structures of the first class constraints H_1 and H_0 , it is already clear that the theory with $a > 0$ describes two massless bosonic fields χ and F interacting with the gravity.

For $a < 0$, the Lorentz variable F represents a ghost particle and the theory is not physically acceptable.

In one approach to quantization, the two constraints are imposed on physical states. The pure gravity theory without matter admits exact classical solutions.¹¹ By choosing boundary conditions compatible with the classical solutions in an open space, the exact quantum wave functional satisfying the two constraints has been found.¹² But in our case, this approach is very difficult, since our constraints are much more complicated and general classical solutions are not known. Alternatively, we shall consider the space to be closed and choose gauge conditions corresponding to the two first class constraints and quantize only independent modes. Note that although H_0 and H_1 form closed algebras at this classical level, a Schwinger term may appear in $[H_1, H_0]$ after quantization.¹³ Then the constraints H_1 and H_0 should be replaced by their positive frequency parts which again form the same closed algebras and the general covariance remains intact.

If $\Delta = \mu = 0$, we can make linear approximations for H_1 and H_0 :

$$H_1 \simeq 2P'_\phi = 0 \\ H_0 \simeq 2\zeta'' + 2P'_F + 2bP'_X = 0$$

This suggests that the gauge conditions should be taken as:

$$\dot{\phi}' = 0 \\ P'_\zeta = 0 \quad (14)$$

The consistency of the gauge conditions with the dynamics determines the arbitrary Lagrangian multiplier η_1 and η_0 and hence uniquely specifies the evolution of the system. The consistency equations are:

$$0 = \dot{\phi}' = \left\{ \phi', \int dx [\eta_0 H_0 + \eta_1 H_1] \right\}_{P.B.} = 2\eta_1'' + (\phi'\eta_1)' - (P_\zeta \eta_0)' \\ 0 = \dot{P}'_\zeta = \left\{ P'_\zeta, \int dx [\eta_0 H_0 + \eta_1 H_1] \right\}_{P.B.} = (P'_\zeta \eta_1)'' - (\phi'\eta_0)'' - 2\eta_0''' + (2\Delta e^\phi \eta_0)'.$$

The above equations are satisfied with $\eta_1 = 0$ and $\eta_0 = 1$. Thus, the gauge conditions (14) are appropriate.

With the conditions (14) we have eliminated the propagating degrees of freedom corresponding to $\phi(x, t)$ and $P_\zeta(x, t)$, leaving only two global (zero-momentum) degrees of freedom $\phi(t)$ and $P_\zeta(t)$ with their conjugate variables defined as:

$$\Pi_\phi(t) = \int dx P_\phi(x, t), \\ Z(t) = \int dx \zeta(x, t).$$

In terms of these variables,

$$\begin{aligned} I_b = & \int dt \left(\dot{\phi} \Pi_\phi + \dot{Z} P_Z \right) + \int d^2x \left(\dot{\chi} P_\chi + \dot{F} P_F \right) \\ & - \int dt \left[-2\Lambda e^\phi Z - P_Z \Pi_\phi - \mu e^\phi + \frac{1}{2}(a+b^2)P_Z^2 \right] \\ & - \int d^2x \left[\frac{1}{2} P_\chi^2 + \frac{1}{2} \chi'^2 + \frac{1}{2a} P_F^2 + \frac{a}{2} F'^2 + 2\Lambda e^\phi (c\chi - bcF) \right], \end{aligned}$$

where the gauge conditions are used and $0 \leq x \leq 1$ is implied.

We can now proceed to quantize the theory by identifying $\{F, P_F\}$ and $\{\chi, P_\chi\}$ as quantum field variables and $\{\phi, \Pi_\phi\}$ and $\{Z, P_Z\}$ as quantum mechanical variables, and replacing their Poisson brackets by operator commutators. The quantum Hamiltonian of the system is:

$$\begin{aligned} H = & -2\Lambda e^\phi Z - P_Z \Pi_\phi - \mu e^\phi + \frac{1}{2}(a+b^2)P_Z^2 \\ & + \int d^2x \left[\frac{1}{2} P_\chi^2 + \frac{1}{2} \chi'^2 + \frac{1}{2a} P_F^2 + \frac{a}{2} F'^2 + 2\Lambda e^\phi (c\chi - bcF) \right] \end{aligned} \quad (15)$$

The constraints are not completely solved, but take the integrated forms:

$$\tilde{H}_1 = \int dx H_1 = \int dx (\chi' P_\chi + F' P_F) = 0 \quad (16)$$

$$\tilde{H}_1 = \int dx H_0 = H = 0 \quad (17)$$

For $\Lambda = 0$, the two bosonic fields are completely decoupled from the global variables representing the gravity. The theory contains two massless free bosonic fields F and χ , and is unitary for $a > 0$. If $a < 0$, the theory contains a ghost field and is inconsistent.

For $\Lambda \neq 0$, the two bosonic fields are interacting with the global variables. The field equations for F and χ in the conformal gauge $\eta_1 = 0$ and $\eta_0 = 1$ are:

$$\begin{aligned} \square \chi + 2c\Lambda e^\phi &= 0 \\ \square F - 2bc\Lambda e^\phi &= 0 \end{aligned}$$

For $\beta \equiv n_R - n_L \neq 0$, the classical solutions must break the translation invariances. It is difficult to analyze such finds of systems. But we can still conclude that the free part of the theory has a (no) ghost field if $a < 0$ ($a > 0$).

The two constraints (16) and (17), although strange, appear similarly when two scalar fields are coupled to the gravity with the following action:

$$\begin{aligned} I_s = & \int d^2x \left[\sqrt{-g} N(R + 2\Lambda) + \frac{1}{2} \sqrt{-g} g^{\mu\nu} (\partial_\mu \varphi_1 \cdot \partial_\nu \varphi_1 + \partial_\mu \varphi_2 \cdot \partial_\nu \varphi_2) \right] \\ = & \int d^2x \left[\dot{\varphi}_1 P_1 + \dot{\varphi}_2 P_2 + \dot{\phi} P_\phi + \dot{N} P_N - \eta_0 h_0 - \eta_1 h_1 \right] \end{aligned}$$

The two constraints are:

$$\begin{aligned} h_1 = & -2P'_\phi + \phi' P_\phi + N' P_N + \phi'_1 P_1 + \phi'_2 P_2 = 0 \\ h_0 = & -2\Lambda N e^\phi + 2N'' - N' \phi' - P_N P_\phi + \left[\frac{1}{2} P_1^2 + \frac{1}{2} \phi_1'^2 + \frac{1}{2} P_2^2 + \frac{1}{2} \phi_2'^2 \right] = 0 \end{aligned}$$

which, after the gauge has been fixed as: $\phi' = 0$, $P'_N = 0$, $P'_N = 0$, $\eta_1 = 0$ and $\eta_0 = 1$, take the integrated forms:

$$\begin{aligned} \tilde{h}_1 = & \int dx h_1 = \int dx [\phi'_1 P_1 + \phi'_2 P_2] = 0 \\ \tilde{h}_0 = & \int dx h_0 = -2\Lambda N e^\phi - P_N \Pi_\phi + \int dx \left[\frac{1}{2} P_1^2 + \frac{1}{2} \phi_1'^2 + \frac{1}{2} P_2^2 + \frac{1}{2} \phi_2'^2 \right] = 0 \end{aligned}$$

where $n(t) = \int dx N(x, t)$ and $\Pi_\phi(t) = \int dx P_\phi(x, t)$. Thus, our effective theory with $\Lambda = 0$ and $a > 0$ is equivalent to the gravity coupled with two massless scalar fields, except some terms in the Hamiltonian involving only the global variables which are decoupled anyway.

When the arbitrary parameter a takes the critical value $a = 0$, $\dot{F}$ does not appear in I_b and an additional constraint is generated:

$$P_F = \frac{\delta I_b}{\delta \dot{F}} = 0 \quad (18)$$

The secondary constraint from (18) is:

$$2bc \Lambda \eta_0 e^\phi = 0.$$

Since η_0 must always be positive, this constraint cannot be met for $\beta = n_R - n_L \neq 0$ unless $\Lambda = 0$. Thus, the theory is inconsistent for $\Lambda \neq 0$ and $a = 0$.

For $\Lambda = 0$ and $a = 0$, the Lorentz variable F disappears from the action (11) completely. This can be understood from the effective action (6). For $\Lambda = 0$ and $a = 0$, although the action:

$$I_e = \int d^2x \left[N \sqrt{-g} R + \frac{1}{192\pi} \sqrt{-g} R \cdot \frac{1}{\sqrt{-g} \nabla^2} (\alpha \sqrt{-g} R + \beta \partial_\mu \sqrt{-g} g^{\mu\nu}) + \mu \sqrt{-g} \right]$$

is not invariant under local Lorentz transformation $\omega_\mu \rightarrow \omega_\mu + \partial_\mu \tilde{\theta}$, it is invariant, however, under

$$\omega_\mu \rightarrow \omega_\mu + \partial_\mu \tilde{\theta}, \quad N \rightarrow N - \frac{\beta}{192\pi} \tilde{\theta}.$$

Because of this new gauge symmetry, the Lorentz variable F is decoupled from the physical theory. The system consists of only one free massless boson representing matter degrees of freedom, together with the global variables which are decoupled. The theory is consistent.

Similar phenomenon happens also in the chiral Schwinger model.² There, when the coefficient of a counterterm takes the critical value $\alpha = 1$, new constraints appear. As a result, although the theory is still gauge non-invariant, the degrees of freedom corresponding to the gauge field are completely eliminated and only matter degrees of freedom are left.

Finally, let's consider the possibility of adding a Wess-Zumino term in the action so that the Lorentz anomaly is cancelled. Faddeev has suggested that adding a Wess-Zumino term into the action may change the second class Gauss' constraints in an anomalous gauge theory to the first class ones, thus restoring the gauge invariance.³ This approach is equivalent to the one where no Wess-Zumino field is introduced but gauge invariance is abandoned, in the sense that in both cases, some degrees of freedom, being either the Wess-Zumino field or the gauge degrees of freedom, although decoupled at classical level, become physical after quantization. Another way of seeing the equivalence is that in the approach with Wess-Zumino field introduced, the Wess-Zumino term vanishes after a unitary gauge has been chosen. These ideas have been applied to the chiral Schwinger model.¹⁴

In our case, the Wess-Zumino term to be added in the action is:

$$W[N, g_{\mu\nu}, \omega_\mu, \theta] = I_e[N, g_{\mu\nu}, \omega_\mu] - I_e[N, g_{\mu\nu}, \omega_\mu] \\ = \int d^2x \left[\frac{\alpha}{2} \sqrt{-g} g^{\mu\nu} \partial_\mu \theta \cdot \partial_\nu \theta + \alpha \sqrt{-g} g^{\mu\nu} \omega_\mu \partial_\nu \theta \right. \\ \left. + \frac{\beta}{192\pi} \theta \sqrt{-g} R \right],$$

where $\omega_\mu^\theta \equiv \omega_\mu + \partial_\mu \theta$ and I_e is the effective action(6). With the addition of $W[N, g_{\mu\nu}, \omega_\mu, \theta]$, the effective action becomes just $I_e[N, g_{\mu\nu}, \omega_\mu^\theta]$ which is, of course, invariant under local Lorentz transformation:

$$\omega_\mu \rightarrow \omega_\mu + \partial_\mu \tilde{\theta}, \quad \theta \rightarrow \theta - \tilde{\theta}.$$

As a consequence, the Lorentz variable F decouples. This effective action can be localized in the same way. If we decompose the metric and the dreibein according to (9) and (10) and choose the gauge $F = 0$, the localized effective action has exactly the same form as (11) with the Lorentz variable F replaced by the Wess-Zumino field θ . Thus, an identical physical content is obtained. In fact, we can observe from the Wess-Zumino action $W[N, g_{\mu\nu}, \omega_\mu, \theta]$ that the Wess-Zumino field θ describes a massless scalar particle interacting with gravity for $\alpha > 0$, while for $\alpha < 0$ it is a ghost. For $\alpha = 0$, θ is no longer dynamical but only serves as a Lagrangian multiplier, with corresponding constraint $\frac{\beta}{192\pi} \sqrt{-g} R = 0$, which is consistent with the field equation (2) only if $\Lambda = 0$.

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